DIFFERENTIAL GEOMETRY OF LAGRANGIAN SUBMANIFOLDS AND RELATED VARIATIONAL PROBLEMS

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Abstract. In this article we shall provide a survey on my recent works and their environs on differential geometry of Lagrangian submanifolds in specific symplectic Kähler manifolds. This article is mainly based on the author's joint work with Hui Ma (Tsinghua Univ., Peking).

The volume minimizing problem of Lagrangian submanifolds in Kähler manifolds under Hamiltonian deformations was investigated first by Y. G. Oh about the beginning of 1990's. It is fundamental and interesting as a geometric variational problem related to Lagrangian submanifolds. In this article we shall discuss several nice classes of Lagrangian submanifolds and their Hamiltonian stability problems in specific Kähler manifolds such as complex space forms (complex Euclidean spaces, complex projective spaces, complex hyperbolic spaces), Hermitian symmetric spaces and especially complex hyperquadrics, and so on. The relationship of minimal Lagrangian submanifold in complex hyperquadrics with isoparametric hypersurfaces in spheres will be emphasized. We shall mention our recent results on a classification of compact homogeneous Lagrangian submanifolds in complex hyperquadrics and the Hamilitonian stability of compact minimal Lagrangian submanifold embedded in complex hyperquadrics obtained as the Gauss images of homogeneous isoparametric hypersurfaces in spheres.

Introduction

The purpose of this article is to give a survey on my recent works and their environs on differential geometry of Lagrangian submanifolds in specific symplectic Kähler manifolds. This article is mainly based on the author's joint work with Hui Ma (Tsinghua Univ., Peking) on Lagrangian submanifolds in complex hyperquadrics ([18]).

Let (M, ω) be a $2n$-dimensional symplectic manifold with a symplectic form ω. A *Lagrangian immersion* $\varphi : L \longrightarrow M$ is a smooth immersion of an n-dimensional smooth manifold L into M satisfying $\varphi^*\omega = 0$. More generally a submanifold $\varphi : S \longrightarrow M$ (whose dimension is not necessary equal to n) immersed in a symplectic manifold M satisfying $\varphi^*\omega = 0$ is called an *isotropic* submanifold of a sympletic manifold M and thus a Lagrangian submanifold is an isotropic submanifold of maximal dimension in a symplectic manifold.

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Lagrangian submanifolds are the most fundamental objects in symplectic geometry (cf. [29],[30],[31]).

On the other hand, let (M, J, g) be a $2n$-dimensional almost Hermitian manifold with an almost complex structure J and an almost Hermitian metric g. A *totally real immersion* $\varphi : L \longrightarrow M$ is defined as a smooth immersion satisfying the condition $g(J\varphi_* X, \varphi_* Y) = 0$ for each $X, Y \in TL$. Totally real submanifolds in not only Kähler manifolds but also non-Kähler Hermitian or almost Hermitian manifolds (eg. the nearly K̈hler 6-dimensional sphere) have also been extensively studied in Riemannian geometry by so many authors (cf. [10], [11]).

Suppose that (M, J, g) is a Kähler manifold with a complex structure J and a Kähler metric g. The Kähler form is a non-degenerate closed 2-form on M defined by $\omega(X, Y) := g(JX, Y)$ for each $X, Y \in TL$ and thus ω defines a symplectic structure of M. The Lagrangian submanifold in a Kähler manifold is nothing but a *totally real submanifold* of maximal dimension in a Kähler manifold, which is one of very interesting objects in Riemannian geometry of submanifolds (cf. [12], [11]). It is a fruitful subject to study Lagrangian submanifolds in Kähler manifolds from both viewpoint of symplectic geometry and Riemannian geometry.

Each one-parameter smooth family of Lagrangian submanifolds in a symplectic manifold, that is, a *Lagrangian deformation*, is induced by a *closed* 1-form on L. The *Hamiltonian deformation* of a Lagrangian submanifold is defined as a Lagrangian deformation induced by an *exact* 1-form on L. For example, any smooth variation of a closed curve on the 2-dimensional standard sphere S^2 is a Lagrangian deformation, and but a Hamiltonian deformation is a smooth variation consisting of closed curves preserving the areas of two domains bisecting S^2. It is very basic a characterization of the Hamiltonian deformation in terms of the notion of an *isomonodromy deformation* (Lemma 1.1).

It is a natural and interesting question to find the best Lagrangian submanifold under all Hamiltonian deformations. The *minimal submanifold* is defined as a submanifold in a Riemannian manifold which has extremal volume under every smooth variation and it is one of main subjects in Riemannian geometry as a generalization of minimal surfaces (cf. [10]). The volume minimizing problem of Lagrangian submanifolds under Hamiltonian deformations were introduced and investigated first by Y.-G. Oh about the beginning of 1990's ([25],[26],[27],[28]).

A Lagrangian submanifolds in a Kähler manifold which has extremal volume under every Hamiltonian deformation is called *Hamiltonian minimal*, and moreover a Hamiltonian minimal Lagrangian submanifold in a Kähler manifold is called *Hamiltonian stable* if the second variation of the volume is non-negative under every Hamiltonian deformation. For example, each great or small circle on S^2 is a 1-dimensional compact length minimizing Lagrangian submanifold under every Hamiltonian deformation (by the isoperimetric inequality on S^2). In higher dimension it is known that the real projective subspace $\mathbf{R}P^n$ of the complex projective space $\mathbf{C}P^n$ is a compact totally geodesic Lagrangian submanifold which is Hamiltonian stable ([25]) and more strongly is globally volume minimizing under every Hamiltonian deformation (Y. G. Oh and B. Kleiner). The class of compact Hamiltonian stable minimal or H-minimal Lagrangian submanifolds seems to be a very restricted class. More examples of compact Hamiltonian stable minimal or H-minimal Lagrangian submanifolds and compact homogeneous Lagrangian submanifolds embedded in complex projective spaces, more generally complex space forms will be described and discussed. At present all known compact Hamiltonian stable minimal or H-minimal Lagrangian submanifolds in complex projective spaces (more generally, in Hermitian symmetric spaces etc.) are compact homogeneous Lagrangian submanifolds, that is, Lagrangian orbits of compact Lie subgroups.

