I studied the Sobolev embeddings, related inequalities, and the variational problems. On the operator of this embeddings, in the critical case, lack of compactness is known by the invariance of scaling. Concerning the variational problem, elliptic equations, properties of solutions, and so on, there are many special phenomena in this case by the lack of compactness. I study these new phenomena in the critical case. I studied the following:

L^p-Lyapunov inequalities L^p-Lyapunov inequalities is a necessarily condition of existence of solution on a linear elliptic equation. This problem is related to the Sobolev embedding, and semilinear elliptic problem. In the critical case of the Neumann problem, I showed the inequality on the open problem. In addition, I showed the Lyapunov inequality in the case of the nonlinear Neumann problem. This work is joint work with F. Takahashi.

Minimization problems on Hardy-Sobolev inequality I studied the Neumann problem on the Hardy-Sobolev inequality. This inequality is related to the embedding from Sobolev spaces to the Lebesgue spaces with weighted function. By singularity of the weighted function, the embedding becomes noncompact. Location of the singularity on the domain is important on the variational problem. As the previous work, we know that there is a minimizer when the singularity is located on the boundary, and the mean curvature at the singularity is positive. I studied other cases and I showed that the scale of the domain is related to the existence and nonexistence of minimizer essentially in addition to the location of the singularity and the mean curvature. Based on these results, I studied the related elliptic equation with C.-H. Hsia and G. Hwang.

On compactness of Sobolev embedding involving variable exponent
Vanishing phenomenon is one of the factor of lack of compactness on the Sobolev embeddings in the whole space case. I studied the borderline of compactness on the variable exponent in the Lebesgue space. In addition, we show existence of solution of quasilinear elliptic equation with a variable exponent. This is a joint work with M. Sano.

Variational problem on Trudinger-Moser inequality The existence of a maximizer of the maximization problem on the classical Trudinger-Moser is know. However, we cannot find the details of concentration phenomena so far. Firstly, Mancini-Martinazzi (2017) showed the energy estimate of the concentration sequences explicitly. After this, on the shape of the functional P.-D. Thizy (2018) and Ibrahim-Masmoudi-Nakanishi-Sani (2019) showed the borderline of existence and nonexistence of maximizer. At the same time, I showed one of the functional on the borderline of existence and nonexistence of maximizer.