The lifts of surfaces in neutral 4-manifolds into the 2-Grassmann bundles

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Contents

1. Minimal surfaces in Riemannian 4-manifolds

N: an oriented Riemannian 4-dimensional manifold with its metric h.

$\Longrightarrow$ For $a \in N$, the eigenvalues of $* : \bigwedge^2 T_a N \longrightarrow \bigwedge^2 T_a N$ are ± 1, and the corresponding eigenspaces are of dimension 3.

We have a bundle decomposition

$$\bigwedge^2 TN = \bigwedge^2_+ TN \oplus \bigwedge^2_- TN$$

(notice the double covering $SO(4) \longrightarrow SO(3) \times SO(3)$).

We see that $\bigwedge^2_\pm TN$ are locally generated by

$$\frac{1}{\sqrt{2}}(\theta_{12} \pm \theta_{34}), \quad \frac{1}{\sqrt{2}}(\theta_{13} \pm \theta_{42}), \quad \frac{1}{\sqrt{2}}(\theta_{14} \pm \theta_{23}),$$

where $\theta_{ij} := e_i \wedge e_j$ and (e_1, e_2, e_3, e_4) is a local ordered orthonormal frame field of TN giving the orientation of N.

- If N is hyperKähler, then one of $\bigwedge_{\pm}^2 TN$ is a product bundle.
- If $N = E^4$, then both of $\bigwedge_{\pm}^2 TN$ are product bundles.

The twistor spaces associated with N are the sphere bundles in $\bigwedge_{\pm}^2 TN$:

$$U\left(\bigwedge_{+}^2 TN\right) := \left\{\Theta \in \bigwedge_{+}^2 TN \mid \hat{h}(\Theta, \Theta) = 1\right\},$$

$$U\left(\bigwedge_{-}^2 TN\right) := \left\{\Theta \in \bigwedge_{-}^2 TN \mid \hat{h}(\Theta, \Theta) = 1\right\}.$$

M: a Riemann surface,

$F : M \longrightarrow N$: a conformal immersion of a Riemann surface M into N.

$\Theta_{F,\pm}$: sections of $U\left(\bigwedge^2_{\pm} F^*TN\right)$ defined by $\Theta_{F,\pm} := \dfrac{1}{\sqrt{2}}(\xi_1 \wedge \xi_2 \pm \xi_3 \wedge \xi_4)$,

where ξ_1, ξ_2, ξ_3, ξ_4 form a local orthonormal frame field of F^*TN s.t.

- $(\xi_1, \xi_2, \xi_3, \xi_4)$ gives the orientation of N,
- $\xi_1, \xi_2 \in dF(TM)$ so that (ξ_1, ξ_2) gives the orientation of M.

$I_{F,\pm}$: the complex structures of F^*TN corresponding to $\Theta_{F,\pm}$.

Then $\Theta_{F,\pm} = \dfrac{1}{\sqrt{2}}(e \wedge I_{F,\pm}(e) + e^{\perp} \wedge I_{F,\pm}(e^{\perp}))$,

where e (respectively, $e^{\perp}$) is a unit tangent (respectively, normal) vector of F.

If N is hyperKähler so that $\bigwedge^2_{+} TN$ (respectively, $\bigwedge^2_{-} TN$) is a product bundle, then we can consider $\Theta_{F,+}$ (respectively, $\Theta_{F,-}$) to be a map from M into $\mathbb{C}P^1$.

Theorem (A, 2020)
Suppose that N *is hyperKähler and that* $F : M \longrightarrow N$ *is minimal. Then one of* $\Theta_{F,+}, \Theta_{F,-}$ *is a holomorphic map from* M *into* $\mathbb{C}P^1$.

In particular, we have the following corollary, which is a well-known theorem (see pp. 16–22 in D. A. Hoffman and R. Osserman, *The geometry of the generalized Gauss map*, Memoirs of AMS **236**, 1980).

Corollary
$F : M \longrightarrow E^4$: *a conformal and minimal immersion of* M *into* E^4. *Then the Gauss map* $\mathcal{G}_F : M \longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ *of* F *is holomorphic.*

Proof of the theorem

Suppose that $\bigwedge_+^2 TN$ is a product bundle.

Then we can suppose that

$$\Theta_{+,1} := \frac{1}{\sqrt{2}}(\theta_{12} + \theta_{34}), \ \ \Theta_{+,2} := \frac{1}{\sqrt{2}}(\theta_{13} + \theta_{42}), \ \ \Theta_{+,3} := \frac{1}{\sqrt{2}}(\theta_{14} + \theta_{23})$$

are horizontal. These sections form an orthonormal frame field of $\bigwedge_+^2 TN$.

$g_{F,+}$: a $\mathbb{C}P^1$-valued function satisfying

$$\Theta_{F,+} = \frac{1 - |g_{F,+}|^2}{1 + |g_{F,+}|^2}\Theta_{+,1} + \frac{2\mathrm{Re}\, g_{F,+}}{1 + |g_{F,+}|^2}\Theta_{+,2} + \frac{2\mathrm{Im}\, g_{F,+}}{1 + |g_{F,+}|^2}\Theta_{+,3}.$$

w: a local complex coordinate of M.

If we set $dF\left(\frac{\partial}{\partial w}\right) = \sum_{i=1}^{4} \psi^i e_i$, then we obtain $g_{F,+} = \sqrt{-1}\frac{\psi^1 + \sqrt{-1}\psi^2}{\psi^3 - \sqrt{-1}\psi^4}$.

Suppose that $F : M \longrightarrow N$ is minimal. Then $\nabla_{\partial/\partial\overline{w}} dF\left(\frac{\partial}{\partial w}\right) = 0$.

We set $\nabla e_i = \sum_{j=1}^{4} \omega_i^j e_j \quad (i = 1, 2, 3, 4)$.