The n-dimensional complex hyperquadric $Q_n(\mathbf{C})$ is a compact complex algebraic hypersurface defined by the quadratic equation in the $(n+1)$-dimensional complex projective space, which is isometric to the real Grassmann manifold of oriented 2-planes and is a compact Hermitian symmetric space of rank 2. hyper The *Gauss map* of any oriented hypersurface in the unit sphere $S^{n+1}(1)$ is always a Lagrangian immersion into $Q_n(\mathbf{C})$ and, as an application of Lemma 1.1, there is a correspondence between deformations of an oriented hypersurface in $S^{n+1}(1)$ and Hamiltonian deformations of its Gauss map into $Q_n(\mathbf{C})$ (Proposition 2.1). Geometry of compact Lagrangian submanifolds in complex hyperquadrics can be investigated from the viewpoint of the theory of isoparametric hypersurfaces, i.e., hypersurfaces with constant principal curvatures, in the unit spheres. Using homogeneous isoparametric hypersurfaces in the unit spheres and the moment map technique, we provided a classification theorem of compact homogeneous Lagrangian submanifolds in complex hyperquadrics and we obtain a new example of a one-parameter family of compact homogeneous Lagrangian submanifolds in complex hyperquadrics (Thorem 3.1, the case (iv)). Moreover we determined the Hamiltonian stability of compact minimal Lagrangian submanifolds embedded in complex hyperquadrics which are obtained as Gauss images of isoparametric hypersurfaces in spheres with $g(=1,2,3,6)$ distinct principal curvatures (Theorems 4.1 and 4.2). Very recently we obtain both Hamiltonian stable and Hamiltonian unstable examples in the case of $g=4$ (Theorem 4.3).

This article is organized as follows : In Section 1 we recall the fundamental properties of of Lagrangian submanifolds in symplectic manifolds such as Lagrangian deformations, Hamiltonian deformations and the moment maps. Moreover we discuss Lagrangian submanifolds in Kähler manifolds and their Hamiltonian minimality and Hamiltonian stability. And we review several examples of compact Hamiltonian stable H-minimal or minimal Lagrangian submanifolds. In Section 2 we discuss the relationship of Lagrangian submanifolds in complex hyperquadrics with isoparametric hypersurface geometry in $S^{n+1}(1)$. In Section 3 we state a classification of compact homogeneous Lagrangian submanifolds in complex hyperquadrics. In Section 4 we mention the Hamiltonian Stability results of the Gauss images of isoparametric hypersurfaces in spheres. In Section 5 we suggest some further problems related to this work. In Appendix, we describe the proof of Lemma 1.1 explicitly.

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1. Lagrangian Submanifolds and Hamiltonian Stability

Let (M, ω) be a $2n$-dimensional symplectic manifold with a symplectic form ω. A *Lagrangian immersion* $\varphi : L \longrightarrow M$ is a smooth immersion of an n-dimensional smooth manifold L into M satisfying $\varphi^*\omega = 0$. There is a canonical linear isomorphism between $\varphi^{-1}TM/\varphi_*TL$ and T^*L of L : $\varphi^{-1}TM/\varphi_*TL \ni v \longmapsto \alpha_v := \omega(v, \cdot) \in T^*L$. A *Lagrangian deformation* is a smooth family of Lagrangian immersions $\varphi_t : L \longrightarrow M$ with $\varphi = \varphi_0$. Denote its variational vector field by $V_t := \frac{\partial \varphi_t}{\partial t} \in C^\infty(\varphi^{-1}TM)$ and the corresponding 1-forms on L by $\alpha_{V_t} \in \Omega^1(L)$. The Lagrangian deformation is characterized by the condition that α_{V_t} is closed, that is $\alpha_{V_t} \in Z^1(L)$, for each t. Furthermore, if α_{V_t} is exact, that is $\alpha_{V_t} \in B^1(L)$, for each t, then $\{\varphi_t\}$ is called a *Hamiltonian deformation* of $\varphi = \varphi_0$.

Hamiltonian deformations in a class of Lagrangian deformations can be characterized in terms of the notion of *isomonodromy deformation* as follows :

Lemma 1.1 ([41], [18]). *Suppose that $[\frac{1}{2\pi}\omega] \in H^2(M; \mathbf{R})$ is an integral cohomology class, and thus there is a complex line bundle $\mathcal{L}$ over M with a $U(1)$-connection whose curvature is $\sqrt{-1}\omega$. Let $\{\varphi_t\}$ be a Legrangian deformation of $\varphi_0 = \varphi : L \to M$. Then $\{\varphi_t\}$ is a Hamiltonian deformation if and only if $\{\varphi_t\}$ provides an isomonodromy deformation of the induced flat $U(1)$-connections in $\varphi_t^{-1}\mathcal{L}$.*

The proof of this lemma will be explained in detail in Appendix.

It is known that all Lagrangian orbits of a compact Hamiltonian group action K on a compact symplectic manifold (M, ω) with moment map μ appears as the level set $\mu(\alpha)$ for some $\alpha \in \mathfrak{z}(\mathfrak{k}^*) := \{\alpha \in \mathfrak{k}^* \mid \mathrm{Ad}^*(a)\alpha = \alpha \text{ for all } a \in K\}$. It is a natural question to study and classify Lagrangian orbits of the Hamiltonian group actions on specific symplectic manifolds.

We shall assume that (M, ω, J, g) is a Kähler manifold and L is a Lagrangian submanifold immersed in M. Let B be the second fundamental form of L in

M and H be the mean curvature vector field of L in M. The *mean curvature form* α_H of L is a smooth 1-form on L defined by

$$\alpha_H(X) := \omega(H, X) = g(JH, X)$$

for each $X \in TL$. A symmetric 3-tensor field S on L is defined by

$$S(X, Y, Z) := \omega(B(X, Y), Z) = g(JB(X, Y), Z)$$

for each $X, Y, Z \in TL$. It is known that α_H satisfies the identity ([13]), which follows from the Codazzi equation :

$$d\alpha_H = \varphi^* \rho_M, \tag{1.1}$$

where ρ_M denotes the Ricci form of the Kähler manifold M.

