RESEARCH OUTLINE

The theory of Kashiwara's *crystal bases* gives combinatorial models for representations of symmetrizable Kac–Moody Lie algebras, which has led to deep connections to other areas of mathematics and physics. One important connection is to statistical mechanics, where crystals arise in the study of integrable systems, such as *solvable lattice models* and quantum spin chains, with a particular emphasis the crystal bases associated to *Kirillov–Reshetikhin (KR) modules*. Recently, I have been generalizing of crystals to areas such as *K-theoretic Schubert calculus*.

Circa 2000, G. Hatayama *et al.* [4,5] conjectured a generalization of the work of Kerov, Kirillov, and Reshetikhin about a generating series identity relating the representations of affine quantum groups with the XXZ Heisenburg spin chain, which is now known as the $X = M$ conjecture. To describe this conjecture, we first note that KR modules are a certain class of nice finite dimensional representations of affine quantum groups that have been well-studied by numerous authors. One outstanding conjecture regarding KR modules has been they admit crystal bases, which has been solved except for a few nodes in exceptional types by myself and K. Naoi [11]. The $X = M$ conjecture can be proven by constructing a bijection between a combinatorial object called rigged configurations and the elements of a tensor product of KR crystals. I have conjectured a bijection in all affine types, which I have been proven in all nonexceptional types with M. Okado and A. Schilling [14] and numerous special cases in the exceptional types [16,17]. I have applied this bijection to the study of soliton cellular automata [8] (Xuan Liu was then an undergraduate), a generalization of the Takahashi–Satsuma box-ball system (an ultradiscrete version of the Korteweg–de Vries (KdV) equation). In a series of papers with B. Salisbury, we have been studying the crystal structure of rigged configurations and generalizing it to all Kac–Moody Lie algebras and Borcherds algebras [15], where we show they are a natural model for crystals. E, Aas, D. Grinberg, and I also used a KR crystal interpretation of the probabilistic model TASEP on a ring in [1] to prove some conjectures.

Beyond the $X = M$ conjecture, KR modules generate an interesting class of affine quantum group representations, which D. Hernandez and B. Leclerc conjecture categorifies a cluster algebra [7]. One approach to proving this conjecture involves studying the R-matrix, the morphism that interchanges the tensor product of two KR modules. S.-j. Oh and I have studied [12,13] when the R-matrix is not an isomorphism, which gives a relation between KR modules called Dorey's rule, and can be applied to prove the categorification conjecture.

Schubert calculus can be described as the study of the Grassmannian, the set of k-dimensional planes in n-dimensional space. The cohomology ring and those for the full flag variety, which is the quotient of invertible $n \times n$ matrices by invertible upper triangular matrices, has been well-studied by relating them to symmetric functions and basis for all polynomials, respectively by using certain subvarities called Schubert varieties. We enrich this theory by instead looking at the K-theory ring, which yields Grothendieck polynomials. For the Grassmannian case, these become symmetric functions and have natural generalizations, all of which can be described combinatorially. I and coauthors show the associated combinatorics admit crystal structures [6,9] and we conjecture a generalization of crystals in an effort to categorify the ring structure [9]. One natural generalization was recently shown by me and K. Motegi to be related with a random matrix process [10].

The R-matrix and KR modules can also be used as a simple model for ice called the six-vertex model, a prototypical example of a solvable lattice model. We can draw this diagrammatically as a square grid with labelings on edges that satisfy local conditions around each vertex and obtain generating functions describing the system using the Yang–Baxter equations. I and coauthors utilize this approach to understand Grothendieck polynomials and their "partial" versions called Lascoux polynomials [2,3], giving the first proven combinatorial rule for Lascoux polynomials.

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