Future works

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N.Ghoussoub-F.Robert [$**$] proves the existence of a minimizer for the best constant of the Sobolev-Hardy inequality under the assumptions that the origin 0 lies on the boundary of a bounded domain Ω, and the mean curvature of $\partial\Omega$ at 0 is negative. Through the corresponding Euler-Lagrange equation, this result implies the existence of a positive solution to the following elliptic equation :

$$\begin{cases} \Delta u + \frac{u^{2^*-1}}{|x|^s} = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $n \geq 3$, $0 < s < 2$, $2^* := \frac{2(n-s)}{n-2}$ and Ω is a smooth bounded domain in $\mathbb{R}^n$. Based on their result, in the papers [5, 8], we succeed in constructing a positive solution to the following elliptic equation which includes two kinds of critical terms :

$$\begin{cases} \Delta u + \mu \frac{u^{2^*-1}}{|x|^s} + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $n \geq 3$, $0 < s < 2$, $2^* = \frac{2(n-s)}{n-2}$, Ω is a smooth bounded domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$, and the mean curvature of $\partial\Omega$ at 0 is negative. In the paper [8], we consider some minimizing problem for a Sobolev-Hardy type inequality, and find its minimizer which yields a positive solution $u \in H_0^1(\Omega)$ to (2). However, this procedure only allows us to get a solution to (2) for **some** $\mu > 0$. On the other hand, in [5], we treat the variation corresponding to (2) for any fixed $\mu > 0$, and apply the mountain pass lemma without the Palais-Smale condition to get a positive solution under the assumption of the negative mean curvature at 0 for **any** $\mu > 0$. Thus we can regard the result in [5] as a refinement of the result in [8]. Taking account of the above known results, I plan the future works below :

(i) In both of (1) and (2), we considered the existence problem under the Dirichlet boundary condition. Our next interest is in investigating this problem under the Neumann boundary condition. That is, the equation (1) will be replaced to the following :

$$\begin{cases} \Delta u - u + \frac{u^{2^*-1}}{|x|^s} = 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3}$$

One way to try to get a positive solution to the equation (3) is to solve the following minimizing problem :

$$\inf \left\{ \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 dx \,\middle|\, u \in H^1(\Omega) \text{ and } \int_\Omega \frac{|u|^{2^*}}{|x|^s} dx = 1 \right\},$$

where the function space should be taken as $H^1(\Omega)$ having no restriction on the boundary, which implies the Neumann boundary condition to (3).

(ii) We next try to treat the existence problems in (1) and (2) with the critical exponent $s = 2$, which comes from the Hardy inequality with the weight $\frac{1}{|x|^2}$. However, we easily find a difficulty to consider this case compared to the case of $s < 2$ on the regularity for positive solutions. Indeed, K.S.Chou-C.W.Chu [$*$] and N.Ghoussoub-F.Robert [$**$] show the boundedness of positive solutions u to (1) and (2), and then the standard elliptic theory and the Sobolev embedding yield $u \in C^1(\overline{\Omega})$. On the other hand, such a procedure is also available to the case $s = 2$ and we get $u \in L^\infty(\Omega)$. However, the elliptic estimate with the Sobolev embedding cannot guarantee the differentiability or even the continuity for solutions at the origin due to the critical weight $\frac{1}{|x|^2}$, which is a difference between the cases $s = 2$ and $s < 2$. Hence, we need to investigate the behavior of solutions near the origin for the case $s = 2$.

• Related references

[$*$] K.S.Chou, C.W.Chu, *On the best constant for a weighted Sobolev-Hardy inequality*, J. London Math. Soc. (2) **48** 137–151 (1993).

[$**$] N.Ghoussoub, F.Robert, *The effect of curvature on the best constant in the Hardy-Sobolev inequalities*, Geom. Funct. Anal. **16** 1201-1245 (2006).