RESEARCH PLAN

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1. OVERVIEW

My research is in the area of automorphic forms and representation theory. One of the fundamental problems in the theory of automorphic representations is the definition of local L and ε factors. There are three different methods to define these factors. The first one follows the classical approach of integral representations. It defines the L-factors directly as a generator of the fractional ideal generated by a certain family of the local zeta integrals. The second one is the Langlands-Shahidi method, which defines the local factors in a large number of cases when the representation is generic. The third one is to define them as the corresponding Artin factors via the conjectured local Langlands correspondence.

The first approach is better suited for obtaining control of the analytic behavior of the local zeta integrals, which is important to analyze the global L-function via integral representation and via Eisenstein series. On the other hand, the second method defines the γ-factors in the first place by using a global argument. In the tempered case the γ-factor determines the L and ε factors at the same time, and they are defined in general by requiring that they behave well under the Langlands classification. Both the first and second methods fall far short of the goal of constructing all local factors of representations. When both the methods are available, we need to produce a full local theory of integral representation to reconcile the two definitions. This has to be done case by case. The theory in the case of the Rankin-Selberg integral representation for $GL(n) \times GL(m)$ is very deep and is the paradigm for the study of automorphic L-functions. However, most of local theory has remained unfinished, mostly due to its difficulty and partly because of attraction of finding new integral representations.

The local Langlands correspondence is a non-abelian generalization of the local class field theory. It has been an important conjecture for a long time, but it was proven for $GL(n)$ by Harris-Taylor and by Henniart, and it will be established for quasi-split classical groups by the pending work of Arthur. The local Langlands correspondence for $GL(n)$ is formulated in terms of L and ε factors defined on both representation and Galois theoretic sides, and it asserts that the three definitions agree. For general reductive groups the well-known approach to characterize the local Langlands correspondence is in terms of stability of distribution characters. It is expected that the local factors defined this way should be the same as the representation theoretically defined ones.

2. Current research projects

My current research involves the following connected projects:

- a full local theory of local doubling L and ε factors;
- characterization of the nonvanishing of local and global theta liftings.

We now briefly review the doubling method. Let F be a number field with adele ring $\mathbb{A}$. For definiteness let $G = O(n)$, $G^\square = O(n,n)$ and P the parabolic subgroup of $G^\square$ whose Levi factor is $GL(n)$. For an idele class character χ we denote by $I(s,\chi)$ the representation of $G^\square(\mathbb{A})$ obtained by inducing the one dimensional representation $(\chi|\ |^s) \circ \det$ of $P(\mathbb{A})$. For a pair (ϕ_1, ϕ_2) of cusp forms on G and a section $f^{(s)}$ of $I(s,\chi)$ we consider the integral

$$\int_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})} \phi_1(g_1)\phi_2(g_2)E((g_1,g_2); f^{(s)})dg_1 dg_2,$$

which unfolds to the product of local zeta integrals

$$\int_{G(F_v)} \mathcal{P}_v(\pi_v(g)\phi_{1,v} \otimes \phi_{2,v}) f_v^{(s)}((g,e))dg$$

and yields at least the partial standard L-function for $O(n)\times GL(1)$ up to some normalizing factors. Here $\mathcal{P} = \prod_v \mathcal{P}_v$ is the usual pairing on the forms on $G(F)\backslash G(\mathbb{A})$ and we have assumed that ϕ_1, ϕ_2 and $f^{(s)}$ are all factorizable. We must take ϕ_1 and ϕ_2 in the same automorphic representation for this integral to be nonzero. This method does not depend on the existence of Whittaker models and works for all representations of all classical groups.

This method has been introduced by Piatetski-Shapiro and Rallis in [6, 5]. Following their work, Harris, Kudla and Sweet [2] defined the local factors in the case where G is a unitary group over a nonarchimedean local field, and they related them to the nonvanishing of certain local theta liftings at least for supercuspidal representations. The L-factor is here defined to be a g.c.d of local zeta integrals as $f_v^{(s)}$ ranges over good sections. The regularized Siegel-Weil formula discovered by Kudla and Rallis [3] identifies residues of the Eisenstein series with regularized theta integrals and thus yields an inner product formula which relates the residues of the standard L-functions to the Petersson inner products of certain global theta liftings. Wee Teck Gan and Takeda [1] derived an inner product formula which involves not only residues but also values of the standard L-functions at the critical points in the case of orthogonal groups. I also obtained some extensions of the Siegel-Weil formula in [7, 8]. In earlier works the results of this sort are stated in terms of the partial standard L-functions due to the lack of a theory of local factors. However, since the local zeta integrals are not necessarily holomorphic at the point in question, we have to consider the complete L-functions, where missibg L-factors are defined as a g.c.d of local zeta integrals, in order to characterize the nonvanishing of global theta liftings completely. I prove that the L-factor defined as a g.c.d agrees to the standard Langlands L-factor for good places in [9]. This is indispensable in global application and a starting point of my projects.

On the other hand, Lapid and Rallis [4] systematically developed the theory of γ-factors for the doubling method. Then they define L and ε factors from the γ-factor, using the Langlands classification, in the same way as was explained before. It is expected that the two definitions of the local factors coincide. The two γ-factors defined in [2] and [4] are essentially the same as they are essentially given as the proportionality constants for local functional equations. As has already been mentioned, I proved that the two L-factors coincide in many cases including the unramified case so far, and further progress is now being made.

References

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