$\Longrightarrow$
- $\omega_j^i = -\omega_i^j$,
- $\omega_2^3 = -\omega_1^4, \ \omega_2^4 = \omega_1^3, \ \omega_3^4 = -\omega_1^2$,
- $\frac{\partial \psi^i}{\partial \overline{w}} + \sum_{j \neq i} \psi^j \omega_j^i \left(\frac{\partial}{\partial \overline{w}}\right) = 0 \quad (i = 1, 2, 3, 4)$.

Using these, we can obtain $\frac{\partial g_{F,+}}{\partial \overline{w}} = 0$. □

$F: M \longrightarrow N$: a conformal and minimal immersion of M into N,

$\Psi := dF(\partial/\partial w)$.

$\Longrightarrow$ Ψdw gives a section of $F^*TN \otimes \mathbb{C} \otimes T^*M$ on M.

$\overline{\nabla}$: the connection of $F^*TN \otimes \mathbb{C} \otimes T^*M$ given by the Levi-Civita connection ∇ of h.

$\Longrightarrow$ $\overline{\nabla}_{\partial/\partial w}(\Psi dw) = \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right) dw$ (σ: the 2nd fundamental form of F).

We see that

$$Q := h\left(\sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right), \ \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right)\right) dw \otimes dw \otimes dw \otimes dw$$

does not depend on the choice of a local complex coordinate w and we can define a complex quartic differential Q on M.

If N is a 4-dimensional Riemannian space form, then we see by the equations of Codazzi that Q is holomorphic.

Theorem *The following are mutually equivalent:*

(a) *at each point of* M*, principal curvatures do not depend on the choice of a unit normal vector of* F;

(b) $h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = h(\sigma(T_1, T_2), \sigma(T_1, T_2))$,
$h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0$ *for* $T_1 := dF(\partial/\partial u)$, $T_2 := dF(\partial/\partial v)$;

(c) $Q \equiv 0$;

(d) *one of* $\Theta_{F,+}$, $\Theta_{F,-}$ *is horizontal w.r.t. the connection* $\hat{\nabla}$ *of* $\bigwedge^2 F^*TN$ *induced by* ∇;

(e) *one of* $I_{F,\pm}$ *is parallel w.r.t.* ∇;

(f) *we have one of* $I_{F,\pm}\sigma(T_1, T_1) = \sigma(T_1, T_2)$.

We say that a minimal immersion F is *isotropic* if one of (a) $\sim$ (f) in the above theorem holds.

We easily see

- (a), (b), (c) and (f) are mutually equivalent,
- (d) and (e) are equivalent.

In addition, (a) and (d) are equivalent (Friedrich).

Suppose $N = S^4$.

Bryant showed that an isotropic minimal surface (superminimal surface) is given by the composition of

- the twistor map

$$\mathbb{C}P^3 \longrightarrow S^4\,(= \mathbb{H}P^1), \quad \boldsymbol{a}\mathbb{C} \longmapsto \boldsymbol{a}\mathbb{H} \quad (\boldsymbol{a} \in \mathbb{C}^4 \setminus \{\mathbf{0}\} = \mathbb{H}^2 \setminus \{\mathbf{0}\})$$

  associated with S^4,
- a holomorphic immersion $\hat{F} : M \longrightarrow \mathbb{C}P^3$ which is horizontal in the twistor space $\mathbb{C}P^3$ $(= Sp(2)/U(2) \cong SO(5)/U(2))$.

Suppose $N = E^4$.
Then a conformal immersion $F : M \longrightarrow E^4$ is an isotropic minimal immersion
if and only if
the composition of F with an isometry of E^4 is a holomorphic immersion into
$\mathbb{C}^2 = E^4$.

Suppose that N is hyperKähler.
Then a conformal immersion $F : M \longrightarrow N$ is an isotropic minimal immersion
compatible with the orientation of N
if and only if
F is a complex curve w.r.t. a complex structure given by
the hyperKähler structure of N.

Suppose that N is a Kähler surface.

Then a conformal immersion $F : M \longrightarrow N$ is an isotropic minimal immersion

which is compatible with the orientation of N and

equipped with at least one complex point

if and only if

F is a complex curve w.r.t. the complex structure given by

the Kähler structure of N.

R: the curvature tensor of ∇:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$\hat{R}$: the curvature tensor of $\hat{\nabla}$.

$\Longrightarrow \hat{R}(X_1, X_2)(Y_1 \wedge Y_2) = (R(X_1, X_2)Y_1) \wedge Y_2 + Y_1 \wedge R(X_1, X_2)Y_2.$

(e_1, e_2): a local ordered orthonormal frame field of TM giving the orientation of M.

- If one of $\Theta_{F,\pm}$ is horizontal, then $\hat{R}(e_1, e_2)\Theta_{F,+} = 0$ or $\hat{R}(e_1, e_2)\Theta_{F,-} = 0$.
- If $\Theta_{F,\pm}$ are horizontal, then $\hat{R}(e_1, e_2)\Theta_{F,\pm} = 0$ and F is totally geodesic.

Theorem (A, 2020)
*$F : M \longrightarrow N$: a conformal and minimal immersion s.t. $\hat{R}(e_1, e_2)\Theta_{F,\pm} = 0$.
Then Q is holomorphic.
In addition, if $\hat{\nabla}\Theta_{F,\pm} \neq 0$, then we can choose (e_1, e_2, e_3, e_4) satisfying*
(a) *the connection forms ω, $\omega^{\perp}$ given by $\omega := h(\nabla e_1, e_2)$, $\omega^{\perp} := h(\nabla e_3, e_4)$ satisfy $d * \omega = 0$ and $d * \omega^{\perp} = 0$;*
(b) *the 2nd fundamental form of F is constructed by a solution of an over-determind system s.t. the compatibility condition is given by $d * \omega = 0$ and $d * \omega^{\perp} = 0$.*

Remark If N is a space form, then $\hat{R}(e_1, e_2)\Theta_{F,\pm} = 0$.

Remark The condition $d * \omega = 0$ means that on a neighborhood of each point of M, there exists a local complex coordinate $w = u + \sqrt{-1}v$ satisfying $e_1 = e^{-\lambda} dF(\partial/\partial u)$, $e_2 = e^{-\lambda} dF(\partial/\partial v)$ for a function λ.

