Research Results

I have studied the topology of Hessenberg varieties. Hessenberg varieties are subvarieties(singular in general) of a flag variety and its topology is associated with many research areas. The following varieties are the special cases of Hessenberg varieties:

(1) Springer varieties (related with geometric representation of the Weyl group)
(2) Peterson variety (related with quantum cohomology of flag varieties)
(3) Permutohedral variety (the toric variety whose fan consists of Weyl chambers)

We calculated the torus equivariant cohomology rings of Springer varieties and regular nilpotent Hessenberg varieties in type A where regular nilpotent Hessenberg varieties are a generalization of Peterson variety. On the other hand, regular semisimple Hessenberg varieties are a generalization of Permutohedral variety. We denote a regular nilpotent Hessenberg variety and a regular semisimple Hessenberg variety by $\mathrm{Hess}(N,h)$ and $\mathrm{Hess}(S,h)$ respectively where each of them has one parameter h which is called a Hessenberg function. The family of $\mathrm{Hess}(N,h)$ (or the family of $\mathrm{Hess}(S,h)$) is a family of subvarieties of a flag variety. We obtained the following ring isomorphism for the cohomology rings[1] of $\mathrm{Hess}(N,h)$ and $\mathrm{Hess}(S,h)$:

$$H^*(\mathrm{Hess}(N,h)) \cong H^*(\mathrm{Hess}(S,h))^{\mathfrak{S}_n}$$

where the right hand side is the invariant subring of $H^*(\mathrm{Hess}(S,h))$ under the action of the n-th symmetric group $\mathfrak{S}_n$, and the $\mathfrak{S}_n$-action on the cohomology $H^*(\mathrm{Hess}(S,h))$ is introduced by Tymoczko using GKM graph.

Moreover, we obtained the surprised result that the cohomology ring of a regular nilpotent Hessenberg variety in any Lie types can be described in terms of hyperplane arrangements. From this result we can see that the above ring isomorphism is true in any Lie types. Furthermore, we could give the proof of the announcement of Dale Peterson and Sommers-Tymoczko conjecture, and an explicit presentation of the cohomology rings of regular nilpotent Hessenberg varieties in types B, C, G_2. We also obtaind that the fundamental relation in the explicit presentation of the cohomology ring of a regular nilpotent Hessenberg variety in type A can be written as an alternating sum of certain Schubert polynomials.

On the other hand, it is known that the $\mathfrak{S}_n$-action on the cohomology $H^*(\mathrm{Hess}(S,h))$ of a regular semisimple Hessenberg variety in type A introduced by Tymoczko has a beautiful connection with the chromatic quasisymmetric function in graph theory. We obtained an explicit presentation of the cohomology rings of the special cases of regular semisimple Hessenberg varieties, and we can see that Stanley-Stembridge conjecture in graph theory is true for the special case from this result together with the above connection between the $\mathfrak{S}_n$-action on $H^*(\mathrm{Hess}(S,h))$ and the chromatic quasisymmetric function. Also, we can see that the volume polynomial of a regular semisimple Hessenberg variety can be described by the sum of the volume of certain faces of the Gelfand-Zetlin polytope, and we obtain an explicit combinatorial formula for the volume polynomial of a regular semisimple Hessenberg variety in terms of tableaux.

[1] The cohomology is the singular cohomology with rational coefficients.