Research plan (April 2020 ∼)

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The study of elliptic operators in the framework of global setting brought deep understanding in the manifold theory and analysis. In my coming research activity, I am going to study global phenomena of mainly sub-elliptic operators including elliptic cases.

Laplace-Beltrami operator and Dirac operator are defined by means of geometric structure of manifolds, the first one is a second order elliptic differential operator and the second one is a first order elliptic differential operator respectively. In the study here sub-Laplacian and sub-Riemannian structure are the main subjects. A sub-Laplacian is a sub-elliptic second order differential operator and is defined based on the sub-Riemannian structure.

This geometric structure (= sub-Riemannian structure) is a case that there is a sub-bundle in the tangent bundle which is bracket generating.

The opposite structure, that is foliation structure, was studied from many years ago. On the other hand it might be natural to study the sub-Riemannian structure from the point of the control theory, however until recently it was not so much studied of such structure from the geometric and analytic point of view, so that I am expecting that to pursue the research in this subject has a meaning.

Contact manifolds and nil-potent Lie groups are well known examples of manifolds carrying a sub-Riemannian structure and it happens often that the total space of a Riemannian submersion has both structures, foliation and sub-Riemannian. Hence there are ample examples of such manifolds to be studied.

Although we can define a transversely elliptic operator on foliated manifolds, we can not define a differential operator based on the foliation structure. On the other hand there is an intrinsically defined second order differential operator (we call it sub-Laplacian) on sub-Riemannian manifolds reflecting the sub-Riemannian structure. Hence, again there must be enough meaning to study the sub-Laplacian in contrast with the Laplacian for the Riemannian case.

Moreover, since this operator satisfies the "*sub-elliptic estimate*" (= sub-ellipticity) proved by Hörmander, the spectral behaviour is somehow similar to elliptic operators. However there is non trivial characteristic variety so that we can not treat them in the framework of K-theory. This may include difficulty in the study and at the same time we can expect the possibility of new phenomena other than obtained by the property of ellipticity. Under these circumstances I am going to continue the research on these topics under the direction to find a new phenomena which will be not included in the elliptic operator theory, and together including the problems whether named classical manifolds have this structure and analytic properties of this type operator, like Weyl low or an explicit construction of the heat kernel.

Meanwhile I am going to continue the research with collaborators by visiting them or inviting from Europe with whom I was doing the joint research in this subject in these years.

Based on the title of the research

From elliptic operators to sub-elliptic operators

I am planing the research on the following problems**:**

(a) Nil-potent Lie groups are examples of sub-Riemannian manifolds carrying a "good" sub-Riemannian structure (i.e., equi-regular sub-Riemannian structure). Among them I was studying a class of groups attached to Clifford algebras (= pseudo H-type algebras and groups) in these years. Still there are many remaining problems, especially I am going to classify "*integral lattices*". Also by comparing the spectral zeta functions of Laplacian and sub-Laplacians on their compact nil-manifolds I will determine their zeta regularized determinants by making clear how they are expressed in terms of classical special functions and seek a possibility of their functional equation.

In particular, I am going to study the inverse spectral problem of the sub-Laplacian which must be interesting since the residues of the spectral zeta function of the sub-Laplacian relate with the values of Riemann zeta function at integral points.

(b) *Weyl low* is a historical fact and played an important role in the development of elliptic operators and Fourier integral operator theory, however to prove a similar result for sub-Laplacian, it is not possible to apply the Fourier integral operator theory directly in the similar way as was done for the elliptic case by looking the proof for it. Hence to prove Weyl low for sub-Laplacians, at the moment I am going to treat the concrete cases with the explicit data of the eigenvalues, especially for the cases of above nil-manifolds in (a) I will see how it looks like based on the explicit data.

So far as I know, there is only one result for a sub-Laplacian by R. S. Strichartz, *Spectral Asymptotics on Compact Heisenberg Manifolds* J. Geom. Anal. (2016) 26, 2450–2458.

(c) I am expecting that Weinstein's eigenvalue theorem is valid also for sub-Laplacians, so that I try to establish a complete proof of such the theorem. Also in relation with this theorem it will be interesting to find various Lagrangian submanifolds satisfying Maslov quantization condition other than those realized as common hyper surfaces in the completely integrable geodesic flow cases(or completely integral bi-characteristic flow of sub-Laplacian, that is they are tori). So mainly I am going to deal with the non-completely integrable cases of bi-characteristic flows and discuss the functorial behaviour of Lagrangian submanifolds under submersion.

(d) Construction of a Calabi-Yao structure on the cotangent bundle or the punctured cotangent bundle and a construction of a quantization operator by the method of geometric quantization.

This problem is relating with the last problem (c) and concentrate in the case of $P^2(\mathbb{C}_a)$, the Cayley projective plain. The structure guarantees the existence of bi-characteristic flow invariant measure required in the Malsov quatization condition, if the structure is bi-characteristic flow invariant. The manifold carrying such a structure is highly restricted and we know that $T_0^*(P^2(\mathbb{C}_a))$ has a Kähler structure and that we want to show the holomorphical triviality of its canonical line bundle. If it is correct, we may expect to be able to construct an operator similar to Bargmann in the case of $L_2(\mathbb{R}^n)$ to Fock space.

(e) Study of the heat kernel on higher step Grushin operator.

Engel group is a 4 dimensional and 3-step nilpotent Lie group and even this case it is not possible to determine the spectrum of Laplacian and sub-Laplacian on its compact nil-manifold explicitly, however we have an action function in terms of Jacobi elliptic function for a Grushin operator of two variables defined on a quotient space of this group. So based on this result and various properties of elliptic functions we try to construct the heat kernel for 2 variables 3-step Grushin type operator.

(f) There exist invariant sub-Riemannian structures on Lie groups, however in general it is not clear of the existence of such a structure on symmetric spaces. In this research we try to show or construct such a structure on symmetric spaces and we will make efforts to classify such structures on symmetric spaces by using the Lie algebra structure.