Proof of the theorem

Since F is minimal, we have $\nabla_{\partial/\partial\overline{w}}\Psi = 0$.

Since $\hat{R}(e_1, e_2)\Theta_{F,\pm} = 0$, we have $\hat{R}\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial\overline{w}}\right)\left(\frac{\partial}{\partial w} \wedge \frac{\partial}{\partial\overline{w}}\right) = 0$.

Therefore we obtain $\nabla^{\perp}_{\partial/\partial\overline{w}}\sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right) = 0$ and this means that Q is holomorphic.

Suppose $\hat{\nabla}\Theta_{F,\pm} \neq 0$.

Then principal curvatures of F at each point depend on the choice of a unit normal vector.

e_3: a locally defined unit normal vector field which gives the maximum of the absolute values of principal curvatures of F at each point.

Then the maximum is positive and therefore we can suppose that e_1, e_2 give principal directions of F w.r.t. e_3.

e_4: a unit normal vector field perpendicular to e_3.
$\sigma_{ij}^k := h(\sigma(e_i, e_j), e_k) \ \ (i, j = 1, 2,\ k = 3, 4)$.
$\Longrightarrow \ \sigma_{11}^k + \sigma_{22}^k = 0 \ \ (k = 3, 4), \quad \sigma_{12}^3 = 0, \quad \sigma_{11}^4 = 0.$

$f_\pm := \sigma_{11}^3 \pm \sigma_{12}^4 \ \Longrightarrow \ f_\pm \neq 0.$
$p^j := 2\omega(e_j), \ \ q^j := (-1)^{3-j}\omega^\perp(e_{3-j}) \ \ (j = 1, 2).$
Then $\hat{R}(e_1, e_2)\Theta_{F,\pm} = 0$ mean

$$e_1(\log|f_\pm|) = -p^2 \pm q^1, \quad e_2(\log|f_\pm|) = p^1 \pm q^2.$$

Since ∇ is torsion-free, we obtain $2[e_1, e_2] + p^1 e_1 + p^2 e_2 = 0$.
Therefore we obtain

- $e_1(p^1) + e_2(p^2) = 0$, i.e., $d * \omega = 0$,
- $e_2(q^1) - e_1(q^2) = \dfrac{1}{2}(p^1 q^1 + p^2 q^2)$, i.e., $d * \omega^\perp = 0$.

□

2. Space-like surfaces with zero mean curvature vector in Lorentzian 4-manifolds and Willmore surfaces in 3-dimensional space forms

N: an oriented Lorentzian 4-dimensional manifold with its metric h,

$F : M \longrightarrow N$: a space-like and conformal immersion of M into N with zero mean curvature vector.

$$\Longrightarrow \ \overline{\overline{\nabla}}_{\partial/\partial w}(\Psi dw) = \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right) dw \qquad \left(\Psi := dF\left(\frac{\partial}{\partial w}\right)\right).$$

We can define a complex quartic differential Q on M by

$$Q := h\left(\sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right), \ \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right)\right) dw \otimes dw \otimes dw \otimes dw.$$

We see that $Q \equiv 0$ if and only if
the 2nd fundamental form is light-like or zero, that is,
the shape operator of a light-like normal vector field vanishes.

If N is a 4-dimensional Lorentzian space form,
then we see by the equations of Codazzi that Q is holomorphic, and
$Q \equiv 0$ means that a light-like normal vector field is contained in a constant direction.

Remark L_0: the constant sectional curvature of N.

- $L_0 = 0 \implies N = E^4_1$.
- $L_0 > 0 \implies N = S^4_1(L_0) = \left\{ x \in E^5_1 \;\middle|\; \langle x, x \rangle_{4,1} = \dfrac{1}{L_0} \right\}$.
- $L_0 < 0 \implies N = H^4_1(L_0) = \left\{ x \in E^5_2 \;\middle|\; \langle x, x \rangle_{3,2} = \dfrac{1}{L_0} \right\}$.

$\iota : M \longrightarrow S^3 = \{x \in E^5_1 \mid \langle x, x\rangle_{4,1} = 0,\ x^5 = 1\}$: a conformal immersion,

e_3: a unit normal vector field of ι in S^3,

H: the mean curvature of ι w.r.t. e_3.

$\Longrightarrow$ $\gamma_\iota := e_3 + H\iota$ is a map from M into the de Sitter 4-space

$S^4_1 = \{x \in E^5_1 \mid \langle x, x\rangle_{4,1} = 1\}$.

$\mathrm{Reg}\,(\iota)$: the set of non-umbilical points of ι.

$\Longrightarrow$ $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ is a space-like immersion s.t. the induced metric g is given by

$g = \varepsilon^2 g^M$, where $\varepsilon := \sqrt{H^2 - K^M + 1}$, and K^M is the curvature of the induced metric g^M by ι.

We call $\gamma_\iota : M \longrightarrow S^4_1$ the *conformal Gauss map* of ι.

We see that ι is a light-like normal vector field of $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ and that the trace of the shape operator of $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ w.r.t. ι vanishes.

ν: a light-like normal vector field of $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ s.t. $\langle \nu, \iota \rangle_{4,1} = -1$.

$\Longrightarrow$ The trace of the shape operator of $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ w.r.t. ν is given by $-(\Delta H + 2H)$ (Δ: the Laplacian on $\mathrm{Reg}\,(\iota)$ w.r.t. g).

Since $\Delta H + 2H = \dfrac{1}{\varepsilon^2}(\Delta^M H + 2\varepsilon^2 H)$, we obtain

Theorem (Bryant) *An immersion ι is Willmore if and only if the mean curvature vector of $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ vanishes.*

$\iota : M \longrightarrow S^3$: a conformal immersion,

$\Xi := 2\sigma^M \otimes \mathrm{Hess}^M_H + (H^2+1)\sigma^M \otimes \sigma^M - 2dH \otimes \nabla^M \sigma^M$, where

- σ^M: the 2nd fundamental form of ι,
- H: the mean curvature of ι,
- Hess^M_H: the Hessian of H w.r.t. the Levi-Civita connection ∇^M of g^M.