The volume minimizing problem of Lagrangian submanifolds under Hamiltonian deformations was investigated first by Y. G. Oh ([25],[26],[27],[28]). A Lagrangian immersion φ is called *Hamiltonian minimal* (H-*minimal*) if the first variation of the volume vanishes under every Hamiltonian deformation of φ. By the first variational formula for volume of submanifolds, the Hamiltonian minimality equation is

$$\delta\alpha_H = 0, \tag{1.2}$$

where δ denotes the codifferential operator of d relative to the induced metric on L. An H-minimal Lagrangian immersion φ is called *Hamiltonian stable* (H-*stable*) if the second variation of the volume is nonnegative under every Hamiltonian deformation $\{\varphi_t\}$ of φ. The second variational formula is as follows ([27]) :

$$\begin{aligned} &\frac{d^2}{dt^2}\mathrm{Vol}(L, \varphi_t^* g)|_{t=0} \\ &= \int_L \left(\langle \Delta_L^1 \alpha, \alpha \rangle - \langle \bar{R}\alpha, \alpha \rangle - 2\langle \alpha \otimes \alpha \otimes \alpha_H, S \rangle + \langle \alpha_H, \alpha \rangle^2 \right) dv, \end{aligned} \tag{1.3}$$

where we set $\alpha = \alpha_{V_0} \in B_1(L)$. Here

$$\langle \bar{R}\alpha, \alpha \rangle := \sum_{i,j=1}^{n} \mathrm{Ric}^M(e_i, e_j)\alpha(e_i)\alpha(e_j),$$

where $\{e_i\}$ is a local orthonormal frame on L and

$$S(X, Y, Z) := \omega(B(X, Y), Z) = g(JB(X, Y), Z)$$

for each $X, Y, Z \in TL$, which is a symmetric 3-tensor field on L defined by the second fundamental form B of L in M.

Problem 1.1. Investigate and classify compact Hamiltonian stable H-minimal Lagrangian submanifolds in specific Kähler manifolds.

Assume that M is an Einstein-Kähler manifold M with Einstein constant κ and L is a compact minimal Lagrangian submanifold immersed in M. Then L is Hamiltonian stable if and only if the first eigenvalue λ_1 of the Laplacian of L on functions satisfies $\lambda_1 \geq \kappa$. See also [9].

Proposition 1.1 ([34], [1]). *Assume that M is a compact homogeneous Einstein-Kähler manifold M with Einstein constant $\kappa > 0$ and L is a compact minimal Lagrangian submanifold immersed in M. Then the first eigenvalue λ_1 of the Laplacian of L on functions satisfies $\lambda_1 \leq \kappa$. Moreover $\lambda_1 = \kappa$ if and only if L is Hamiltonian stable. (It is possible to relax slightly the assumption on the homogeneity of M.)*

Problem 1.2. Investigate and classify compact minimal Lagrangian submanifolds satisfying $\lambda_1 = \kappa$ in such Kähler manifolds.

Not so many examples of compact Hamiltonian stable Lagrangian submanifolds are known for us.

Example 1.1.
(1) Circles on plane $S^1 \subset \mathbf{C}$. In fact, it is Hamiltonian volume minimizing (by the isoperimetric inequality).
(2) Great circles and small circles $S^1 \subset S^2 = \mathbf{C}P^1$. In fact, it is Hamiltonian volume minimizing (by the isoperimetric inequality).
(3) Real projective subspaces $\mathbf{R}P^n \subset \mathbf{C}P^n$ (Y. G .Oh), in fact Hamiltonian volume minimizing (Y. G. Oh-B. Kleiner).
(4) Clifford tori : A product of $n+1$ circles $S^1(r_0) \times \cdots \times S^1(r_n) \subset \mathbf{C}^{n+1}$ and the quotient space by the S^1-action $T^n \subset \mathbf{C}P^n$ (Y. G. Oh). Note that $T^n \subset \mathbf{C}P^n$ is minimal if and only if $r_0 = \cdots = r_n$.
(5) Totally geodesic Lagrangian torus $T^2 = S^1 \times S^1 \subset S^2 \times S^2 (\cong Q_2(\mathbf{C}))$, in fact Hamiltonian volume minimizing (H.Iriyeh-H.Ono-T.Sakai [16]).

Theorem 1.1 (F. Urbano [40], S. Chang [8]). *A minimal Lagrangian minimal torus L in $\mathbf{C}P^2$ is Hamiltonian stable if and only if L is a minimal Clifford torus.*

Example 1.2 ([1]). There are compact irreducible symmetric spaces standardly embedded in complex projective spaces as compact minimal Lagrangian submanifolds. They all are Hamiltonian stable :
(a) $SU(p)/SO(p)\mathbf{Z}_p \subset \mathbf{C}P^{(p-1)(p+2)/2}$,
(b) $SU(p)/\mathbf{Z}_p \subset \mathbf{C}P^{p^2-1}$,
(c) $SU(2p)/Sp(p)\mathbf{Z}_{2p} \subset \mathbf{C}P^{(p-1)(2p+1)}$,
(d) $E_6/F_4\mathbf{Z}_3 \subset \mathbf{C}P^{26}$.

These examples all satisfy the parallel property $\nabla S = 0$ of the second fundamental form. It is trivial that the condition $\nabla S = 0$ implies the Hamiltonian minimality. However here we should note the following result :

Proposition 1.2. *Any compact Hamiltonian minimal Lagrangian submanifold L with nonnegative sectional curvature immersed in complex space forms satisfies $\nabla S = 0$.*

Proof. Since the mean cuvature form of L is a harmonic 1-form on a compact Riemannian manifold L with nonnegative Ricci curvature, the mean cuvature form of L is parallel, that is, L has parallel mean curavture vector. Hence, by the result of [32] and [39], L has parallel second fundamental form, that is, $\nabla S = 0$. $\square$

By using the classification theory of parallel Lagrangian submanifolds in complex space forms (H. Naitoh-M. Takeuchi [22], [23], [24]), we extended the above Hamiltonian stability results to

Theorem 1.2 ([2],[3]). *Any compact Lagrangian submanifold L embedded in $\mathbf{C}P^n$, $\mathbf{C}^n$ or $\mathbf{C}H^n$ with $\nabla S = 0$ is Hamiltonian stable.*

But there exists a compact Hamiltonian stable minimal Lagrangian submanifold embedded in a complex projective space with $\nabla S \neq 0$ as follows:

Example 1.3 ([5], [33]). Let ρ_3 be the irreducible unitary representation of $SU(2)$ on the vector space V_3 of complex homogeneous polynomials with two variables z_0, z_1. Then

(e) $\rho_3(SU(2))[z_0^3 + z_1^3] \subset \mathbf{C}P^3$ is a 3-dimensional compact embedded Hamiltonian stable minimal Lagrangian submanifold with $\nabla S \neq 0$.

