I studied the Sobolev embeddings, related inequalities, and the variational problems. It is known that the operator of the critical Sobolev embedding is not compact. By the lack of compactness, there are many phenomena in the variational problem, elliptic equations, properties of solutions, and so on. I study these new phenomena in the critical case. I studied the following:

(1) L^p-Lyapunov inequalities

L^p-Lyapunov inequalities is a necessary condition of existence of solution on a linear elliptic equation. This problem is related to the Sobolev embedding and corresponding semilinear elliptic problem. I obtained the L^p-Lyapunov inequality in the critical case of the Neumann problem. In addition, I showed the L^p-Lyapunov inequality in the case of the nonlinear Neumann problem. This work was joint work with F. Takahashi.

(2) Minimization problems on Hardy-Sobolev inequality

I studied the Neumann problem on the Hardy-Sobolev inequality. This inequality is related to the embedding from Sobolev spaces to the Lebesgue spaces with weighted function. By singularity of the weighted function and an invariance of a scaling of functions, the embedding is non-compact. Concerning this problem, some topological properties of bounded domain plays a important role. As the previous work, it is known that there is a minimizer when the singularity is located on the boundary and the mean curvature at the singularity is positive. I studied other cases and I showed that the scale of the domain is related to the existence and nonexistence of minimizer. Under some conditions, I clarified the necessary and sufficient condition on existence of a minimizer. Based on these results, I studied the related elliptic equation with C.-H. Hsia and G. Hwang.

(3) On compactness of Sobolev embedding involving variable exponent

Vanishing phenomenon is one of the factor of the lack of compactness on the Sobolev embeddings in the whole space case. I studied the borderline of compactness on the embedding from Sobolev space to Lebesgue space with a variable exponent. In addition, I showed existence of solution of quasilinear elliptic equation with a variable exponent. This is a joint work with M. Sano.

(4) Variational problem on Trudinger-Moser inequality

It is known that there exists a extremal for the classical Trudinger-Moser inequality. By Mancini-Martinazzi（2017）, it was shown that the asymptotic expansion of the Dirichlet energy of some concentrating sequence, which is strictly related to the variational problem. After their study, P.-D. Thizy (2018) and Ibrahim-Masmoudi-Nakanishi-Sani (2020) showed the threshold nonlinearity dividing existence and nonexistence of maximizer. At the same time, I also showed one of the threshold nonlinearity dividing existence and nonexistence of a maximizer.