We consider Ξ to be a complex 4-linear function on the complexification of the tangent space of M at each point.

Proposition (Bryant)

If ι is Willmore, then a complex quartic differential

$$\tilde{Q} := \Xi\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right) dw \otimes dw \otimes dw \otimes dw$$

is holomorphic.

Theorem (A) *M: a Riemann surface,*
$\iota : M \longrightarrow S^3$: a conformal and Willmore immersion.
Then the holomorphic quartic differential Q for a conformal immersion $F := \gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ coincides with $\tilde{Q}$ on $\mathrm{Reg}\,(\iota)$ up to a nonzero constant.

Remark We can have analogous discussions
for $\iota : M \longrightarrow H^3 = \{x \in E^5_1 \mid \langle x, x\rangle_{4,1} = 0,\ x^1 = 1,\ x^5 > 0\}$
or $E^3 = \{x \in E^5_1 \mid \langle x, x\rangle_{4,1} = 0,\ x^5 = x^1 + 1\}$.

$\iota : M \longrightarrow S^3_1 = \{x \in E^5_2 \mid \langle x, x \rangle_{3,2} = 0,\ x^5 = 1\}$:
a space-like and conformal immersion,
e_3: a normal vector field of ι in S^3_1 s.t. $\langle e_3, e_3 \rangle_{3,2} = -1$,
H: the mean curvature of ι w.r.t. e_3.

$\Longrightarrow$ • $\gamma_\iota := -e_3 + H\iota$ is a map from M into the anti-de Sitter 4-space $H^4_1 = \{x \in E^5_2 \mid \langle x, x \rangle_{3,2} = -1\}$,

• $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ is a space-like immersion s.t. $g = \varepsilon^2 g^M$ $\left(\varepsilon := \sqrt{H^2 + K^M - \delta}\right)$.

We call $\gamma_\iota : M \longrightarrow H^4_1$ the *conformal Gauss map* of ι.

We can show that an immersion ι is Willmore if and only if the mean curvature vector of $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ vanishes.

$$\Xi := 2\sigma^M \otimes \mathrm{Hess}_H^M - (H^2 - \delta)\sigma^M \otimes \sigma^M - 2dH \otimes \nabla^M \sigma^M.$$

Proposition (A) *If* ι *is Willmore, then* $\tilde{Q}$ *is holomorphic.*

Theorem (A) M: *a Riemann surface,*
$\iota : M \longrightarrow S_1^3$: *a conformal and Willmore immersion.*
Then the holomorphic quartic differential Q *for a conformal immersion*
$F := \gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ *coincides with* $\tilde{Q}$ *on* $\mathrm{Reg}\,(\iota)$ *up to a nonzero constant.*

Remark We can have analogous discussions
for $\iota : M \longrightarrow H_1^3 = \{x \in E_2^5 \mid \langle x, x \rangle_{3,2} = 0,\ x^1 = 1\}$
or $E_1^3 = \{x \in E_2^5 \mid \langle x, x \rangle_{3,2} = 0,\ x^5 = x^1 + 1\}$.

$\iota : M \longrightarrow L^+ := \{x \in E^4_1 \mid \langle x, x\rangle_{3,1} = 0,\ x^4 > 0\}$:
a space-like and conformal immersion,
ξ: a light-like normal vector field of ι in E^4_1 s.t. $\langle \xi, \iota\rangle_{3,1} = -1$,
H: the mean curvature of ι w.r.t. a normal vector field ι.

$\Longrightarrow$ • $\gamma_\iota := -\xi + H\iota$ is a map from M into E^4_1,
• $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ is a space-like immersion s.t. $g = \varepsilon^2 g^M$ ($\varepsilon := \sqrt{H^2 - K}$).

We call $\gamma_\iota : M \longrightarrow E^4_1$ the *conformal Gauss map* of ι.

Remark We see that H is determined by the induced metric g^M.

Theorem (A) *An immersion* ι *satisfies* $\Delta^M H - 2\varepsilon^2 = 0$ *if and only if the mean curvature vector of* $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ *vanishes.*

Remark The Euler-Lagrange equation for Willmore surfaces in L^+ is given by $\Delta^M H + 2H^2 = 0$.

$$\Xi := \sigma^M \otimes \mathrm{Hess}_H^M - H\sigma^M \otimes \sigma^M - dH \otimes \nabla^M \sigma^M,$$

where σ^M is the 2nd fundamental form of ι w.r.t. a normal vector field ι.

Proposition (A) *If* ι *satisfies* $\Delta^M H - 2\varepsilon^2 = 0$, *then* $\tilde{Q}$ *is holomorphic.*

Theorem (A) M: *a Riemann surface,*
$\iota : M \longrightarrow L^+ \subset E_1^4$: *a conformal immersion s.t.* $\Delta^M H - 2\varepsilon^2 = 0$.
Then the holomorphic quartic differential Q *for a conformal immersion* $F := \gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ *coincides with* $\tilde{Q}$ *on* $\mathrm{Reg}\,(\iota)$ *up to a nonzero constant.*

3. Space-like surfaces with zero mean curvature vector in neutral 4-manifolds

(N, h): an oriented neutral 4-dimensional manifold.

$\Longrightarrow$ The metric h induces an indefinite metric $\hat{h}$ of $\bigwedge^2 TN$ defined by

$$\hat{h}(x_i \wedge x_j, x_k \wedge x_l) = h(x_i, x_k)h(x_j, x_l) - h(x_i, x_l)h(x_j, x_k).$$

(e_1, e_2, e_3, e_4): a local ordered pseudo-orthonormal frame field of TN giving the orientation of N.

$$\Theta_{\pm,1} := \frac{1}{\sqrt{2}}(\theta_{12} \pm \theta_{34}), \ \ \Theta_{\pm,2} := \frac{1}{\sqrt{2}}(\theta_{13} \pm \theta_{42}), \ \ \Theta_{\pm,3} := \frac{1}{\sqrt{2}}(\theta_{14} \pm \theta_{23}).$$

$\Longrightarrow$ $\Theta_{\pm,1}$, $\Theta_{\pm,2}$, $\Theta_{\pm,3}$ are mutually orthogonal and satisfy

$$\hat{h}(\Theta_{\pm,1}, \Theta_{\pm,1}) = 1, \quad \hat{h}(\Theta_{\pm,2}, \Theta_{\pm,2}) = \hat{h}(\Theta_{\pm,3}, \Theta_{\pm,3}) = -1.$$

Therefore $\hat{h}$ has signature $(2, 4)$.