By using the classification theory of prehomogeneous vector spaces due to M. Sato and T. Kimura [17], L. Bedulli and A. Gori [4] provided a classification of compact homogeneous Lagrangian submanifolds in $\mathbf{C}P^n$ which are obtained as Lagrangian orbits of compact connected simple Lie subgroups of $SU(n+1)$:

$$\begin{aligned} \text{16 examples} &= [\text{5 examples with} \nabla S = 0 \ : \ \mathbf{R}P^n, \text{(a)} \sim \text{(d)}] \\ &\quad +[\text{11 examples with} \nabla S \neq 0 \ni \text{(e)}]. \end{aligned}$$

Problem 1.3. Is it true that any compact minimal Lagrangian submanifold embedded in complex projective space $\mathbf{C}P^n$ satisfies $\lambda_1 = \kappa$?

However there exist compact Hamiltonian *unstable* minimal Lagrangian submanifolds embedded in compact Hermitian symmetric spaces of rank greater than or equal to 2. The Hamiltonian stability of all compact totally geodesic Lagrangian submanifolds embedded in compact irreducible Hermitian symmetric spaces (with Einstein constant 1/2) are known as follows ([38],[1]) :

M	L	Einstein	λ_1	H-stable	stable
$G_{p,q}(\mathbf{C}), p \leq q$	$G_{p,q}(\mathbf{R})$	Yes	$\frac{1}{2}$	Yes	No
$G_{2p,2q}(\mathbf{C}), p \leq q$	$G_{p,q}(\mathbf{H})$	Yes	$\frac{1}{2}$	Yes	Yes
$G_{m,m}(\mathbf{C})$	$U(m)$	No	$\frac{1}{2}$	Yes	No
$SO(2m)/U(m)$	$SO(m), m \geq 5$	Yes	$\frac{1}{2}$	Yes	No
$SO(4m)/U(2m), m \geq 3$	$U(2m)/Sp(m)$	No	$\frac{m}{4m-2}$	No	No
$Sp(2m)/U(2m)$	$Sp(m), m \geq 2$	Yes	$\frac{1}{2}$	Yes	Yes
$Sp(m)/U(m)$	$U(m)/O(m)$	No	$\frac{1}{2}$	Yes	No
$Q_{p+q-2}(\mathbf{C}), q-p \geq 3$	$Q_{p,q}(\mathbf{R}),\ p \geq 2$	No	$\frac{p}{p+q-2}$	No	No
$Q_{p+q-2}(\mathbf{C}), 0 \leq q-p < 3$	$Q_{p,q}(\mathbf{R}), p \geq 2$	No	$\frac{1}{2}$	Yes	No
$Q_{q-1}(\mathbf{C}),\ q \geq 3$	$Q_{1,q}(\mathbf{R})$	Yes	$\frac{1}{2}$	Yes	Yes
$E_6/T \cdot Spin(10)$	$P_2(\mathbf{K})$	Yes	$\frac{1}{2}$	Yes	Yes
$E_6/T \cdot Spin(10)$	$G_{2,2}(\mathbf{H})/\mathbf{Z}_2$	Yes	$\frac{1}{2}$	Yes	No
$E_7/T \cdot E_6$	$SU(8)/Sp(4)\mathbf{Z}_2$	Yes	$\frac{1}{2}$	Yes	No
$E_7/T \cdot E_6$	$T \cdot E_6/F_4$	No	$\frac{1}{6}$	No	No

More generally, by the result of [6] Lagrangian submanifolds with $\nabla S = 0$ in Hermitian symmetric spaces are already classified and thus we know that any compact Lagrangian submanifold with $\nabla S = 0$ in a compact irreducible Hermitian symmetric space of rank ≥ 2 is an (extrinsic) symmetric submanifold and it is obtained by a Lagrangian deformation of a totally geodesic Lagrangian submanifold, which is a certain one-parameter family of Lagrangian submanifolds with $\nabla S = 0$ associated with an irreducible symmetric R-space of type $U(r)$.

Each Lagrangian submanifold in the above examples is a *homogeneous Lagrangian submanifold* in the sense that it is obtained as a Lagrangian orbit of a compact Lie subgroup in the holomorphic isometry group of the ambient Kähler manifolds.

Problem 1.4. Construct and classify compact homogeneous Lagrangian submanifolds in Hermitian symmetric spaces, more generally in generalized flag manifolds equipped with invariant symplectic(-Kähler) metrics.

Very recently R. Miyaoka has provided an interesting construction of compact homogeneous Lagrangian submanifolds in a generalized flag manifold G_2/T^2 ([20]).

2. Lagrangian submanifolds in complex hyperquadrics and hypersurface geometry in $S^{n+1}(1)$

We shall discuss Lagrangian submanifolds in complex hyperquadrics

$$\widetilde{\mathrm{G}r}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C}) \cong SO(n+2)/SO(2) \times SO(n),$$

which is a compact irreducible Hermitian symmetric space of rank 2. Here $\widetilde{Gr}_2(\mathbf{R}^{n+2})$ denotes the real Grassmann manifold of oriented 2-planes in $\mathbf{R}^{n+2}$ and $Q_n(\mathbf{C})$ denotes the complex hypersurface of $\mathbf{C}P^{n+1}$ defined by the algebraic equation $z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0$. Let $N^n \subset S^{n+1}(1) \subset \mathbf{R}^{n+2}$ be an oriented hypersurface immersed in the unit standard sphere. Denote by $\mathbf{x}$ its position vector of points p of N^n and by $\mathbf{n}$ the unit normal vector field of N^n in $S^{n+1}(1)$. Then we can define its "Gauss map "as

$$\mathcal{G} : N^n \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \cong [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] \in \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C}).$$

