Summary of my research

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It is famous that the nilpotent orbits in the Lie algebra $\mathfrak{gl}_n(\Bbbk)$ of the general linear group $\mathrm{GL}_n(\Bbbk)$ over an algebraically closed field $\Bbbk$ are classified by the partitions of n, by the theory of Jordan normal form. Let G be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ and $\mathfrak{g}$ the Lie algebra of G. In 1976, P. Bala and R. W. Carter proved that there exists a bijection between the nilpotent $\mathrm{Ad}(G)$-orbits in $\mathfrak{g}$ and conjugacy classes of pairs (L, P_L) where L is a Levi subgroup (of a parabolic subgroup) of G and P_L is a distinguished parabolic subgroup of L, if $\mathrm{ch}\,\Bbbk = 0$ or if $\mathrm{ch}\,\Bbbk \gg 0$, by using the $\mathfrak{sl}_2$-theory. This classification theorem is called the Bala-Carter theorem, and it is known well. Afterward, K. Pommerening showed that the Bala-Carter theorem can be extended to good characteristic. This fact is a basic result in the representations of algebraic groups, and it is especially important in the theory of nilpotent orbits in good characteristic. For example, the following results has been shown by using Pommerening's theorem: the existence of cocharacters associated to the nilpotent elements in $\mathfrak{g}$, the existence for good transverse slices to the nilpotent orbits in $\mathfrak{g}$, and the dimension theorem between the variety of Borel subalgebras containing X and the nilpotent orbits $\mathrm{Ad}(G)(X)$ for each nilpotent element X in $\mathfrak{g}$. His proof, however, needed case-by-case computations in each root system, and some details of these computations are omitted. Therefore some noncomputational conceptual proof was expected, but it was not solved for a long time.

In 2003, A. Premet gave a conceptual proof of Pommerening's theorem by using the Kempf-Rousseau theory. At the same time, he proved the existence for good transverse slices to the nilpotent orbits in $\mathfrak{g}$. The proof is considerably difficult compared with proof of the Bala-Carter theorem; Premet's proof uses the invariant theory, the theory of finite reductive groups, and the Bala-Carter theorem.

So we researched whether there was more concise proof. To be thought it is the most reasonable as a proof method of using the Kempf-Rousseau theory is to prove the following; for any distinguished nilpotent element X the optimal parabolic subgroup $P(X)$ is distinguished and the Richardson orbit corresponding to $P(X)$ contains X. However, two weighty subjects are generated.

- For each nilpotent element X in $\mathfrak{g}$, is there an optimal cocharacter for X such that X is a $m(X)$-weight vector for λ? (Here $m(X)$ is the certain positive integer determined by X.)
- For each nilpotent element X in $\mathfrak{g}$, do we have $m(X) \leq 2$?

These are true facts by the proof of Premet. Therefore anticipating that there should exist a direct proof of this result, and we have obtained an even simpler proof of Pommerening's theorem. We also simplified Premet's proof of the existence for good transverse slices to the nilpotent orbits in $\mathfrak{g}$. The main point that became concise is as follows.

- The Bala-Carter theorem need not be used.
- The theory of finite reductive groups need not be used.
- Because we only consider "distinguished" nilpotent elements, a part of proof has been simplified more.