$\bigwedge^2_+ TN$, $\bigwedge^2_- TN$: $SO(2,2)$-invariant subbundles of $\bigwedge^2 TN$ with rank 3 s.t.
all the elements of $\bigwedge^2_+ TN$ are $SU(1,1)$-invariant
(notice the double covering
$SO_0(2,2) \longrightarrow SO_0(1,2) \times SO_0(1,2)$).

$\Longrightarrow$ Each fiber of $\bigwedge^2_+ TN$ (resp. $\bigwedge^2_- TN$) is spanned by
$\Theta_{-,1}$, $\Theta_{+,2}$, $\Theta_{+,3}$ (resp. $\Theta_{+,1}$, $\Theta_{-,2}$, $\Theta_{-,3}$).

In particular, we see

- $\bigwedge^2 TN = \bigwedge^2_+ TN \oplus \bigwedge^2_- TN$,
- $\bigwedge^2_+ TN \perp \bigwedge^2_- TN$ w.r.t. $\hat{h}$,
- The restriction of $\hat{h}$ on each of $\bigwedge^2_+ TN$, $\bigwedge^2_- TN$ has signature $(1,2)$.

- If N is neutral hyperKähler, then one of $\bigwedge^2_\pm TN$ is a product bundle.
- If $N = E^4_2$, then both of $\bigwedge^2_\pm TN$ are product bundles.

The space-like twistor spaces associated with N are fiber bundles in $\bigwedge^2_\pm TN$ given by

$$U_+\left(\bigwedge^2_+ TN\right) := \left\{\Theta \in \bigwedge^2_+ TN \mid \hat{h}(\Theta, \Theta) = 1\right\},$$
$$U_+\left(\bigwedge^2_- TN\right) := \left\{\Theta \in \bigwedge^2_- TN \mid \hat{h}(\Theta, \Theta) = 1\right\}.$$

M: a Riemann surface,

$F : M \longrightarrow N$: a space-like and conformal immersion of M into N.

$\Theta_{F,\pm}$: sections of $U_+\left(\bigwedge^2_\pm F^*TN\right)$ defined by $\Theta_{F,\pm} := \dfrac{1}{\sqrt{2}}(\xi_1 \wedge \xi_2 \mp \xi_3 \wedge \xi_4)$,

where ξ_1, ξ_2, ξ_3, ξ_4 form a local pseudo-orthonormal frame field of F^*TN s.t.

- $(\xi_1, \xi_2, \xi_3, \xi_4)$ gives the orientation of N,
- $\xi_1, \xi_2 \in dF(TM)$ so that (ξ_1, ξ_2) gives the orientation of M.

$I_{F,\pm}$: the complex structures of F^*TN corresponding to $\Theta_{F,\pm}$.

Then $\Theta_{F,\pm} = \dfrac{1}{\sqrt{2}}(e \wedge I_{F,\pm}(e) - e^\perp \wedge I_{F,\pm}(e^\perp))$,

where

- e is a unit tangent vector of F,
- $e^\perp$ is a normal vector of F with $h(e^\perp, e^\perp) = -1$.

If N is neutral hyperKähler so that $\bigwedge^2_+ TN$ (respectively, $\bigwedge^2_- TN$) is a product bundle, then we can consider $\Theta_{F,+}$ (respectively, $\Theta_{F,-}$) to be a map from M into $\mathbb{C}H^1$.

Theorem (A, 2020) *Suppose*

- *N is neutral hyperKähler,*
- *$F : M \longrightarrow N$ has zero mean curvature vector.*

Then one of $\Theta_{F,+}, \Theta_{F,-}$ is a holomorphic map from M into $\mathbb{C}H^1$.

Corollary (A, 2020)

$F : M \longrightarrow E^4_2$: *a space-like and conformal immersion with zero mean curvature vector.*

Then the Gauss map $\mathcal{G}_F : M \longrightarrow \mathbb{C}H^1 \times \mathbb{C}H^1$ of F is holomorphic.

M: a Riemann surface,

$F : M \longrightarrow N$: a space-like and conformal immersion of M into N with zero mean curvature vector.

$$\Longrightarrow \ \overline{\nabla}_{\partial/\partial w}(\Psi dw) = \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right) dw \quad \left(\Psi = dF\left(\frac{\partial}{\partial w}\right)\right).$$

We can define a complex quartic differential Q on M by

$$Q := h\left(\sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right), \ \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right)\right) dw \otimes dw \otimes dw \otimes dw.$$

If N is a 4-dimensional neutral space form,

then we see by the equations of Codazzi that Q is holomorphic.

Theorem *The following are mutually equivalent:*

(a) *at each point of M, principal curvatures do not depend on the choice of a normal vector $e^{\perp}$ of F with $h(e^{\perp}, e^{\perp}) = -1$;*

(b) $h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = h(\sigma(T_1, T_2), \sigma(T_1, T_2))$, $h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0$ *for* $T_1 := dF(\partial/\partial u)$, $T_2 := dF(\partial/\partial v)$;

(c) $Q \equiv 0$;

(d) *one of* $\Theta_{F,+}$, $\Theta_{F,-}$ *is horizontal w.r.t.* $\hat{\nabla}$;

(e) *one of* $I_{F,\pm}$ *is parallel w.r.t.* ∇;

(f) *we have one of* $I_{F,\pm}\sigma(T_1, T_1) = \sigma(T_1, T_2)$.