Here $\mathbf{x}(p) \wedge \mathbf{n}(p)$ denotes an oriented 2-plane in $\mathbf{R}^{n+2}$ spanned by two orthonormal vectors $\mathbf{x}(p)$ and $\mathbf{n}(p)$. Then $\mathcal{G}$ is a Lagrangian immersion. We should note that the induced metric on N^n from $Q_n(\mathbf{C})$ is different from the original metric on N^n in $S^{n+1}(1)$. Using Lemma 1.1, we obtain

Proposition 2.1 ([18])**.** *Any Hamiltonian deformation of $\mathcal{G}$ can be obtained as a smooth family of the Gauss maps given by a deformation of N^n consisting of oriented hypersurfaces in $S^{n+1}(1)$ and the converse also holds.*

The mean curvature formula was given by B. Palmer as follows :

Lemma 2.1 ([36])**.**

$$\alpha_H = d\left(\mathrm{Im}\left(\log \prod_{i=1}^{n}(1 + \sqrt{-1}\kappa_i)\right)\right),$$

where H denotes the mean curavture vector field of $\mathcal{G}$ and κ_i $(i = 1, \cdots, n)$ denotes the principal curvatures of $N^n \subset S^{n+1}(1)$.

In case $n = 2$, if $N^2 \subset S^3(1)$ is a minimal surface, then $\mathcal{G} : N^2 \longrightarrow \widetilde{Gr}_2(\mathbf{R}^4) \cong Q_2(\mathbf{C}) \cong S^2 \times S^2$ is a minimal Lagrangian immersion. See also [7].

In general case n, suppose that $N^n \subset S^{n+1}(1)$ is a compact oriented hypersurface with constant principal curvatures, so called an "isoparametric hypersurface "in the unit standard sphere. Isoparametric hypersurfaces were originated and investigated by Elie Cartan. By the great results of H. F. Münzner [21], an isoparametric hypersurface is always real algebraic and the number g of distinct principal curvatures must be $g = 1, 2, 3, 4, 6$. Then the "Gauss image "of a minimal Lagrangian immersion $\mathcal{G} : N^n \longrightarrow Q_n(\mathbf{C})$. is a compact embedded minimal Lagrangian submanifold $L = \mathcal{G}(N^n) = N^n/\mathbf{Z}_g \subset Q_n(\mathbf{C})$. We can observe that

Proposition 2.2 ([18])**.** *An isoparametric hypersurface $N^n \subset S^{n+1}(1)$ is homogeneous (i.e. an orbit of a compact Lie subgroup $K \subset SO(n+2)$ on $S^{n+1}(1)$) if and only if its Gauss image $\mathcal{G}(N^n)$ is a homogeneous Lagrangian submanifold in $Q_n(\mathbf{C})$ (i.e. an orbit of a compact Lie subgroup $K \subset SO(n+2)$ on $Q_n(\mathbf{C})$) .*

All homogeneous isoparametric hypersurfaces in the standard spheres have already been classified and in the case of $g = 4$ NON-HOMOGENEOUS isoparametric hypersurfaces in spheres were discovered first by H. Ozeki and M. Takeuchi [35] and well-developed by D. Ferus, H. Karcher and H. F. Münzner [14]. By results of W.-Y. Hsiang-H. B. Lawson, Jr. [15] and R. Takagi-T. Takahashi [37], it is well-known that all compact homogeneous isoparametric hypersurfaces $N^n \subset S^{n+1}(1)$ can be obtained as principal orbits of Riemannian symmetric pairs (U, K) of compact type and of rank 2.

Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition as a symmetric Lie algebra and $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$. For each regular element H of $\mathfrak{a} \cap S^{n+1}(1)$, we have a homogeneous isoparametric hypersurface in the standard unit sphere $N^n := (\mathrm{Ad}K)H \subset S^{n+1}(1) \subset \mathbf{R}^{n+2} \cong \mathfrak{p}$. Its Gauss image is $\mathcal{G}(N^n) = (\mathrm{Ad}K)[\mathfrak{a}] \subset \widetilde{Gr}_2(\mathfrak{p}) \cong Q_n(\mathbf{C})$. Then the moment map μ of the action of K on $Q_n(\mathbf{C})$ induced by the adjoint action of K on $\mathfrak{p}$ is given as follows : $\mu : Q_n(\mathbf{C}) \cong \widetilde{\mathrm{Gr}}_2(\mathfrak{p}) \ni [\mathbf{a} + \sqrt{-1}\mathbf{b}] = [V] \longmapsto \in [\mathbf{a}, \mathbf{b}] \in \mathfrak{k} \cong \mathfrak{k}^*$, where $\{\mathbf{a}, \mathbf{b}\}$ is an orthonormal basis of $V \subset \mathfrak{p}$ compatible with its orientation. Hence we have $\mathcal{G}(N^n) = \mu^{-1}(0)$.

g	Type	(U, K)	$\dim N$	m_1, m_2	$N = K/K_0$
1	$S^1\times$ BDII	$(S^1 \times SO(n+2), SO(n+1))$ $(n \geq 1)$	n	n	S^n
2	BDII $\times$BDII	$(SO(p+2) \times SO(n+2-p),$ $SO(p+1) \times SO(n+1-p))$ $(1 \leq p \leq n-1)$	n	$p, n-p$	$S^p \times S^{n-p}$
3	AI_2	$(SU(3), SO(3))$	3	$1, 1$	$\frac{SO(3)}{\mathbf{Z}_2+\mathbf{Z}_2}$
3	$\mathfrak{a}_2$	$(SU(3) \times SU(3), SU(3))$	6	$2, 2$	$\frac{SU(3)}{T^2}$
3	AII_2	$(SU(6), Sp(3))$	12	$4, 4$	$\frac{Sp(3)}{Sp(1)^3}$
3	EIV	(E_6, F_4)	24	$8, 8$	$\frac{F_4}{Spin(8)}$
4	$\mathfrak{b}_2$	$(SO(5) \times SO(5), SO(5))$	8	$2, 2$	$\frac{SO(5)}{T^2}$
4	AIII_2	$(SU(m+2), S(U(m) \times U(2)))$ $(m \geq 2)$	$4m-2$	$2,$ $2m-3$	$\frac{S(U(m)\times U(2))}{SU(m-2)\times T^2}$
4	BDI_2	$(SO(m+2), SO(m) \times SO(2))$ $(m \geq 3)$	$2m-2$	$1,$ $m-2$	$\frac{SO(m)\times SO(2)}{SO(m-2)\times \mathbf{Z}_2}$
4	CII_2	$(Sp(m+2), Sp(m) \times Sp(2))$ $(m \geq 2)$	$8m-2$	$4,$ $4m-5$	$\frac{Sp(m)\times Sp(2)}{Sp(m-2)\times Sp(1)^2}$
4	DIII_2	$(SO(10), U(5))$	18	$4, 5$	$\frac{U(5)}{SU(2)\times SU(2)\times T^1}$
4	EIII	$(E_6, Spin(10) \cdot T)$	30	$6, 9$	$\frac{Spin(10)\cdot T}{SU(4)\cdot T}$
6	$\mathfrak{g}_2$	$(G_2 \times G_2, G_2)$	12	$2, 2$	$\frac{G_2}{T^2}$
6	G	$(G_2, SO(4))$	6	$1, 1$	$\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}$

