Research Achievements

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Many differential equations which describe various natural phenomena have the variational structure. Furthermore to see the dynamics, we frequently need to know characterizations, shapes and asymptotic behaviors of solutions. My research theme is to analyze the existence, characterizations, shapes and asymptotic profiles of solutions in various differential equations which have the variational structure.

1: Research on a fourth order nonlinear elliptic problem with singular forcing terms

I considered the following fourth order nonlinear elliptic problem:

$$\Delta^2 u - c_1 \Delta u + c_2 u = u^p + \kappa \sum_{i=1}^{m} \alpha_i \delta_{a_i} \text{ in } \mathcal{D}'(\mathbb{R}^N), \ u(x) > 0, \text{ in } \mathbb{R}^N, \ u(x) \to 0 \text{ as } |x| \to \infty, \qquad (1)$$

where $N \geq 5$, $\kappa > 0$, $m \in \mathbb{N}$, $\alpha_i > 0$ and $c_1, c_2 > 0$. We denote by δ_{a_i} the delta-function supported at $a_i \in \mathbb{R}^N$. In [7], I studied the existence, non-existence of positive solutions of (1) and their properties under the assumptions $1 < p < \frac{N}{N-4}$ and $c_1^2 - 4c_2 \geq 0$. In fourth order elliptic problems, the maximum principle does not hold in general. Thus it is rather hard to obtain the positivity or the decay rate of solutions at infinity. I overcome this difficulty by suitably using properties of the fundamental solution.

2: Research on a non-homogeneous nonlinear elliptic equation

I studied the following nonlinear elliptic problem:

$$-\Delta u + u = g(u) + f(x), \ x \in \mathbb{R}^N, \qquad (2)$$

where $N \geq 3$, $f(x) \geq 0$ and $f(x) \not\equiv 0$. In [6], I showed that for a wide class of nonlinearities, problem (2) has at least two positive solutions when $\|f\|_{L^2(\mathbb{R}^N)}$ is small.

When we study non-homogeneous nonlinear elliptic problems, we see that the variational structures become more complicated than those of homogeneous problems. In previous works, they obtained the multiplicity of positive solutions by assuming the convexity or certain monotonicity of the nonlinear term. Especially when $g(u) = |u|^{p-1}u$ or g is convex, Zhu (1991), Cao-Zhou (1996), Deng-Li (1997) obtained the existence of two positive solutions.

Analyzing the graph of an energy functional which corresponds to (2), we can readily see that if $\|f\|_{L^2(\mathbb{R}^N)}$ is small, then there is a local minimizer. Using the convexity of the nonlinear term, we can prove that the local minimizer is non-degenerate. The non-degeneracy of the local minimizer enables us to rewrite problem (2) and obtain another positive solution without needing any advanced energy estimates. However when the nonlinear term is general, it is very hard to prove that the local minimizer is non-degenerate. In [6], I analyzed properties of the energy functional more precisely. Then I could obtain the multiplicity of positive solutions without using the non-degeneracy of the local minimizer. In [6], we don't require neither the convexity nor the monotonicity of g. Especially the feature of my result is that we can apply our result to the case g is a FitzHugh-Nagumo type nonlinearity which appears in mathematical biology.

To prove the existence of a second positive solution, we apply the Mountain Pass Theorem whose paths start from the local minimizer. Then we need some estimate of the mountain pass value. This estimate is said to be an interaction estimate. When g is convex, we can easily show this estimate. The key of my result is the following. Using decay rates of solutions at infinity, I obtained precise asymptotic expansions. Then I succeeded to prove the required energy estimate even if the nonlinear term g is general, especially non-convex or sigh-changing.

3: Research on solutions with a vortex to nonlinear Schrödinger equations in $\mathbb{R}^2$

I studied the following nonlinear Schrödinger equation in $\mathbb{R}^2$:

$$-\epsilon^2 \Delta u + (\frac{\epsilon^2 n^2}{|x|^2} + V(x))u = f(u), \ x \in \mathbb{R}^2, \tag{3}$$

where $n \in \mathbb{N}$ and the potential $V(x)$ is a radial function. This problem appears in the study of Bose-Einstein condensation. Especially the natural number n represents an eigenvalue of the angular momentum.

When $f(u) = |u|^{p-1}u$, D'aprile (2002) showed that a radial least energy solution of (3) concentrates around the origin. However she did not give its asymptotic profile. In [3], I obtained precise asymptotic profiles of radial mountain pass solutions of (3).

By the restriction to the class of radial functions, we can see that the radial mountain pass solution $u_\epsilon(x)$ of (3) has a zero (vortex) at the origin. The important part is to prove that a weak limit of the rescaled function of u_ϵ is a nontrivial solution of the corresponding limiting problem. The key of my result is to obtain a uniform pointwise estimate near the origin. More precisely, I analyzed the corresponding ODE and obtained an integral form of u_ϵ. Using this form, I proved that u_ϵ behaves like $u_\epsilon \sim |x|^n$ uniformly near the origin.

