Research Results

I have studied the topology of Hessenberg varieties. Hessenberg varieties are subvarieties of a flag variety and its topology is associated with many research areas. The following varieties are the special cases of Hessenberg varieties which are associated with many research areas:

1. Springer varieties (geometric representation of a symmetric group)
2. Peterson varieties (quantum cohomology of the flag varieties)
3. Regular nilpotent Hessenberg varieties (hyperplane arrangements)
4. Regular semisimple Hessenberg varieties (graph theory)

First, I calculated the equivariant cohomology rings of varieties in the above 1, 2, 3 (List of Papers [1-1], [1-2], [1-3], [1-4], [2-2]). In particular, we obtained an explicit presentation of the cohomology rings of regular nilpotent Hessenberg varieties. From this result, we also obtained the connection with regular semisimple Hessenberg varieties (List of Papers [2-2]) and the connection with hyperplane arrangements (List of Papers [2-1]). I explain these results as below.

A type A Hessenberg variety $\mathrm{Hess}(X,h)$ is a subvariety of a flag variety determined by two data (i) a linear operator $X : \mathbb{C}^n \to \mathbb{C}^n$ and (ii) a Hessenberg function $h : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ (which is a weakly increasing function satisfying $h(i) \geq i$ for any i). If X is regular nilpotent N(resp. regular semisimple S), $\mathrm{Hess}(N,h)$ is called a regular nilpotent Hessenberg variety ($\mathrm{Hess}(S,h)$ is called a regular semisimple Hessenberg variety). Then, we obtained the following ring isomorphism:

$$H^*(\mathrm{Hess}(N,h)) \cong H^*(\mathrm{Hess}(S,h))^{\mathfrak{S}_n}$$

where $\mathfrak{S}_n$ is the n-th symmetric group and the $\mathfrak{S}_n$-action on $H^*(\mathrm{Hess}(S,h))$ is introduced by Tymoczko using GKM graph.

We consider a type A Hessenberg variety in the above. We can define a Hessenberg variety as a subvariety of a flag variety G/B in any types. Then, a Hessenberg variety $\mathrm{Hess}(X,I)$ is a subvariety of a flag variety G/B determined by two data (i) $X \in \mathfrak{g}$ and (ii) lower ideal $I \subset \Phi^+$ where $\mathfrak{g}$ is the Lie algebra of G and Φ^+ is a set of all positive roots. On the other hand, from a lower ideal I we can define an ideal subarrangement $\mathcal{A}_I$ of a Weyl arrangement, and we consider its logarithmic derivation module $D(\mathcal{A}_I)$ which is a module over $\mathcal{R} := \mathrm{Sym}\ \mathfrak{t}^*$ the symmetric algebra of a dual space of the Lie algebra of the maximal torus. We define an ideal $\mathfrak{a}(I)$ of a ring $\mathcal{R}$ from $D(\mathcal{A}_I)$, and we obtained the result that its quotient ring $\mathcal{R}/\mathfrak{a}(I)$ is isomorphic to the cohomology ring $H^*(\mathrm{Hess}(N,I))$ of the regular nilpotent Hessenberg variety as rings. Moreover, we obtained the quotient ring $\mathcal{R}/\mathfrak{a}(I)$ is isomorphic to the W-invariant subring $H^*(\mathrm{Hess}(S,I))^W$ of the cohomology ring of the regular semisimple Hessenberg variety as rings where W is the Weyl group. In summary we obtained the following ring isomorphism:

$$H^*(\mathrm{Hess}(N,I)) \cong \mathcal{R}/\mathfrak{a}(I) \cong H^*(\mathrm{Hess}(S,I))^W$$