3. Classification of compact homogeneous Lagrangian submanifolds in complex hyperquadrics

Theorem 3.1 ([18]). *Let L be a compact homogenoeus Lagrangian submanifold in $Q_n(\mathbf{C})$. Then there is uniquely a compact homogeneous isoparametric hypersurface N^n in $S^{n+1}(1)$ corresponding to a compact Riemannian symmetric pair (U, K) of rank 2 such that L is obtained from N^n in the following way (1) or (2) :*

(1) *$L = \mathcal{G}(N^n) = \mu^{-1}(0) \subset Q_n(\mathbf{C})$, which is a compact homogeneous minimal Lagrangian submanifold.*
(2) *$L = \mu^{-1}(\xi) \subset Q_n(\mathbf{C})$ for some $\xi \in \mathfrak{c}(\mathfrak{k})$ and so L can be obtained as a Lagrangian deformation of $\mathcal{G}(N^n)$.*

Actually there exists such a non-trivial Lagrangian deformation of $\mathcal{G}(N^n)$ only when (U, K) is one of

(i) *$(S^1 \times SO(3), SO(2))$,*
(ii) *$(SO(3) \times SO(3), SO(2) \times SO(2))$,*
(iii) *$(SO(3) \times SO(n + 1), SO(2) \times SO(n))$ $(n \geq 3)$,*
(iv) *$(SO(m + 2), SO(2) \times SO(m))$ $(n = 2m - 2, m \geq 3)$.*

(i) : If (U, K) is $(S^1 \times SO(3), SO(2))$, then L is a small or great circle in $Q_1(\mathbf{C}) \cong S^2$.

(ii) : If (U, K) is $(SO(3) \times SO(3), SO(2) \times SO(2))$, then L is a product of small or great circles of S^2 in $Q_2(\mathbf{C}) \cong S^2 \times S^2$.

(iii) : If (U, K) is $(SO(3) \times SO(n + 1), SO(2) \times SO(n))$ $(n \geq 2)$, then

$$L = K \cdot [W_\lambda] \subset Q_n(\mathbf{C}) \quad \text{for some } \lambda \in S^1 \setminus \{\pm\sqrt{-1}\},$$

where $K \cdot [W_\lambda]$ $(\lambda \in S^1)$ is the S^1-family of Lagrangian or isotropic K-orbits satisfying

(1) $K \cdot [W_1] = K \cdot [W_{-1}] = \mathcal{G}(N^n)$ is a totally geodesic Lagrangian submanifold in $Q_n(\mathbf{C})$.
(2) For each $\lambda \in S^1 \setminus \{\pm\sqrt{-1}\}$,
$$K \cdot [W_\lambda] \cong (S^1 \times S^{n-1})/\mathbf{Z}_2 \cong Q_{2,n}(\mathbf{R})$$
is a Lagrangian orbit in $Q_n(\mathbf{C})$ with $\nabla S = 0$.
(3) $K \cdot [W_{\pm\sqrt{-1}}]$ are isotropic orbits in $Q_n(\mathbf{C})$ with $\dim K \cdot [W_{\pm\sqrt{-1}}] = 0$.

(iv) : If (U, K) is $(SO(m + 2), SO(2) \times SO(m))$ $(n = 2m - 2)$, then

$$L = K \cdot [W_\lambda] \subset Q_n(\mathbf{C}) \quad \text{for some } \lambda \in S^1 \setminus \{\pm\sqrt{-1}\},$$

where $K \cdot [W_\lambda]$ $(\lambda \in S^1)$ is the S^1-family of Lagrangian or isotropic orbits satisfying

(1) $K \cdot [W_1] = K \cdot [W_{-1}] = \mathcal{G}(N^n)$ is a minimal (NOT totally geodesic) Lagrangian submanifold in $Q_n(\mathbf{C})$.

(2) For each $\lambda \in S^1 \setminus \{\pm\sqrt{-1}\}$,

$$K \cdot [W_\lambda] \cong (SO(2) \times SO(m))/(\mathbf{Z}_2 \times \mathbf{Z}_4 \times SO(m-2))$$

is a Lagrangian orbit in $Q_n(\mathbf{C})$ with $\nabla S \neq 0$.

(3) $K \cdot [W_{\pm\sqrt{-1}}] \cong SO(m)/S(O(1) \times O(m-1)) \cong \mathbf{R}P^{m-1}$ are isotropic orbits in $Q_n(\mathbf{C})$ with $\dim K \cdot [W_{\pm\sqrt{-1}}] = m-1$.

4. Hamiltonian Stability of the Gauss images of isoparametric hypersurfaces

Let N^n be a compact isoparametric hypersurface embedded in $S^{n+1}(1)$. Palmer [36] showed that its Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbf{C})$ is Hamiltonian stable if and only if $N^n = S^n \subset S^{n+1}(1)$ $(g = 1)$.

Question. Hamiltonian stability of its Gauss image $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$?

$g = 1 : \mathcal{G}(N^n) \cong N^n \cong S^n$ is Hamiltonian stable. More strongly, it is stable as a minimal submanifold and homologically volume-minimizing because it is a calibrated submanifold.

