Research Plan

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Measure-theoretic entropy and linear response formula for random β-transformations

Let $\beta > 1$ and $p \in (0,1)$. Let us denote by $[\beta]$ the greatest integer less than β. As stated in summary of research, each random β-transformation K_β, defined on $\{0,1\}^{\mathbb{N}} \times [0, [\beta]/(\beta-1)]$, has a unique invariant probability measure $\hat{\mu}_{\beta,p}$ absolutely continuous with respect to the product measure $m_p \otimes \lambda_\beta$, where m_p is the $(1-p,p)$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ and λ_β is the normalized Lebesgue measure on J_β. Furthermore, the measure $\hat{\mu}_{\beta,p}$ is of the form $\hat{\mu}_{\beta,p} = m_p \otimes \mu_{\beta,p}$. In [1], Dajani and de Vries calculated the measure theoretic entropy $h_{\hat{\mu}_{\beta,p}}(K_\beta)$ in the case where β is some special algebraic integer and conjectured an explicit formula for the entropy $h_{\hat{\mu}_{\beta,p}}(K_\beta)$ in general cases. In this research, I will attempt to verify the explicit formula and will investigate the asymptotic behavior of the entropy $h_{\hat{\mu}_{\beta,p}}(K_\beta)$ for parameters (β, p). In particular, it is interesting to consider the minimum and maximum values problem for the function $p \mapsto f_{\hat{\mu}_{\beta,p}}(K_\beta)$, which we can consider since the function is smooth due to the analyticity of the function $p \mapsto f_{\beta,p}$, where $f_{\beta,p}$ is the density function of $\mu_{\beta,p}$. One of our methods to study the behavior of the entropy $h_{\hat{\mu}_{\beta,p}}(K_\beta)$ for a parameter p is to find a sort of a linear response formula for the function $p \mapsto f_{\beta,p}$, which give a representation of the N-th derivative $\dfrac{\partial^N f_{\beta,p}}{\partial^N p}$, and to apply it to the minimum and maximum values problem. Therefore, I will aim to give a linear response formula for the function $p \mapsto f_{\beta,p}$ and to connect it to the explicit formula for the entropy $h_{\hat{\mu}_{\beta,p}}(K_\beta)$. Furthermore, I will attempt to extend the linear response formula to more general classes of random dynamical systems.

Bernoulli convolutions and β-expansions

Let $1 < \beta \leqq 2$ and $p \in (0,1)$. Let us denote by m_p the $(p, 1-p)$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$. We define the function $g_\beta : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ by

$$g_\beta((a_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}$$

for $(a_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$. Then, in the view of β-expansions, the Bernoulli convolution $\nu_{\beta,p}$ is defined as the distribution of f_β with respect to m_p i.e. $\nu_{\beta,p} = m_p \circ g_\beta^{-1}$. It is known that the Bernoulli convolution is a self-similar measure on $\mathbb{R}$ whose support is $[0, [\beta]/(\beta-1)]$ and either absolutely continuous or singular with respect to the Lebesgue measure on $\mathbb{R}$ for each (β, p). In the case of $\beta = 2$, the distribution function of $\nu_{\beta,p}$ is known as the Lebesgue singular function for a parameter p and its value on $x \in [0,1]$ is given via the decimal expansion of x. In this research, I will aim to investigate Bernoulli convolutions and β-expansions in a similar analogy of the case of $\beta = 2$, and to relate the algebraic properties of β to the properties of the corresponding Bernoulli convolution. Since the distribution function of $\nu_{\beta,p}$ satisfies a similar functional equation which the Lebesgue singular function satisfies, I will attempt to extend some results known about the Lebesgue singular function to the Bernoulli convolution. For example, I will try to give the value of the distribution function at $x \in [0, [\beta]/(\beta-1)]$ by using β-expansions of $x \in [0, [\beta]/(\beta-1)]$.

References

[1] K. Dajani and M. de Vries, *Invariant densities for random β-expansions*, J. Eur. Math. Soc. **9** (2007), 157–176.