Research Results

From the middle of the nineteen-eighties injective harmonic mappings in the unit disk $\mathbb{D}$ of the complex plane have been actively investigated not only as a research object of researchers on minimal surfaces, but also as mappings analogous to conformal mappings. According to the progress of researches on harmonic mappings, the interests of the research on quasiconformality of injective harmonic mappings have been gradually increasing. From the latter half of the nineteen-nineties I have studied jointly with D.Partyka (The John Paul II Catholic University of Lublin, Poland) quasiconformality and some properties, say, Heinz type inequalities and Schwarz type inequalities, of harmonic mappings.

Let f be a sense-preserving homeomorphic self-mapping of the unit circle $\mathbb{T}$ and let F be the harmonic extension (the Poisson integral) of f.Then by Radó-Kneser-Choquet theorem, F is a diffeomorphic self-mapping of the unit disk $\mathbb{D}$. We discussed necessary and sufficient conditions for this harmonic extension to be quasiconformal in the following papers:
A note on non-quasiconformal harmonic extensions, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. 23(1997), 51-63.
Quasiconformality of harmonic extensions, J. Comput. Appl. Math. 105(1999), 425-436.
These results were deeply improved by the following paper by M.Pavlović: Boundary correspondence under harmonic quasiconformal homeomorphisms of the unit disk, Ann. Acad. Sci. Fenn. Math.27(2002), 365-372.
He showed that a necessary and sufficient condition for such a harmonic extension F to be quasiconformal is that F is bi-Lipschitz. Moreover, he additionally showed that another necessary and sufficient condition for F to be quasiconformal is that f is absolutely continuous, both $\dot{f}$ and the Hilbert transformation of $\dot{f}$ belong to L^∞ and the essential infimum of the absolute values of $\dot{f}$ on the unit circle is positive, where $\dot{f}$ is the derivative with respect to θ which exists a.e. $z = e^{i\theta}$. If the range is a bounded convex domain in the complex plane, then we could give an example which shows that these results are not valid as they stand.Whether analogous results hold or not was an open question.We then discussed a deformation of a quasiconformal harmonic mapping of the unit disk $\mathbb{D}$ onto an α-convex domain which is a generalization of the convex domain, and used a deformation of harmonic mappings by one parameter family in the following papers:

On a result of Clunie and Sheil-Small, Ann. Univ. Mariae Curie-Sklodowska, Sectio A.66 (2012),no.2, 81-92.
A simple deformation of quasiconformal harmonic mappings in the unit disk, Ann. Acad. Sci. Fenn. Math.37(2012),539-556.
By using results in these papers, even if the range is a bounded convex domain in the complex plane, if the Lipschitz property of a harmonic mapping is assumed, then we see that we can obtain a result which is viewed as a generalization of the above result by Pavlović in the following paper:
Quasiconformal and Lipschitz harmonic mappings of the unit disk onto bounded convex domains, Ann. Acad. Sci. Fenn. Math. Vol. 39 (2014), 811-830.
Moreover, some new properties for quasiconformal harmonic mappings were discussed in the following paper:
Distortion of the area measure for one-to-one harmonic mappings of the unit disk onto itself, Sci. Bull.Chelm Math. Comput. Sci. 2(2007), 39–48.

On the other hands we had investigated generalized forms of Heinz's inequality (Heinz 1959). Asymptotically sharp variants of Heinz inequality for K-quasiconformal harmonic mappings were discussed in the following papers:
On Heinz's inequality, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform.36(2002), 27-34.
On an asymptotically sharp variant of Heinz's inequality, Ann. Acad. Sci. Fenn. Math. 30(2005), 167-182.
On a variant of Heinz's inequality for harmonic mappings of the unit disk onto bounded convex domains, Bull. Soc. Sci. Lett. Lódź 59 (2009), 25– 36, Sér. Rech. Déform. 59.
Moreover, for harmonic mappings which are Poisson integrals of functions of bounded variation on the unit circle $\mathbb{T}$ we showed several Heinz type inequalities in the following paper:
Heinz type inequalities for Poisson integrals, Computational Methods and Function Theory, 14(2014), 219-236.
On asymptotically sharp Schwarz type inequalities and bi-Lipschitz type inequalities for K-quasiconformal harmonic mappings we published the following papers:
Three variants of Schwarz's lemma for harmonic mappings,Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. 51(2006), 23-36.
On bi-Lipschitz type inequalities for quasiconformal harmonic mappings, Ann. Acad. Sci. Fenn. Math. 32(2007), 579-594.

Moreover, on the boundary behavior of a μ-conformal mapping which is a generalization of a quasiconformal mapping I jointly with T.Sugawa (Tôhoku Univ.) and V. Gutlyanskii(NAS of Ukraine) published the following paper:
On μ-conformal homeomorphisms and boundary correspondence, Complex Variables and Elliptic Equations 58 (2013),no.7, 947-962.
Sense-preserving injective harmonic mappings are also μ-conformal mappings.

Let $F = H + \overline{G}$ be a sense-preserving injective harmonic mapping of $\mathbb{D}$. Here H, G are holomorphic functions and G satisfies $G(0) = 0$. Suppose that H is convex, that is, H is a conformal mapping and $H(\mathbb{D})$ is a convex domain. In this case I together with D. Partyka and J. Zhu (Huaqiao University, P. R. China) investigated necessary and sufficient conditions for F to be quasiconformal and necessary and sufficient conditions for F to be quasiconformal and Lipschitz. Then we showed several results, some of them are anologous to and some others are different from coresponding results in the case of $F(\mathbb{D})$ is a convex domain in the paper
Quasiconformal harmonic mappings with the convex holomorphic part, Ann. Acad. Sci. Fenn. Math. 43(2018), 401-418.

Furthermore, let

$$F_\epsilon := H + \epsilon\overline{G}, \quad \epsilon \in \mathbb{C}, \ \epsilon\|\mu_F\|_\infty \le 1,$$

$$(\mu_F(z) := \frac{\overline{\partial} F(z)}{\partial F(z)}, \ z \in \mathbb{D})$$

Under the assumption such that F or H which is assumed to be injective satisfies certain conditions, we shall investigate injectivity, quasiconformality, close-to-convexity of F_ϵ and bi-Lipschitzness of $F_\epsilon \circ H^{-1}$.