$g = 2 : N^n = S^{m_1} \times S^{m_2}$ Clifford hypersurface $(n = m_1 + m_2, 1 \leq m_1 \leq m_2)$ and $\mathcal{G}(N^n) = Q_{m_1+1,m_2+1}(\mathbf{R}) \cong (S^{m_1} \times S^{m_2})/\mathbf{Z}_2 \subset Q_n(\mathbf{C})$. If $m_2 - m_1 \geq 3$, then $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is NOT Hamiltonian stable. Otherwise $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is Hamiltonian stable. The spherical harmonics of degree 2 on the sphere of smaller dimension give volume-decreasing Hamiltonian deformations of $\mathcal{G}(N^n)$.

Theorem 4.1 (Hui Ma-O.[18])**.** *If $g = 3$, then $L = \mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is (strictly) Hamiltonian stable.*

Remark. In case $g = 3$, each induced metric from $Q_n(\mathbf{C})$ is a normal homogeneous metric. It does not hold at all in cases $g = 4, 6$

Theorem 4.2 (Hui Ma-O.)**.** *If $g = 6$ and N^n is homogeneous, then $L = \mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is (strictly) Hamiltonian stable.*

More recently, in the case when $g = 4$ and N^n is homogeneous, we obtain

Theorem 4.3 (Hui Ma-O.)**.**

$$\mathcal{G}(N^n) = SO(5)/T^2 \cdot \mathbf{Z}_4$$

is (strictly) Hamiltonian stable, but

$$\mathcal{G}(N^n) = (SO(2) \times SO(m))/(\mathbf{Z}_2 \times SO(m-2)) \cdot \mathbf{Z}_4 \quad (m \geq 3)$$

is NOT Hamiltonian stable if and only if $m \geq 6$, i.e. $m_2 - m_1 = (m-2) - 1 \geq 3$.

In the forthcoming joint paper with Dr. Hui Ma, we will mention further results on the Hamiltonian stability of their Gauss images in the case when $g = 4$ and N^n is homogeneous.

5. Further Problems

(1) Hamiltonian stability in the case when N is a non-homogeneous isoparametric hypersurface with $g = 4$?

(2) Investigate the properties of minimal Lagrangian submanifolds $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ for (homogeneous or non-homogeneous) isoparametric hypersurfaces $N^n \subset S^{n+1}(1)$?

(3) How about geometry of Lagrangian submanifolds in other compact Hermitian symmetric spaces such as a classification of compact homogeneous Lagrangian submanifolds, Hamiltonian stability problems and so on ? The case when M is a complex Grassmann manifold $\mathrm{Gr}_2(\mathbf{C}^{n+2})$ of complex 2-dimensional vector subspaces of $\mathbf{C}^{n+2}$ seems to be very interesting to be studied next.

6. Appendix : Proof of Lemma 1.1

Here for readers we shall describe an elementary computation for the proof of Lemma 1.1. This argument is motivated by and based on [41].

Let $\varphi : L^n \to (M, \omega)$ be a Lagrangian immersion. Suppose that $\varphi_t : L^n \to (M, \omega)$ $(t \in I)$ is a smooth family of Lagrangian immersions with $\varphi_0 = \varphi$. For each $t \in I$, set $V_t := \frac{\partial \varphi_t}{\partial t} \in C^\infty(\varphi_t^* TM)$ as the variational vector field of $\{\varphi_t\}$. Define $\Phi : I \times L \to M$ by $\Phi(t, p) := \varphi_t(p) \in M$ for each $(t, p) \in I \times M$.

For each $[\gamma] \in \pi_1(L)$, using an expression $\gamma = \gamma(v)$ $(0 \leq v \leq 1)$, we define

$$\begin{aligned}(\mathcal{A}(t))([\gamma]) :&= \int_0^t \int_0^1 \omega(V_u, (\varphi_t)_* \dot{\gamma}(v)) du dv, \\ &= \int_0^t \left(\int_\gamma \alpha_{V_u} \right) du = \int_{[0,1]\times\gamma} \Phi^* \omega.\end{aligned}$$

Note that $(\mathcal{A}(t))([\gamma])$ is independent of the choice of representatives in $[\gamma] \in \pi_1(L)$, because of the Lagrangian property of φ_t. Then we see that $\mathcal{A}(t) : \pi_1(L) \to \mathbf{R}$ defines a group homomorphism and $\mathcal{A}(0) = 0$. Hence we obtain

Lemma 6.1. *The following conditions are equivalent each other :*

(1) $\mathcal{A}(t) \equiv 0$.
(2) $\mathcal{A}(t)$ *is independent of* $t \in I$.
(3) $\int_\gamma \alpha_{V_t} = 0$ *for each* $t \in I$ *and each* $[\gamma] \in \pi_1(L)$.
(4) α_{V_t} *is exact for each* $t \in I$.
(5) $\{\varphi_t\}$ *is a Hamiltonian deformation.*

For each t, let $\rho_t : \pi_1(L) \to U(1)$ denote the holonomy homomorphism of the induced flat $U(1)$-connection $\nabla^t = \varphi_t^{-1} \nabla$ in $\varphi_t^{-1} \mathcal{L}$. Then we shall prove the formula

$$\rho_t([\gamma])^{-1} \frac{d}{dt} \rho_t([\gamma]) = -\sqrt{-1} \int_\gamma \alpha_{V_t} \tag{6.1}$$

for each $[\gamma] \in \pi_1(L)$.

Let $[\gamma] \in \pi_1(L)$ be an arbitrary element of the fundamental group of L. We express γ as $\gamma = \gamma(v)$ $(0 \leq v \leq 1)$ with $\gamma(0) = \gamma(1) = p \in L$.

Let s_U be a local orthonormal frame field of $\mathcal{L}$ defined in a neighborhood U of M. The connection form θ_U on U, which is a 1-form on U with values in $\sqrt{-1}\mathbf{R}$, is defined as

$$\nabla s_U = \theta_U \otimes s_U. \tag{6.2}$$

Then the curvature form Θ defined by $d\theta_U = \Theta$ is a 2-form on the whole M with values in $\sqrt{-1}\mathbf{R}$ and it coincides with $\sqrt{-1}\omega$.

