Research results

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Since 19 century, elliptic operators were studied in many directions, especially in relation with physics in which the Dirichlet problem was one of the first problem. Then following the development of the theory of the Hilbert space, the theory developed so quickly including the development of pseudo-differential operators and Fourier integral operator theory base on the revolutionary treatment of the concept of functions (that is, distribution theory where functions are being treated as observable in the sense of physics). I was continuing my research in this framework on the relations of the manifold structure and differential operators existing on the manifold in a natural way. In particular, around these ten to fifteen years I was concentrating in the study of sub-Riemannian manifolds and sub-elliptic operators, which are defined on the manifolds based on this geometric structure. In the explanations below the number [∗] indicates the number of the paper(refereed) in the publication list.

The studies are classified as

(1) Study of spectral flow and Maslov index
(2) Study of geometric quantization
(3) Study of Laplacian and sub-Laplacian
  (3-1) Study of heat kernel and spectrum
  (3-2) Study of sub-Riemannian structure

(1): Based on the studies by Floer who found a phenomenon in the transversely intersecting Lagrangian submanifolds and Yoshida's result proving the equality of Maslov index and spectral flow on the partitioned manifold, we generalized the equality "*Spectral flow = Maslov index*" in the framework of the infinite dimensional symplectic Hilbert space, where Maslov index in the infinite dimensional version was formulated as an invariant of homotopy classes of arbitrary paths with fixed end points in the Fredholm-Lagrangian-Grassmannian. This result was published in [37] and I worked up several results in [28], [30], [33], [35], [36], [37] in a paper

Kenro Furutani, *Fredholm-Lagrangian-Grassmannian and the Maslov index*, Journal of Geometry and PhysicsVol. 51, No. 3(2004), pp. 269–331.
The paper [29] in the list is this paper.

In the concrete elliptic boundary value problem, the Cauchy data space is Lagrangian is equivalent to the problem being selfadjoint. Starting from this basic fact I proved a generalization by purely functional analytic way.

(2) The purpose of this study is to construct an operator similar to Bargmann transformation. For this purpose I started to find a manifold M whose punctured cotangent bundle $T^*(M)\setminus\{0\} := T_0^*(M)$ has a Calabi-Yao structure other than spheres. I constructed such a structure on the complex projective space $P^n(\mathbb{C})$ and the quaternion projective $P^n(\mathbb{H})$. Together I showed that their canonical line bundle has a nowhere vanishing global holomorphic section Ω, calculated its pairings $\Omega \wedge \overline{\Omega}$ in terms of the Liouville volume form, and constructed an operator similar to Bargmann transformation. Although the original Bargmann transformation is unitary, for these cases the operators constructed are not unitary. Sphere cases was studied by J. H. Ranwsley.

As for the case $P^2(\mathbb{C}_a)$, the Cayley projective plain, I could only constructed a Kähler structure on $T_0^*(P^2(\mathbb{C}_a))$ and other properties were not completed. These and related results were published in [22], [27], [32], [34], [39] and [40].

(3) Here I aimed the study to construct heat kernels on nil-manifolds in concrete forms as a guideline of the study. As results I had followings:

(i) I determined the zeta-regularized determinant for Laplacians and sub-Laplacians on Heisenberg manifolds, standard sphere and some others. Also I gave a proof of the zeta-regularized determinant for a product type manifold which includes the case of $S^1 \times M$ in a form of infinite product which can be seen as a generalization of Kronecker's second limit formula ([18], [19], [21], [22], [23]).

(ii) In the paper [20] we corrected many of known facts with respect to concrete sub-Laplacians together in 100 pages +. This is the first paper starting the joint research with W. Bauer(Hannover Univ.) and C. Iwasaki(Hyogo Univ.) in this subject.

Also together with three friends (O. Calin, D-C. Chang and C. Iwasaki) we published a book on the subject of various methods for the construction of the heat kernel mainly for sub-Laplacian from Birkhäuser(now it is a part of Springer verlag).

(iii) Study on higher step Grushin operators: We constructed an action function which is one of the main constituent function of the heat kernel of Grushin operators of two independent variables([17]). Also we constructed Green functions of higher step Grushin operators of general number of variables. Here I mean that the Grushin operator is a second order sub-elliptic operator which is defined as a descended operator of an invariant sub-Laplacian on nil-potent Lie groups to a quotient space.

(iv) We proved that there exists a co-dimension 3 sub-Riemannian structure on Gromoll-Meyer exotic 7 sphere ([7]). We also made some study on the sub-Riemannian structure on sphere ([16]) and in [6] studied Popp's measure intrinsically defined by sub-Riemannian structures in the framework of a submersion and especially in the framework of a principal bundle.

(v) Study of nil-potent Lie groups(algebras) as examples carrying a natural sub-Riemannian structure: We studied 2 step nil-potent Lie algebras attached to Clifford algebras (we call these pseudo H-type Lie algebra). The results are their complete classification, existence of lattices and the determination of automorphism groups. The last result is just submitted ([1], [3], [8], [9], [11], [15]).