Current research (Satoshi Takagi)

In the doctor course, I was studying the higher-dimensional Weil-zeta function, which is defined as the generating function of the number of effective cycles of intermediate dimension on a fixed polarized projective variety. According to the definition of Wan[1], this function does not converge in the complex-analytic sense, but only in the p-adic sense. The alternating definition proposed by Moriwaki[2] is given by changing the power of the indeterminates, which turns out to be complex-analytic. I showed that the convergent radius of the Moriwaki's higher-dimensional Weil-zeta function is determined by the self-intersection number of the polarization divisor, when the dimension of effective cycles are of codimension 1 relative to the given variety [3]. Here, I showed Fujita's approximation theorem in positive characteristics[4] in order to give the upper bound of the so-called volumes of the line bundles.

Also, some computations hint us that when the codimension of the cycles is more than 1, this convergent radius gives a piecewise-linear function on the ample cone of the Néron-Severi group, which implies some relations with tropical geometry.

Roughly speaking, tropical geometry is a theory concerning with scheme-like geometric objects constructed from semirings. In other areas of mathematics, some people are proposing to construct geometric objects from commutative monoids. This is the so-called schemes over $\mathbb{F}_1$, the hypothetical field with one element; it is recognized as an important tool to draw the Riemann-zeta function into the arithmetical workfield.

With these aspects in mind, I aimed to construct the theory of $\mathscr{A}$-schemes which includes tropical geometry and schemes over $\mathbb{F}_1$ in 2010[5]. This $\mathscr{A}$-scheme is defined as a $\mathscr{A}$-valued space with nice properties, for a fixed algebraic type $\mathscr{A}$ which satisfies certain conditions (which is fullfilled, for example, when $\mathscr{A}$ is the type of monoids, semirings, and rings). Even if we restrict our attention to the case when $\mathscr{A}$ is the type of rings, this $\mathscr{A}$-scheme has some advantages, compared with the conventional schemes: the category of $\mathscr{A}$-schemes contains that of ordinary schemes as a full subcategory, and the inclusion functor preserves fibre products and open-patchings. Also, there is a spectrum functor between this category and the opposite category of rings. Therefore, we can say that $\mathscr{A}$-schemes are very close to conventional schemes. On the other hand, we can take arbitrary limits and co-limits on $\mathscr{A}$-schemes, which enables us to do various categorical manipulations.

For example, we can construct Zariski-Riemann spaces (ZR-spaces for short) as $\mathscr{A}$-schemes, by just imitating the construction of Stone-Čech compactifications. A ZR-space is a universal compatification of a given non-compact variety, and this enables us to handle asymptotic invariants, such the volumes of line bundles in a uniform way. We can also obtain the Nagata embedding theorem as a Corollary, using ZR-spaces.

References

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