Research Plan

My main research plans are as follows.

(1) Classification of two-generator Kleinian groups. At a workshop held in Budapest in 2002, Agol announced the classification of Kleinian groups generated by two parabolic transformations and the classifications of parabolic generating pairs of such Kleinian groups. However he has not published the proof. I was very much interested in these results, because they look very natural from the view-point of my ongoing research project at that time, with Hirotaka Akiyoshi, Masaaki Wada and Yasushi Yamashita, on Jorgensen's theory and its extension. So, I immediately had contact with Agol and tried to understand his results. Agol eventually encouraged me to write up the results, and so I started a project to give a complete proof to his results. In this year (2019), we completed a draft which gives a proof to the classification theorem of two-parabolic-generator Kleinian groups through joint work with Hirotaka Akiyoshi, Ken'ichi Ohshika, John Parker, and Han Yoshida, and we also completed a draft that gives an alternative proof to the classification theorem of parabolic generating pairs of such Kleinian groups through joint work with Donghi Lee and my students, Shunsuke Aimi and Shunsuke Sakai (arXiv:2001.09564, arXiv:2001.11662). I conjecture that similar results hold for the images of type-preserving representations of the fundamental group of the punctured torus and for the groups generated by two elliptic transformations. I hope to give proofs to these conjectures.

(2) The space of Kleinian groups. The space of groups generated by two parabolic transformations are parametrized by the complex plane minus the origin. It is proved by Ohshika-Miyachi that the subspace, $\mathcal{D}_f$, of free Kleinian groups are equal to the closure of the Riley slice $\mathcal{R}$ and moreover it is homeomorphic to the open annulus. My joint work with Akiyoshi, Wada, and Yamshita suggests that there is a natural tessellation of the Riley slice $\mathcal{R}$, which is essentially isomorphic to the Farey tessellation. On the other hand, the subspace $\mathcal{D}_{nf}$ consisting of non-free Kleinian groups is a discrete space, whose accumulation set is equal to the boundary of $\mathcal{D}_f$. My joint work with Akiyoshi, Wada, and Yamshita also suggests that there are natural paths connecting $\mathcal{D}_{nf}$ with $\mathcal{D}_f$. Motivated by discussion with Gaven Martin, I came to an idea that $\mathcal{D}_{nf}$ has a nice pattern, which I would like to call the *mesh structure*. Through a joint work with Akiyoshi, which started in the last year, we came to a conjecture that the complement of $\mathcal{D}_f$ admits a natural tessellation isomorphic to a 'dual' to the Farey tessellation, which can be described by using the theory of hyperbolic Dehn surgery, and that the mesh structure can be described by using the conjectural tessellation. Moreover, I expect that it might be possible to give a direct proof to the results stated in the research plan (1). I also expect that similar story should hold for the images of type-preserving representations of the fundamental group of the punctured torus and for the groups generated by two elliptic transformations. I hope to study these conjectures.

(3) Similarity between fiber surfaces and Heegaard surfaces of 3-manifolds. Heegaard surfaces and fiber surfaces of 3-manifolds have quite different nature: in fact, the former are incompressible while the latter are compressible. However, the branched fibration theorem which I proved in a paper published in 1981 shows that there is a direct relationship between Heegaard surfaces and fiber surfaces. I expect that this is a consequence of much deeper relation/similarity between them. We can define the 'monodromy group', $\mathcal{M}(M, \Sigma)$, of a Heegaard surface Σ of a 3-manifold M, as an analogy of the cyclic monodromy group generated by the monodromy of a fiber surface. The monodromies of the fiber surfaces constructed from a Heegaard surface by the branched fibration theorem are special elements of $\mathcal{M}(M, \Sigma)$. By using this fact, I hope to study relation between the complexity of a Heegaard surface (in particular, the Hempel distance), and the complexity of the monodromies of the fiber surfaces (in particular, the stable translation length in the curve complex). Through a series of joint works with Donghi Lee, we have proved that the 2-bridge spheres (the simplest Heegaard surfaces) of 2-bridge links have various beautiful properties and in particular that an analogy of McShane's identity holds for 2-bridge spheres. It is natural to ask to what extent these results hold for general Heegaard surfaces; in particular, whether an analogy of McShane's identity holds for Heegaard surfaces, as for fiber surfaces. (An analogy of McShane's identity was established by my joint work with Akiyoshi and Miyachi as a generalization of Bowditch's work.) I would also like extend the theorem proved in my joint work with Ohshika, which says that the orientation-preserving subgroup $\mathcal{M}^+(M, \Sigma)$ admits a natural free product decomposition, to the full monodromy group $\mathcal{M}(M, \Sigma)$.