## Abstract of present research

Our study is polynomial approximation theory as an application of potential theory. On $\mathbb{R}$, every polynomial $P(x)$ blows up as $|x| \to \infty$, we must multiply a weight function $w(x)$. Then, for $1 \le p \le \infty$ and $fw \in L^p(\mathbb{R})$, is there exist a sequence of polynomials $\{P_n\}$ such that

$$\lim_{n\to\infty} \|(f - P_n)w\|_{L^p(\mathbb{R})} = 0? \tag{A}$$

We assume that an exponential weight $w$ belongs to relevant class $\mathcal{F}(C^2+)$. Let $w$ be $w(x) = \exp(-Q(x))$. We consider a function $T(x) := xQ'(x)/Q(x)$, $(x \neq 0)$. If $T$ is bounded, then $w$ is called Freud-type weight, and otherwise, $w$ is called Erdős-type weight. The de la Vallée Poussin mean $v_n(f)$ of $f$ is defined by $v_n(f)(x) := \frac{1}{n}\sum_{j=n+1}^{2n} s_j(f)(x)$, where $s_m(f)(x)$ is the partial sum of Fourier series of $f$ for orthogonal polynomials with respect to $w$. The degree of approximation for $f$ defined by $E_{p,n}(w; f) := \inf_{P\in\mathcal{P}_n} \|(f - P)w\|_{L^p(\mathbb{R})}$. Here, $\mathcal{P}_n$ is the set of all polynomials of degree at most $n$.

1. **Error estimate of approximation**: We assume that $w \in \mathcal{F}(C^2+)$ and suppose that $T(a_n) \le c\,(n/a_n)^{2/3}$ for some $c > 0$. Here, the notation $a_n$ is called MRS number of $w$. Then there exists a constant $C \ge 1$ such that for every $n \in \mathbb{N}$ and when $fw \in L^p(\mathbb{R})$,

   $$\|(f - v_n(f))w\|_{L^p(\mathbb{R})} \le CT^{1/4}(a_n)E_{p,n}(w; f). \tag{B}$$

   H. N. Mhaskar and others show (B) for a Freud-type weight. This is an extensions of the Mhaskar's estimate to Erdős-type weights. On a proof of this theorem, $L^p$ boundedness of the de la Vallée Poussin mean is important.
2. **Convergence of the de la Vallée Poussin mean**: Now, what the conditions

   $$\lim_{n\to\infty} \|(f - v_n(f))w\|_{L^p(\mathbb{R})} = 0 \tag{C}$$

   for an Erdős-type weight? We already know that if $w \in \mathcal{F}(C^2+)$, then $E_{p,n}(w; f) \to 0$ as $n \to \infty$. If $w$ be a Freud-type weight, then (C) holds. But, If $w$ be a Erdős-type weight, by unboundedness of $T$, (C) is not always true. We show the following two conditions: If $w$ belongs to smooth subclass $\mathcal{F}_\lambda(C^3+)$ and $T^{1/4}fw \in L^p(\mathbb{R})$, then (C) holds by using mollification of the weight. On the other hand, if $f$ be an absolutely continuous function with $f'w \in L^p(\mathbb{R})$, then (C) holds by the Jackson-Favard inequality. Then the de la Vallée Poussin mean $v_n(f)$ for a function $f$ gives one of the concrete example of (A). Moreover, if $f$ satisfies the second condition and $f''w \in L^p(\mathbb{R})$, then the de la Vallée Poussin mean $v_n(f)$ of $f$ is not only a good approximation polynomial for $f$, but also its derivatives give an approximation for $f'$.
3. **Uniform convergence of the Fourier partial sum**: By the way, we show the condition uniformly convergence of $s_n(f)$ for a weight in a class $\mathcal{F}_\lambda(C^3+)$: Suppose that $f$ is continuous and has a bounded variation on any compact interval of $\mathbb{R}$. If $f$ satisfies $\int_{\mathbb{R}} w(x)|df(x)| < \infty$, then

   $$\lim_{n\to\infty} \left\|(f - s_n(f))\frac{w}{T^{1/4}}\right\|_{L^\infty(\mathbb{R})} = 0.$$