Future research plans

One of central problems of G_2-geometry is to know whether we can define enumerative invariants of G_2-manifolds by counting G_2-instantons or associative submanifolds. (There is an analogous problem in Spin(7)-geometry, too.) We need a deep understanding of singular G_2-instantons or associative submanifolds for the verification. Recall the following similarities from "My achievements".

Figure ♠ :

$$\begin{tikzcd} & \{\text{dHYM}, G_2, \mathrm{Spin}(7)\text{-dDT conn.}\} \arrow[dl, leftrightarrow, "mirror"'] \arrow[dr, leftrightarrow, "analogue"] & \\ \{\text{calibrated submanifold}\} & & \{\text{HYM}, G_2, \mathrm{Spin}(7)\text{-instanton}\} \end{tikzcd}$$

By (Ⅲ), we see that the moduli space of G_2-dDT connections has properties similar to the moduli spaces of associative submanifolds and G_2-instantons. I suspect that the study of G_2-dDT connections would shed light on this problem. I will study singular $G_2, \mathrm{Spin}(7)$-dDT connections to find a breakthrough to the problem above. Based on this study, I aim to find new properties for the other two in Figure ♠ from the study of $G_2, \mathrm{Spin}(7)$-dDT connections.

[Plan 1] Compactness theorem : I want to know whether the compactness theorem holds for $G_2, \mathrm{Spin}(7)$-dDT connections. This is important for the compactification of the moduli space. It has already been shown for $G_2, \mathrm{Spin}(7)$-instantons and (some) associative submanifolds. From the similarities in the figure ♠, the theorem is expected to hold for $G_2, \mathrm{Spin}(7)$-dDT connections as well.

The problem is to set the appropriate functional first. It is unclear at this point which functional is appropriate, but (i) the Yang-Mills functional and (ii) the mirror of the "3-energy" for submanifolds are considered as candidates. This is because they correspond to the functionals of the above two cases. First, for these functionals, I will investigate whether the basic properties for the compactness theorem (such as the removable singularity theorem) hold. In addition, we investigate various functionals such as "mirror" of functionals for submanifolds. If an appropriate functional is found, I aim to prove the theorem by applying and developing the above two methods. In the case of associatives, the theory is also related to the p-harmonic maps. So I will also study from this direction.

[Plan 2] "Mirror" of mean curvature flow : The "volume" of a connection can be introduced as a "mirror" of the standard volume of a submanifold. (In physics, it is called the Dirac-Born-Infeld (DBI) action.) I would like to study this gradient flow (hereinafter referred to as the "**mirror MCF**"). This seems to be useful for investigating singular $G_2, \mathrm{Spin}(7)$-dDT connections (boundaries of the moduli spaces) as in the case of mean curvature flow (MCF) for submanifolds. There are many studies on MCFs, and similar results are expected for mirror MCFs. First, I would like to show the short-time existence of this flow. To show this, I am considering if the flow can be reduced to a parabolic PDE as in MCFs. If this is proved, the short-time existence is highly possible. I will try to prove it by considering Deturck's trick, the Nash-Moser's implicit function theorem or the techniques developed during the study of deformation theories.

Recently, I found that the $G_2, \mathrm{Spin}(7)$-dDT connections minimize their "volumes" (as calibrated submanifolds are minimal). Then I would like to study the stabilities of $G_2, \mathrm{Spin}(7)$-dDT connections. To prove this, I am considering the analogue of the method of H. Li for Lagrangian MCFs. That is, under the assumption that the first eigenvalue of the linearized equation is sufficiently large, I will show that L^2 norm of the mirror mean curvature decays exponentially etc. I will study with reference to other flows whose stability is known (such as the Laplacian flow on G_2-manifolds) as appropriate.

If [Plan 2] proceeds faster than I expected, I plan to investigate the similar properties of MCFs further. For example, I would like to know whether the solution of the mirror MCF blows up. If it does, I will try to classify singularities according to the way they blow up and investigate their properties.