Abstract of results

1. Study of a birational relation of members in families of $K3$ surfaces.

Define the Picard lattice $\mathrm{Pic}(\mathcal{F}_a)$ of the family $\mathcal{F}_a$ of weighted $K3$ surfaces in a projective space $\mathbf{P}(a)$ to be the Picard lattice of generic members in $\mathcal{F}_a$.

Theorem 1 [Kobayashi-Mase] If the lattices $\mathrm{Pic}(\mathcal{F}_a)$ and $\mathrm{Pic}(\mathcal{F}_b)$ are isometric, then, there exists a common subfamily $\mathcal{F}$ of $\mathcal{F}_a, \mathcal{F}_b$ that induces a birational correspondence between the general members in the family $\mathcal{F}_a$ and those in $\mathcal{F}_b$. The correspondence is given by an explicit map of monomials in the projective spaces $\mathbf{P}(a)$ and $\mathbf{P}(b)$. The Picard lattice $\mathrm{Pic}(\mathcal{F})$ of $\mathcal{F}$ is isometric to the Picard lattices $\mathrm{Pic}(\mathcal{F}_a), \mathrm{Pic}(\mathcal{F}_b)$. ■

By Theorem 1, it is induced that there are essentially 75 families of weighted $K3$ surfaces in the sense that there are birational correspondences of general members among families.

Theorem 2 [Mase] The Picard lattices of families of $K3$ surfaces in smooth toric Fano 3-folds are mutually distinct. ■

By Theorem 2, it is induced that there does not exist a birational correspondence among the families of $K3$ surfaces in smooth toric Fano 3-folds.

Let l and C be a line and a smooth plane cubic on the same plane in $\boldsymbol{P}^3$; X' (resp. X) be a smooth Fano 3-fold obtained by a blow-up of $\mathbf{P}^3$ along l (resp. C); and $\mathcal{F}'$ (resp. $\mathcal{F}$) be the family of $K3$ surfaces in X' (resp. X).

Theorem 3 [Mase] There exists an explicit birational correspondence between the general members in $\mathcal{F}'$ and those in $\mathcal{F}$ induced by a projective transformation. ■

It is expected by Theorem 3 that the Gromov-Witten invariant of $K3$ surfaces in the smooth non-toric Fano 3-fold X can be computed, by using the fact that there are protocols to compute the Gromov-Witten invariant of toric Calabi-Yau manifolds.

2. Study of automorphisms of $K3$ surfaces.

Let X be a smooth toric Fano 3-fold with the polar dual polytope Δ and S be a $K3$ surface in X defined by a polynomial $m_0 + m_1 + \ldots + m_r$, where m_0 is a monomial corresponding to the only lattice point in the interior of Δ, and m_i's $(i = 1, \ldots, r)$ to the vertices of Δ.

Theorem 4 [Mase-Taki] Denote by $\mathrm{Aut}_{\boldsymbol{T}}(S)$ the group of automorphisms of S that are restricted from automorphisms of X. The group $\mathrm{Aut}_{\boldsymbol{T}}(S)$ is determined. ■

Theorem 4 proposes a way of computing a (sub)group of automorphisms of toric hypersurfaces by observing the polytope.