We say that F is *isotropic* if one of (a) $\sim$ (f) in the above theorem holds.

Theorem (A, 2020)

$F : M \longrightarrow N$: *a space-like and conformal immersion*

with zero mean curvature vector and $\hat{R}(e_1, e_2)\Theta_{F,\pm} = 0.$

Then Q *is holomorphic.*

In addition, if $\hat{\nabla}\Theta_{F,\pm} \neq 0$, *then we can choose* (e_1, e_2, e_3, e_4) *satisfying*

(a) *the connection forms* ω, $\omega^{\perp}$ *given by* $\omega := h(\nabla e_1, e_2)$, $\omega^{\perp} := h(\nabla e_3, e_4)$ *satisfy* $d * \omega = 0$ *and* $d * \omega^{\perp} = 0$;

(b) *the 2nd fundamental form of* F *is constructed by a solution of an over-determind system s.t. the compatibility condition is given by* $d * \omega = 0$ *and* $d * \omega^{\perp} = 0$.

Remark If N is a 4-dimensional neutral space form, then $\hat{R}(e_1, e_2)\Theta_{F,\pm} = 0$.

4. Time-like surfaces with zero mean curvature vector in neutral 4-manifolds

The time-like twistor spaces associated with N are fiber bundles in $\bigwedge^2_{\pm} TN$ given by

$$U_-\left(\bigwedge^2_{\varepsilon} TN\right) := \left\{\Theta \in \bigwedge^2_{\varepsilon} TN \mid \hat{h}(\Theta, \Theta) = -1\right\} \quad (\varepsilon = +, -).$$

M: a Lorentz surface (two-dimensional manifold with a holomorphic system of paracomplex coordinate neighborhoods),

$F : M \longrightarrow N$: a time-like and conformal immersion of M into N.

$\Theta_{F,\pm}$: sections of $U_-\left(\bigwedge^2_{\pm} F^*TN\right)$ defined by $\Theta_{F,\pm} := \dfrac{1}{\sqrt{2}}(\xi_1 \wedge \xi_3 \pm \xi_4 \wedge \xi_2)$,

where ξ_1, ξ_2, ξ_3, ξ_4 form a local pseudo-orthonormal frame field of F^*TN (we suppose that ξ_1, ξ_2 are space-like) s.t.

- $(\xi_1, \xi_2, \xi_3, \xi_4)$ gives the orientation of N,
- $\xi_1, \xi_3 \in dF(TM)$ so that (ξ_1, ξ_3) gives the orientation of M.

$J_{F,\pm}$: the paracomplex structures of F^*TN corresponding to $\Theta_{F,\pm}$.

Then $\Theta_{F,\pm} = \dfrac{1}{\sqrt{2}}(e \wedge J_{F,\pm}(e) - e^{\perp} \wedge J_{F,\pm}(e^{\perp}))$,

where

- e is a unit tangent vector of F,
- $e^{\perp}$ is a normal vector of F with $h(e^{\perp}, e^{\perp}) = -1$.

If N is neutral hyperKähler so that $\bigwedge^2_+ TN$ (respectively, $\bigwedge^2_- TN$) is a product bundle, then we can consider $\Theta_{F,+}$ (respectively, $\Theta_{F,-}$) to be a map from M into $\tilde{\mathbb{C}}H^1$ (a hyperboloid of one sheet as a Lorentz surface).

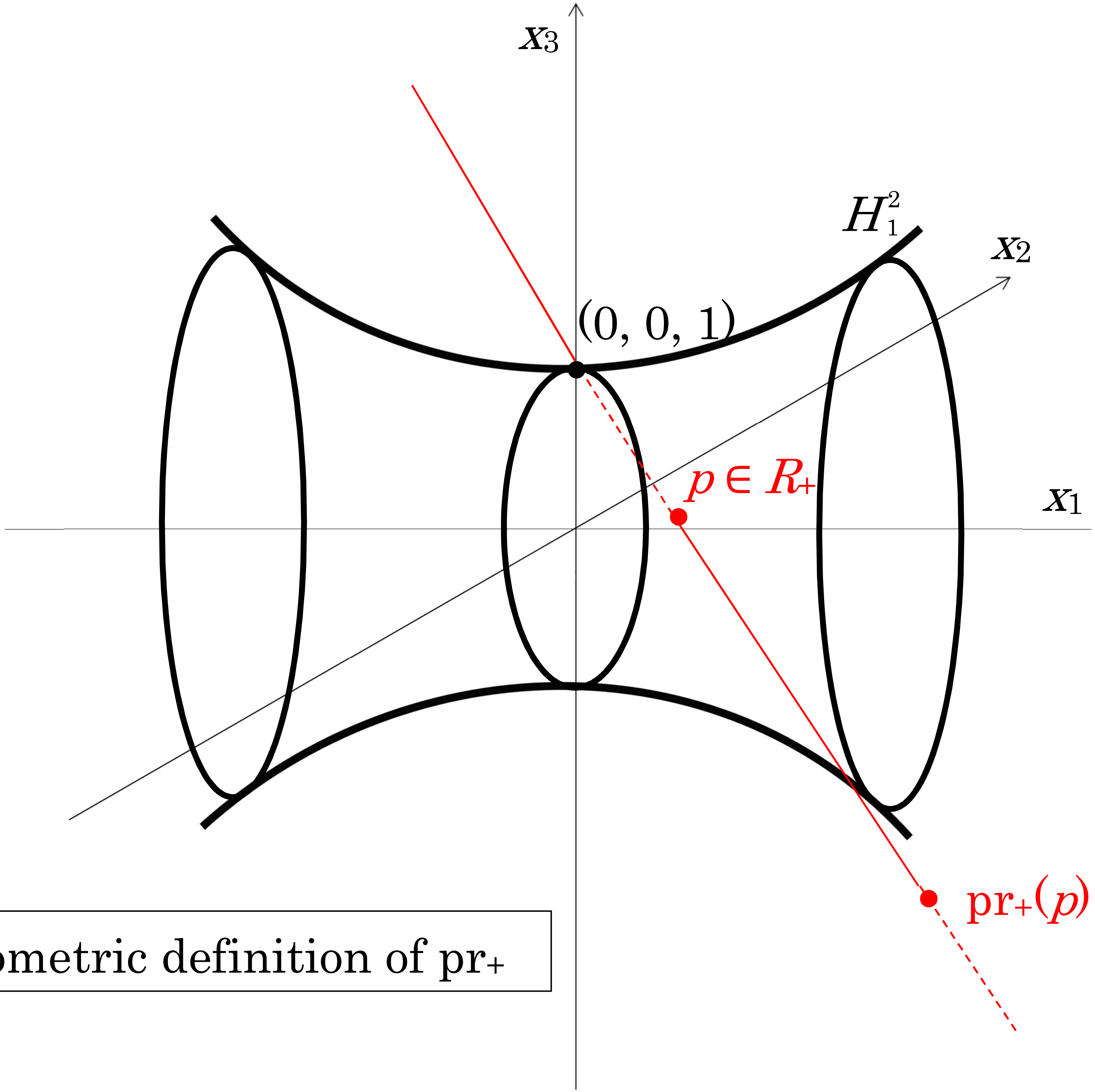
A hyperboloid of one-sheet is given by $H^2_1 = \{x \in E^3_2 \mid \langle x, x\rangle_{1,2} = -1\}$.
Let R_+, R_- be open subsets of H^2_1 defined by

