1. Context: from braid groups to loop braid groups

The *braid groups* B_n, introduced by Hurwitz [16] in 1891 as fundamental groups of configuration spaces of n points in the complex plane, owe their name to Artin [2], which considered them in terms of braid automorphisms of the free group on n generators F_n. Magnus [21] considered the same groups from the point of view of mapping classes, while Markov [23] introduced these groups from a purely group-theoretic point of view. All these points of view have long been known to be equivalent [29].

The *loop braid groups* LB_n, introduced under this name in [3], have a very similar nature. Loop braids have first been considered as *motions* of n unknotted, unlinked circles in $\mathbb{R}^3$ [14, 15], a notion that can be translated in terms of mapping classes. McCool and Savushkina [24, 25] considered them as *basis-conjugating automorphisms* and *permutation conjugacy automorphisms* of F_n. They are also known in the literature as *welded braids*, as defined by Fenn-Rimány-Rourke [11], in the form of elements of a presented group, with diagrams representing them. More recently, in [9] an interpretation of these groups appeared in terms of the fundamental groups of the configuration spaces of n unlinked euclidean circles. Moreover one could give a topological interpretation of loop braids in terms of braided ribbon tubes, *i.e.*, as 2-dimensional submanifolds of $\mathbb{R}^4$ that bound immersed 3-manifolds with only ribbon singularities, and of Gauss braid diagrams – a point of view that allows us to consider loop braid groups as quotients of virtual braid groups.

2. Results

2.1. Unifying the different approaches to loop braid groups.

In addition to appearing in many areas of pure mathematics, thanks to their nature similar to braid groups B_n, loop braid groups (together with their representations) are getting a lot of attention in the mathematical physic's branch of statistics of loop-like objects in three spatial dimensions [3, 10, 20]. Several parallel formulations and notations are being used by researchers working in different fields, this is why a well-established collection of these formalisms was needed. Although the equivalence of some of the points of view was already proved or at least stated in the literature, for example in [4], the explicit isomorphisms between some of them were not yet to be found. In my thesis I introduced the several interpretations of loop braid groups appeared in the literature, and provided a complete proof of the equivalence of these formulations. This provided a clear and well-established theoretical setting for these groups, as the one we have in the case of the braid group B_n. Moreover I was able to find some topological applications. This result appears, in a more synthetic form in preprint [26], submitted in May 2016.

2.2. Towards a topological version of Markov's theorem for ribbon torus-links.

Using the understanding of the interpretation of loop braids as ribbon knotted objects, I started working on the adaptation of a classical result of braid theory, beginning with Markov's theorem: two braids (possibly with different numbers of strings) have isotopic closures in $\mathbb{R}^3$ if and only if one can be obtained from the other with a finite number of Markov moves, called conjugation and stabilization. This theorem allows to describe all braids with isotopic closures as knots in $\mathbb{R}^3$. In the classical case many proofs are known ([6, 8, 18, 22, 28]): some of them are combinatorial, other topological, and these last ones are particularly beautiful in this framework, as they use all of the isomorphisms between the different definitions of braid groups. Sorting out a topological version of Markov's theorem entailed the complication of dealing with the fact that ribbon tubes in $\mathbb{R}^4$ could change orientation (seeing this from the point of view of movements of circles in time, it corresponds to a circle making a "flip" of 180 degrees) whilst 1-dimensional manifolds in $\mathbb{R}^3$ do not present this problem.

Using the approach of [28] and [18] I have been able to prove a version of Markov theorem for ribbon braids with closure in a solid torus in the 4-dimensional space, *i.e.*, a 3-ball times the circle S^1:

Theorem 2.1. *For any $n \geq 1$ and any $\beta, \beta' \in LB_n$, the closed ribbon braids $\hat{\beta}, \hat{\beta}'$ are isotopic in $B^3 \times S^1$ if and only if β and β' are conjugate in LB_n^{ext}.*

To extend it to $\mathbb{R}^4$ means to prove the invariance for stabilization. Following the approach of [18] would mean to use the diagrammatic description of loop braids given by [19], but this would not cast a light on the topological structure. Besides, it shall be recalled that for welded braid diagrams and welded links there already is a combinatorial version of Markov's theorem, due to Kamada [17]. So in my future plans I will consider Bennequin's proof and use contact structures to prove a complete topological version of Markov's theorem for loop braids and extended loop braids.

2.3. **Alexander invariants for ribbon tangles and planar algebras.** In a joint work with Vincent Florens of the University of Pau, we have been working on an Alexander invariant defined on ribbon tangles. From this invariant we can extract a functorial generalization of the Alexander polynomial, which goes from the category of ribbon cobordisms to the category of graded modules. This functor is an extension of the Alexander functor for classical tangles defined in [7] and [13]. When considered on braid-like objects, this functor coincides with the exterior powers of the Burau representation extended to ribbon braids. This invariant has a profund topological meaning, being a generalization of the classical Alexander polynomial of links to ribbon tangles, but lacked a simple way of computation. We found a combinatorial "easy" way of computing establishing a correspondence with Jana Archibal's multivariable Alexander polynomial for virtual/welded tangles defined in [1]. This second invariant is defined on diagrams being part of a circuit algebra structure. This last one is a generalisation of Jones' planar algebras, or, more explicitly a morphism between the operad of circuit diagrams, which are composed by disks with points marked on the border, a finite number of subdisks removed, and arcs relying the points (the *circuit* comes from the fact that important information carried by these diagrams is how points are paired), and the operad of modules on a certain ring. So before the making of the bridge between the two invariants, this second one lacked of a topological interpretation. One of the main difficulties has been working with ribbon cobordism and modules categories on one side, and tensor algebras with a circuit algebra structure on the other side, and having to translate the two sides in a common formalism: we overcame it defining a circuit algebra structure on the topological objects. This work resulted in a preprint [27], now submitted.

2.4. **Quotients of the virtual braid group.** In a joint work with Valeriy Bardakov and Paolo Bellingeri we studied the group of *unrestricted virtual braids* UVB_n, a quotient of the virtual and welded braid groups, and described its structure, the main result being the following:

Theorem 2.2 ([5, Theorem 2.4])**.** *Let X_n be the right-angled Artin group generated by $x_{i,j}$ for $1 \leq i \neq j \leq n$ where all generators commute except the pairs $x_{i,j}$ and $x_{j,i}$ for $1 \leq i \neq j \leq n$. The group UVB_n is isomorphic to $X_n \rtimes S_n$ where S_n, the group of permutations on n elements, acts by permutation on the indices of generators of X_n.*

Using the insight given by knowing its structure, we showed that any fused link admits as a representative the closure of a *pure unrestricted virtual braid*, and deduced an easy proof of a theorem of Fish and Keyman [12] stating that fused links that have only classical crossings are characterized by their linking numbers. Furthermore we constructed a representation for UVB_n in the group of automorphisms of the free 2-step nilpotent group of rank n which allowed us to defined a notion of group of fused links. This work resulted in a publication [5].

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