Summary of research

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Let $\beta > 1$. Let us denote by $[\beta]$ the greatest integer less than β. We call an expansion of a number $x \in [0, [\beta]/(\beta - 1)]$ of the form

$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}$$

with $\{a_n\}_{n=1}^{\infty} \in \{0, 1, \cdots, [\beta]\}^{\mathbb{N}}$ a β-expansion of x. As is well-known that in the case when β is an integer greater than or equal to 2 a number $x \in [0, [\beta]/(\beta-1)]$ has a unique β-expansion except countably many points in $[0, [\beta]/(\beta - 1)]$. In the case when β is a non-integer, however, almost every x in $[0, [\beta]/(\beta-1)]$ (with respect to the Lebesgue measure) has uncountably many different β-expansions, whose statistical properties are of interest. The statistical properties of β-expansions closely relate to the ergodic properties of dynamical systems which generate β-expansions. In my previous work, I studied the ergodic properties of dynamical systems which generate β-expansions or a sort of β-expansions, and obtained some results about them. We shall state the results in the following.

1. Artin-Mazur zeta functions and lap-counting functions of generalized β-transformations (List of papers 1,2)

The β-transformation $\tau_\beta : [0, 1] \to [0, 1]$ is defined by $\tau_\beta(x) = \beta x \bmod 1$ for $x \in [0, 1]$. As well-known that this transformation generates the greedy β-expansion of a number $x \in [0, 1]$. In [3], Flatto et al. gave a functional equation for the Artin-Mazur zeta function of a β-transformation, which defined on some region in $\mathbb{C}$, by using the generating function ϕ_β for the coefficients sequence of the greedy β-expansion of 1. In general, for a piecewise linear expanding map each of whose branches has the same absolute value of the slope, it is known that the poles of its Artin-Mazur zeta function relate to the ergodic properties of the map. Therefore, it is important to investigate the analytic properties of the function. In my paper 1, I extended the result by Flatto et al. to the class of generalized β-transformations ,introduced by Góra in [4], each of which is obtained by replacing some of the branches of a β-transformation with branches of the constant negative slope. As an application, I investigated the relation between the analytic properties of the Artin-Mazur zeta function and the algebraic properties of β.

In [3], Flatto et al. also gave a functional equation for the lap-counting function of the β-transformation τ_β via the generating function ϕ_β. Furthermore, they showed that the poles of the Artin-Mazur zeta function of τ_β coincide with those of its lap-counting function, including their multiplicity. In my paper 2, I extended the functional equation to the class of generalized β-transformations and showed that the poles of the Artin-Mazur zeta function of a generalized β-transformation coincide with those of its lap-counting function in special cases, which include the case where the map is a negative β-transformation.

2. Invariant density functions of random β-transformations (List of papers 3)

In [1], Dajani and Kraaikamp defined the greedy map $T_{\beta,1}$, which is a naturally extended map of the β-transformation τ_β to $J_\beta := [0, [\beta]/(\beta - 1)]$, and the lazy map $T_{\beta,0}$, which is given by $T_{\beta,0} = l_\beta \circ T_{\beta,1} \circ l_\beta^{-1}$, where l_β is the map: $l_\beta(x) = [\beta]/(\beta - 1) - x$ for $x \in J_\beta$. By using the maps $T_{\beta,1}$ and $T_{\beta,0}$, they introduced a sort of random β-transformation K_β on $\{0,1\}^{\mathbb{N}} \times J_\beta$. We can obtain a β-expansion of $x \in J_\beta$ for each $\omega \in \{0,1\}^{\mathbb{N}}$ via the map K_β, which is called a random β-expansion of x. Since all β-expansions of x are obtained as random β-expansions of x, we can investigate the statistical properties of β-expansions via the ergodic properties of the map K_β. In [2], Dajani and de Vries showed that there exists a unique K_β-invariant probability measure $\hat{\mu}_{\beta,p}$ absolutely continuous with respect to the product measure $m_p \otimes \lambda_\beta$, where m_p is the $(1-p, p)$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ with a parameter $p \in (0,1)$ and λ_β is the normalized Lebesgue measure on J_β. By general theory, we have that the probability measure $\hat{\mu}_{\beta,p}$ is given by the product measure of the form $m_p \otimes \mu_{\beta,p}$ and the dynamical system $(K_\beta, m_p \otimes \mu_{\beta,p})$ is ergodic. In my paper 3, I showed that the dynamical system $(K_\beta, m_p \otimes \mu_{\beta,p})$ is exact. In addition, I gave an explicit formula for the density function $f_{\beta,p}$ of the probability measure $\mu_{\beta,p}$. This explicit formula enables us to evaluate the statistical quantities of random β-expansions. Furthermore, as its application, I showed that the function $p \to f_{\beta,p} \in L^1(\lambda)$ is analytic and the function $\beta \to f_{\beta,p} \in L^1(\lambda)$ is continuous everywhere except on some subset of algebraic integers, , where λ denotes the Lebesgue measure on $\mathbb{R}$.

References

[1] K. Dajani and C. Kraaikamp, *Random β-expansions*, Ergod.Th. & Dynam.Sys. **23** (2003) 461-479

[2] K. Dajani and M. de Vries, *Invariant densities for random β-expansions*, J. Eur. Math. Soc. **9** (2007), 157–176.

[3] L. Flatto, J. C. Lagarias and B. Poonen, *The zeta function of the beta transformation*, Ergod. Th. & Dynam. Sys, **14** (1994), 237–266

[4] P.Góra, *Invariant densities for generalized β-maps*, Ergod. Th. & Dynam. Sys, **27** (2007) 1583–1598