In [5], I considered the case the nonlinear term f is asymptotically linear, that is, $\frac{f(u)}{u} \to c_0 < \infty$, $u \to \infty$ and studied the multiplicity of solutions. Although in the original physical background, n should be a natural number, I also studied the case n is a general real positive number. In this situation, we face the following difficulty. Since the singularity of the potential is strong, it is not trivial that a critical point in a suitable function space does satisfy (3) including the origin in the weak sense. Actually if $n > \frac{1}{2}$, I could obtained the pointwise estimate $u_\epsilon(x) \sim |x|^n$ near the origin. Using this estimate, I showed a solution of (3) can be obtained as a critical point of a suitable energy functional.

A difficult point of asymptotically linear problems is that we can not obtain the boundedness of Palais-Smale sequences in general. Therefore I applied a variant of the Mountain Pass Theorem using Cerami sequences. The most difficult part is to prove the boundedness of Cerami sequences. I overcome this difficulty by using a non-resonance condition and a blow-up type argument.

In [5], I also consider the case the potential V is harmonic, that is, $V(x) = |x|^2$. This case is particularly important in physics. Using the Linking Theorem, I showed if c_0 lies in the k-th spectral gap, then problem (3) has at least k distinct nontrivial radial solutions. To obtain the multiplicity of solutions via the Linking Theorem, we use the fact that the codimension of a closed subspace spanned by all eigenvalues is zero. When $n \geq 1$, we can see that the corresponding Schrödinger operator is self-adjoint by Weyl's criterion. Then the above fact follows from a general theory on self-adjoint operators. However in the case $\frac{1}{2} < n < 1$, we can not readily know the self-adjointness. The key of the proof is the following. When $V(x) = |x|^2$, eigenfunctions can be expressed by confluent hypergeometric functions. Using their precise expressions, I proved the codimension of a closed subspace spanned by all eigenvalues is zero.

4: Research on exterior Neumann problem with an asymptotically linear nonlinearity

Let $B_R(0) \subset \mathbb{R}^N$ be an open ball with the radius R centered at the origin. I studied the following exterior Neumann problem:

$$-\Delta u + u = f(u) \text{ in } \mathbb{R}^N \setminus \overline{B_R(0)}, \ \frac{\partial u}{\partial n} = 0 \text{ on } \partial B_R(0), \tag{4}$$

where $N \geq 3$ and n is the interior unit normal vector on $\partial B_R(0)$.

When $f(u) = |u|^{p-2}u$ $(1 < p < \frac{N+2}{N-2})$, Esteben (1991) showed that there exists a least energy solution of (4) and it is not radially symmetric (symmetry breaking phenomenon occurs). In [4], I considered the case the nonlinear term f is asymptotically linear. I proved the existence of a least energy solution and its symmetry breaking phenomenon.

To show symmetry breaking phenomena, we usually use the homogeneity of nonlinear terms or so-called Nehari manifold. However to use the Nehari manifold, we need to impose strong conditions on the nonlinear term. When the nonlinear term is asymptotically linear, such conditions do not hold. The feature of my result is to give a new idea of the proof, which is based on the Pohozaev identity and is applicable to not only the asymptotically linear case but more general nonlinearities.

5: Research on nonlinear Schrödinger equations arising in nonlinear optics

I studied the following nonlinear Schrödinger equation:

$$-u'' + (\lambda - \chi_A(x))u = V(x)(1 - \chi_A(x))|u|^{p-1}u, \ x \in \mathbb{R}, \tag{5}$$

where$\lambda > 1$, $p > 1$, A is a bounded interval in $\mathbb{R}$ and χ_A is the characteristic function of A. This problem describes the propagation of electromagnetic waves through a medium consisting of layers of dielectric materials. Especially the set A is called the linear medium because the equation becomes linear in A. I considered a symmetry breaking phenomenon of a least energy solution. When $V(x) \equiv 1$ and $A = [-d, d]$、 $d > 0$, Arcoya-Cingolani-Gámez (1999) showed that the least energy solution of (5) is asymmetric. In [1], I extended their result to the case $V(x) \not\equiv 1$ and general symmetric bounded intervals A and proved the symmetry breaking phenomenon occurs.

In [1], I also considered the following singularly perturbed problem:

$$-\epsilon^2 u'' + (\lambda - \chi_A(x))u = V(x)(1 - \chi_A(x))|u|^{p-1}u, \ x \in \mathbb{R}. \tag{6}$$

I studied asymptotic profiles of least energy solutions of (6) under suitable assumptions on V. Singular perturbations in nonlinear Schrödinger equations have been widely studied. However in the study of (6), new phenomena occur even if the situation is the most simplest. Especially compared with previous works, a new limiting equation appears. In [2], I studied the case $A = [a_1, b_1] \cup [a_2, b_2]$, $a_1 < b_1 < a_2 < b_2$ and discussed positive solutions which have multi-peaks in (b_1, a_2). I proved the existence of positive multi-peaked solutions whose asymptotic profiles are original.