(英訳文)

This reserch is a continuation of [Result 8]. As seen in Result (3), at least in the case when the manifold X is toric Fano, we can construct the optimal destabilizer from the asymptotic data of MA^{-1}-flow. We want to extend this to arbitrary Fano manifolds. For this, we need to study the singularity formation along the flow. In this reserch, we will study some cases that are relatively easy to handle:

(A) Optimal destabilizers obtained by jumping of complex structures
(B) Optimal destabilizers obtained by the MA^{-1}-flow on Calabi ansatz

Reserch (A) Optimal destabilizers obtained by jumping of complex structures

Like the Mukai-Umemura 3-fold[*3], there are some examples of no KE manifolds which admit KE metrics just by replacing with other complex structure J_∞ (just as the differentiable structure of X was). In this case, the optimal destabilizer of X should have "mild singularities" in that sense. We try to replace the MA^{-1}-flow with another (but equivalent) flow on the moduli space of complex structures, and study the convergence property: now we fix a Kähler form $\widehat{\omega} \in \mathcal{H}_L$, and take the pullback of any Kähler forms in $\mathcal{H}$ via the diffeomorphisms obtaind by Moser's theorem. In particular, using the diffeomorphisms determined by the MA^{-1}-flow $\{\omega_t\}$, the complex structure on X is pulled back to a flow $\{J_t\}$ on $\mathcal{J}$, where $\mathcal{J}$ denotes the moduli space of $\widehat{\omega}$-compatible complex structures on X. Although $\{\omega_t\}$ and $\{J_t\}$ are equivalent to each other, they have own advantage and disadvantage. Thus we prove the convergence of the flow $\{J_t\}$ by combining their advantages (compensating for disadvantages) and construct the optimal destabilizer.

Strategy on Reserch (A)

First, see the advantage/disadvantage of the two flows displayed in Table 2. Since the flow $\{J_t\}$ is

flow	Advantage	Disadvantage
$\{\omega_t\}$	strongly parabolic	can not converge if there are no KE metrics in $\mathcal{H}_L$
$\{J_t\}$	can converge even if there are no KE metrics in $\mathcal{H}_L$	weakly parabolic

Table 2 Advantage and disadvantage of the two flows

invariant under the action of the Gage group $\mathrm{Symp}(X, \widehat{\omega})$, it is not strongly parabolic and even the short-time existence does not follow from general theory of parabolic equations. Nevertheless, we know that the flow $\{J_t\}$ can be solved for all time thanks to the existence of the long-time solution to the MA^{-1}-flow. On the other hand, it is impossible for the flow $\{\omega_t\}$ to converge if there are no KE metrics on $\mathcal{H}_L$, but we can expect that the flow $\{J_t\}$ converges to some complex structure $J_\infty \in \mathcal{J}$ (in this case, all J_t except for $t = \infty$ can be transformed to each other via diffeomorphisms, but the flow $\{J_t\}$ jumps to a non-trivial structure J_∞ at infinity). A direct computation shows that $\{J_t\}$ is a gradient flow of the Calabi type functional

$$R(J) := \int_X (1 - e^{\rho(J)})^2 \widehat{\omega}^n, \quad J \in \mathcal{J}$$

and J_∞ is one of the critical points of it. Here we associate each $J \in \mathcal{J}$ to a function $\rho(J) \in C^\infty(X, \mathbb{R})$ uniquely determined by the formula

$$\mathrm{Ric}(\widehat{\omega}) - \widehat{\omega} = \sqrt{-1}\partial_J \bar{\partial}_J \rho(J), \quad \int_X e^{\rho(J)} \widehat{\omega}^n = \int_X \widehat{\omega}^n.$$

[*3] A typical example of no KE manifolds obtained by a compactification of $\mathrm{SL}(2, \mathbb{C})/\Gamma$, where Γ is a subgroup of $\mathrm{SL}(2, \mathbb{C})$ of order 60.

In general, for real analytic functionals on Banach spaces, we can control its gradient flow near the critical points by using Simon-Lojasiewictz type inequality [Sim83]. We apply this method to the functional R to show the convergence of the flow $J_t \to J_\infty$ with an initial data J_0 which is sufficiently close to J_∞.

Reserch (B) Optimal destabilizers obtained by the MA^{-1}-flow on Calabi ansatz

From [Result 8], (except for the case when X admits solitons), it is known that one of the easiest example for the MA^{-1} flow forming singularities is the 3-dimensional Calabi ansatz $X := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}_{\mathbb{P}^2}(2)) \to \mathbb{P}^2$. Using the symmetry of X, one can reduce the MA^{-1}-flow/soliton equation to a parabolic PDE/ODE on a closed interval via the fiberwise moment map of the standard $U(1)$-action. Then by using the explicit expression of the PDE/ODE, we study the algebraicity, codimension of singular sets on the limit spaces, and geometric convergence properties of the flow.

Strategy on Reserch (B)

We reduce the soliton equation for the MA^{-1}-flow to a ODE on a closed interval, and denote it by f_∞. Then the positivity of f_∞ corresponds to the positivity of Kähler metrics. Since we assume that X does not admit solitons, the metric corresponding to f_∞ violates the positivity condition. In other words, the function f_∞ has the "positive part" as well as the "negative part", which give a decomposition of X into moment sublevel sets $X = X_{\mathrm{s}} \cup X_{\mathrm{u.s}}$ via the fiberwise moment map. Since the volume is preserved under the flow, the unstable locus $X_{\mathrm{u.s}}$ defined by the euqation $f_\infty \leq 0$ must be lost in the limit, and crease to the stable locus X_{s} defined by $f_\infty > 0$, and the flow should converge in the local C^∞-topology on X_{s}. Then, the stable locus X_{s} should admit a singular metric with non-trivial Lelong numbers as an evidence of collapsed $X_{\mathrm{u.s}}$. In order to prove this, we study the singularity of the metric corresponding to f_∞ along the level set (conical, cusp, etc.), and depending on the situations, we try to solve this degenerate Monge-Ampère equation by adding some auxiliary terms. For instance, Guenancia-Paun [GP16] established a general theory of Monge-Ampère equations with conical singularities, that will be helpful to our reserch.

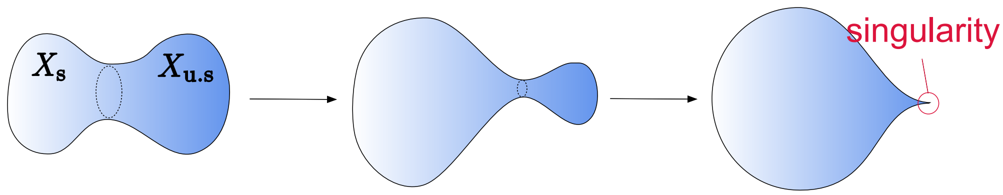

Fig.2 Deformation of Calabi ansatz X along the MA^{-1}-flow.

References

[GP16] H. Guenancia and M. Paun, *Conic singularities metrics with prescribed Ricci curvature: general cone angles along normal crossing divisors*, J. Diff. Geom. **103** (2016), 15–57.

[Sim83] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. **118** (1983), no. 3, 525–571.