Plan of my reseach for the future

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The purpose of my research is to study the following 2 problems.

A knot contained in a spatial graph:
We consider a spatial embedding f of the complete graph K_n with n vertices. We denote the maximum number of crossing numbers of knots contained in $f(K_n)$ by $c(f(K_n))$, and the minimum number of $c(f(K_n))$ in every spatial embedding by $\alpha(n)$. Conway-Gordon proved that every spatial embedding of the complete graph with 7 vertices contain a nontrivial knot. In other word, $\alpha(4) = 3$. We are trying to evaluate the values of $\alpha(n)$.

Problem 1: For an integer $n \geq 4$, $\alpha(2n-1) = \alpha(2n) = 2n-5$?
Since we proved that "every column embedding of the complete graph with $(2n-1)$ vertices or $2n$ vertices contains the torus knot of type $(2n-5, 2)$", we conjectured the following equality.

We are trying to solve this problem by the following two methods. One is to deal with any crossing changes of column embeddings. We are studying in the case of $n = 5$. We proved "for a column embedding of the complete graph with 9 vertices, the spatial graph dealt with one crossing change at a vertex that appears on the xy-plane contains a knot whose crossing number is greater than or equal to 5". Conway-Gordon proved that every spatial embedding of the complete graph with 7 vertices contain a nontrivial knot by using the Arf invariant. Note that we can not evaluate the crossing number of a knot by the Arf invariant. So, in order to evaluate crossing numbers we try to construct an invariant which is not changed by any crossing change. Another is to deal with a generalization of linear embeddings. In particular, we focus on p-bent embeddings. A p-bent embedding is an embedding such that any edges mapped on $p+1$ segments. In case of $p = 1$, a 1-bent embedding of the complete graph is a linear embedding of the bipartite graph. So, the number of vertices of this linear embedding of the bipartite graph is determined by the number of vertices of the complete graph. We try to prove that this embedding contains a knot whose crossing number is greater than or equal to 5 in the case of $p = 1$. Moreover, we will study the case of $p \geq 2$.

The knot table by circular numbers:
We are try to classify knots by column embeddings. For a knot K, we proved the following relations between the circular number $\mathrm{Circ}(K)$ and the crossing number$c(K)$; (1) $\mathrm{Circ}(K) \leq c(K) + 2$, (2) if $\mathrm{Circ}(K) \geq 2$, then $2\{\mathrm{Circ}(K)\}^2 - 3\mathrm{Circ}(K) \geq c(K)$. As a result, it may be meaningful to make the table of knots by circular numbers. In the case of $\mathrm{Circ}(K) = 3$, we proved that K is a trefoil knot. We consider the case of $\mathrm{Circ}(K) \geq 4$.

Problem 2: If a knot K is a prime knot with circular number 4, then is K the 8-figure knot?
Here, a prime knot is a knot that is not a connected sum of knots.

Since the projections of a circle to a plane are ellipses, the combinatorial number of immersions of 4 ellipses is finite. The number of projections involved in 4 ellipses is also finite. Thus, the number of knots associated with circular embedding and such a projection is finite. We believe that we can solve Problem 2 in this way. Since lengths of major and minor axes of ellipses are not fixed, this problem is too complicated. In the case of the circular number $n \geq 4$, it is more difficult. We might approach to the property of circular numbers by making an algorism.