$$R_+ := \{x = (x^1, x^2, x^3) \in H^2_1 \mid x^3 \neq 1\},$$
$$R_- := \{x = (x^1, x^2, x^3) \in H^2_1 \mid x^3 \neq -1\}.$$

$\tilde{\mathbb{C}}$: the paracomplex plane $= \{\tilde{w} = u + jv \mid u, v \in \mathbb{R}\}$
(j: the paraimaginary unit),
$|\tilde{w}|^2 := \tilde{w}\overline{\tilde{w}} = u^2 - v^2$,
$C_\delta := \{\tilde{w} \in \tilde{\mathbb{C}} \mid |\tilde{w}|^2 = \delta\}$ $(\delta = 0, 1)$.

The stereographic projections $\mathrm{pr}_\pm$ are bijective maps from $R_\pm$ onto $\tilde{\mathbb{C}} \setminus C_1$ defined by

$$\mathrm{pr}_\pm^{-1}(\tilde{w}) = \left(\frac{2\mathrm{Re}\,\tilde{w}}{1 - |\tilde{w}|^2},\ \mp\frac{2\mathrm{Im}\,\tilde{w}}{1 - |\tilde{w}|^2},\ \mp\frac{1 + |\tilde{w}|^2}{1 - |\tilde{w}|^2}\right) \quad (\tilde{w} \in \tilde{\mathbb{C}} \setminus C_1).$$



The geometric definition of pr_+

Since $\mathrm{pr}_{\pm}(R_+ \cap R_-) = \tilde{\mathbb{C}} \setminus (C_1 \cup C_0)$, we see that the composition

$$\mathrm{pr}_- \circ \mathrm{pr}_+^{-1} : \mathrm{pr}_+(R_+ \cap R_-) \longrightarrow \mathrm{pr}_-(R_+ \cap R_-)$$

is holomorphic.

Therefore, noticing $R_+ \cup R_- = H_1^2$, we can consider H_1^2 to be a Lorentz surface, which is denoted by $\tilde{\mathbb{C}}H^1$.

Theorem (A, 2020) *Suppose*

- *N is neutral hyperKähler,*
- *$F : M \longrightarrow N$ has zero mean curvature vector.*

Then one of $\Theta_{F,+}, \Theta_{F,-}$ is a holomorphic map from M into $\tilde{\mathbb{C}}H^1$.

Corollary (A, 2020)

$F : M \longrightarrow E_2^4$: a time-like and conformal immersion with zero mean curvature vector,

Then the Gauss map $\mathcal{G}_F : M \longrightarrow \tilde{\mathbb{C}}H^1 \times \tilde{\mathbb{C}}H^1$ of F is holomorphic.

M: a Lorentz surface,

$F : M \longrightarrow N$: a time-like and conformal immersion of M into N
with zero mean curvature vector,

$w = u + jv$: a local paracomplex coordinate of M,

$\Psi := dF\left(\frac{\partial}{\partial w}\right) \quad \left(\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} + j\frac{\partial}{\partial v}\right)\right).$

$\Longrightarrow \ \overline{\nabla}_{\partial/\partial w}(\Psi dw) = \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right) dw.$

We can define a paracomplex quartic differential Q on M by

$$Q := h\left(\sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right), \ \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right)\right) dw \otimes dw \otimes dw \otimes dw.$$

If N is a 4-dimensional neutral space form,
then we see by the equations of Codazzi that Q is holomorphic.

Theorem *The following are equivalent:*

(a) $h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = -h(\sigma(T_1, T_2), \sigma(T_1, T_2))$,

$h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0$ *for* $T_1 := dF(\partial/\partial u)$, $T_2 := dF(\partial/\partial v)$;

(b) $Q \equiv 0$.

We say that F is *isotropic* if one of (a), (b) in the above theorem holds.

Theorem (A, 2020) *The following are mutually equivalent:*

(a) *one of* $\Theta_{F,+}$, $\Theta_{F,-}$ *is horizontal w.r.t.* $\hat{\nabla}$;

(b) *one of* $J_{F,\pm}$ *is parallel w.r.t.* ∇;

(c) *we have one of* $J_{F,\pm}\sigma(T_1, T_1) = \sigma(T_1, T_2)$.

In addition, if F *satisfies one of* (a), (b), (c), *then* F *is isotropic.*

We say that F is *strictly isotropic* if one of (a), (b), (c) in the above theorem holds for the orientation of N.

It is possible that although F is isotropic, none of the covariant derivatives of $\Theta_{F,+}$, $\Theta_{F,-}$ w.r.t. $\hat{\nabla}$ become zero.

Proposition (A, 2020)

If both $\hat{\nabla}\Theta_{F,+}$ *and* $\hat{\nabla}\Theta_{F,-}$ *are light-like, then one of the following holds:*

(a) *the shape operator of a light-like normal vector field vanishes and then* Q *vanishes;*

(b) *the shape operator of any normal vector field is zero or light-like, and then* Q *is zero or null.*

Remark

Suppose that N is a 4-dimensional neutral space form.

- Condition (a) implies that a light-like normal vector field of the surface is contained in a constant direction.
  The conformal Gauss map of a time-like surface in a 3-dimensional Lorentzian space form of Willmore type with $Q \equiv 0$ has this property.
- We can characterize surfaces with condition (b), based on the Gauss-Codazzi-Ricci equations.

M: an oriented two-dimensional manifold,

$\iota : M \longrightarrow N^3_1 = S^3_1$, E^3_1 or H^3_1: a time-like immersion

(we consider S^3_1, E^3_1, H^3_1 to be subsets of E^5_2),

e_3: a unit normal vector field of ι in N^3_1,

H: the mean curvature of ι w.r.t. e_3,

$\gamma_\iota := e_3 + H\iota$,

$\Lambda := H^2 - K^M + \delta$

($\delta = 1, 0$ or -1, K^M: the curvature of the induced metric g^M by ι),

$\mathrm{Reg}\,(\iota)$: the set of nonzero points of Λ.

$\Longrightarrow$ $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ is a time-like immersion of $\mathrm{Reg}\,(\iota)$ into S^4_2 s.t.

the induced metric g by $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ is given by $g = \Lambda g^M$.

We call $\gamma_\iota : M \longrightarrow S^4_2$ the *conformal Gauss map* of $\iota : M \longrightarrow N^3_1$.

- ι is a light-like normal vector field of a time-like immersion $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$,
- the trace of the shape operator of $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ w.r.t. ι is zero,
- if we denote by ν a light-like normal vector field of $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ satisfying $\langle \iota, \nu \rangle_{3,2} = -1$, then the trace of the shape operator of $\gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ w.r.t. ν is given by $-\dfrac{1}{\Lambda}(\Delta^M H + 2\Lambda H)$.

Since $\Lambda \equiv 0$ means that $\Delta^M H = 0$, we obtain

Theorem (A) *An immersion* $\iota : M \longrightarrow N^3_1$ *satisfies* $\Delta^M H + 2\Lambda H = 0$ *if and only if the mean curvature vector of* $\gamma_\iota|_{\mathrm{Reg}\,(\iota)} : \mathrm{Reg}\,(\iota) \longrightarrow S^4_2$ *vanishes.*

We say that ι is *of Willmore type* $\overset{\text{def}}{\Longleftrightarrow}$ $\Delta^M H + 2\Lambda H = 0$.

M: a Lorentz surface,

$\iota : M \longrightarrow N_1^3$: a time-like and conformal immersion,

$\Xi := 2\sigma^M \otimes \mathrm{Hess}_H^M + (H^2 + \delta)\sigma^M \otimes \sigma^M - 2dH \otimes \nabla^M \sigma^M$

(σ^M: the 2nd fundamental form of ι).

Proposition (A) *If $\iota : M \longrightarrow N_1^3$ is of Willmore type, then a paracomplex quartic differential*

$$\tilde{Q} := \Xi\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right) dw \otimes dw \otimes dw \otimes dw$$

is holomorphic ($w = u + jv$: a local paracomplex coordinate of M).

Theorem (A)

$\iota : M \longrightarrow N^3_1$: *a time-like and conformal immersion of Willmore type.*

On $\mathrm{Reg}\,(\iota)$, *the following hold:*

(a) *the null points of the differential* Q *for* $F := \gamma_\iota|_{\mathrm{Reg}\,(\iota)}$ *coincide with the null points of* $\tilde{Q}$, *and a null point of* Q *is just given by a condition that the shape operator of* F *w.r.t.* ν *is light-like;*

(b) *except the null points,* Q *coincides with* $\tilde{Q}$ *up to a nonzero constant;*

(c) $Q \equiv 0$ *if and only if a light-like normal vector field* ν *of* F *is contained in a constant direction.*

Remark

Suppose

- ι as in the above theorem satisfies $\tilde{Q} \equiv 0$;
- $\left(\nabla_{T_1} T_1\right)^\perp \neq \pm \left(\nabla_{T_1} T_2\right)^\perp$ $(T_1 = dF(\partial/\partial u),\, T_2 = dF(\partial/\partial v))$.

$\Longrightarrow$ For $\Theta_{F,\pm}$ with $F = \gamma_\iota|_{\mathrm{Reg}\,(\iota)}$, $\hat{\nabla}\Theta_{F,\pm}$ are light-like.

(e_1, e_3): a local ordered pseudo-orthonormal frame field of TM giving the orientation of M.

Theorem (A, 2020)

$F : M \longrightarrow N$: *a time-like and conformal immersion with zero mean curvature vector and* $\hat{R}(e_1, e_3)\Theta_{F,\pm} = 0$.

Then Q *is holomorphic and the 2nd fundamental form of* F *is constructed by solutions of four families of ordinary differential equations defined along integral curves of light-like vector fields* $e_1 \pm e_3$ *and given by the connection forms* $\omega := -h(\nabla e_1, e_3)$, $\omega^{\perp} := -h(\nabla e_2, e_4)$.

If $\hat{\nabla}\Theta_{F,\pm}$ are zero or light-like, then $\hat{R}(e_1, e_3)\Theta_{F,\pm}$ are zero or light-like.

Theorem (A, 2020)

$F : M \longrightarrow N$: *a time-like and conformal immersion with zero mean curvature vector s.t.* $\hat{R}(e_1, e_3)\Theta_{F,\pm}$ *are zero or light-like.*

Then the 2nd fundamental form of F *is constructed by solutions of suitable two families of ordinary differential equations of the four families in the previous theorem.*

THE FIRST TALK HAS ENDED.