Let $\xi = \xi(v)$ $(0 \leq v \leq 1)$ be a smooth section of $\varphi_t^{-1}\mathcal{L}$ along $\gamma^{-1}(\varphi_t^{-1}(U))$. Note that there is a smooth function $q : (\varphi_t \circ \gamma)^{-1}(U) \to \mathbf{C}$ such that $\xi(v) = q(v)s_{\gamma(v)}$ for each $v \in \gamma^{-1}(\varphi_t^{-1}(U))$. Then the section $\xi = \xi(v)$ is parallel with ∇^t if and only if

$$(\nabla^t)_{\dot\gamma}\xi = 0, \tag{6.3}$$

that is,

$$\left(\frac{dq(v)}{dt} + q(v)\theta(\dot\gamma(v))\right) s_{\gamma(v)} = 0,$$

and thus

$$\frac{dq(v)}{dt} + q(v)\theta(\dot\gamma(v)) = 0. \tag{6.4}$$

Using the equation (6.4), since

$$\begin{aligned}\log\frac{q(v)}{q(0)} = \log(q(v)) - \log(q(0)) &= \int_0^v \frac{d}{dv}\log q(v)dv \\ &= -\int_0^v \theta(\dot\gamma(v))dv,\end{aligned}$$

we have

$$q(v) = q(0)\exp\left(-\int_0^v \theta(\dot\gamma(v))dv\right). \tag{6.5}$$

Next we shall discuss the explicit expression of a holonomy homomorphism $\rho_t : \pi_1(L) \to U(1)$ of the flat $U(1)$-connection $\varphi_t^{-1}\nabla$.

First we choose a contractible neighborhood U_1, U_2 in M such that

$$\bigcup_{|t|\leq\varepsilon} \varphi_t(\gamma([0,\frac{1}{2}])) \subset U_1, \quad \bigcup_{|t|\leq\varepsilon} \varphi_t(\gamma([\frac{1}{2},1])) \subset U_2. \tag{6.6}$$

We express a smooth section ξ as

$$\xi = \xi(t,v) \in (\varphi_t^{-1}\mathcal{L})_{\gamma(v)}$$

for each t with $|t| \leq \varepsilon$ and each v with $0 \leq v \leq 1$.

Let $s^{(1)}$ be a local orthonormal frame field of $\mathcal{L}$ defined on U_1. Over U_1, ξ can be expressed as

$$\xi(t,v) = q^{(1)}(t,v)(s^{(1)} \circ \varphi_t)(\gamma(v)) \in (\varphi_t^{-1}\mathcal{L})_{\gamma(v)} \quad (0 \leq v \leq \frac{1}{2}),$$

where $q^{(1)}(t,v) \in \mathbf{C}$. Let $s^{(2)}$ be a local orthonormal frame field of $\mathcal{L}$ defined on U_2. Over U_2, ξ can be expressed as

$$\xi(t,v) = q^{(2)}(t,v)(s^{(2)} \circ \varphi_t)(\gamma(v)) \in (\varphi_t^{-1}\mathcal{L})_{\gamma(v)} \quad (0 \le v \le \frac{1}{2}),$$

where $q^{(2)}(t,v) \in \mathbf{C}$. Over $U_1 \cap U_2$, if we set $s^{(2)} = q^{21} s^{(1)}$, then we have

$$q^{(2)}(t,v) = q^{21}(t.v) q^{(1)}(t,v).$$

Now we assume that

$$\xi(t,0) = s^{(1)}(\varphi_t(p)) \in (\varphi_t^{-1}\mathcal{L})_p \ (|t| \le \varepsilon), \tag{6.7}$$

that is, $q^{(1)}(t,0) = 1$ $(|t| \le \varepsilon)$, and

$$\nabla^t \xi = \nabla^t \xi(t,v) = 0, \tag{6.8}$$

that is, ξ is parallel with respect to each ∇^t. Then we have

$$\xi(t,1) = \rho_t([\gamma])\xi(t,0) \in (\varphi_t^{-1}\mathcal{L})_p \quad \text{and} \quad \rho_t([\gamma]) \in U(1). \tag{6.9}$$

Over U_1, (6.8) is equivalent to

$$\frac{dq^{(1)}}{dv} + q^{(1)}(\varphi_t^* \theta^{(1)})(\dot{\gamma}(v)) = 0, \tag{6.10}$$

and thus

$$\frac{d}{dv} \log q^{(1)} = -(\varphi_t^* \theta^{(1)})(\dot{\gamma}(v))$$

$$\int_0^{1/2} \frac{d}{dv} \log q^{(1)} dv = -\int_0^{1/2} (\varphi_t^* \theta^{(1)})(\dot{\gamma}(v)) dv.$$

Hence we have

$$\begin{aligned} q^{(1)}(t,1/2) &= q^{(1)}(t,0) \exp\left(-\int_0^{1/2} (\varphi_t^* \theta^{(1)})(\dot{\gamma}(v)) dv \right) \\ &= \exp\left(-\int_0^{1/2} (\varphi_t^* \theta^{(1)})(\dot{\gamma}(v)) dv \right). \end{aligned} \tag{6.11}$$

Over U_2, (6.8) is equivalent to

$$\frac{dq^{(1)}}{dv} + q^{(2)}(\varphi_t^* \theta^{(2)})(\dot{\gamma}(v)) = 0, \tag{6.12}$$

and thus

$$\frac{d}{dv} \log q^{(2)} = -(\varphi_t^* \theta^{(2)})(\dot{\gamma}(v))$$

$$\int_{1/2}^{1} \frac{d}{dv} \log q^{(2)} dv = -\int_{1/2}^{1} (\varphi_t^* \theta^{(2)})(\dot{\gamma}(v)) dv.$$

Hence we have

$$q^{(2)}(t,1) = q^{(2)}(t,1/2) \exp\left(-\int_{1/2}^{1} (\varphi_t^* \theta^{(2)})(\dot{\gamma}(v)) dv \right). \tag{6.13}$$

Since by (6.11) we compute

$$
\begin{aligned}
&q^{(2)}(t,1/2)\\
=&q^{(1)}(t,1/2)(q^{21}(\varphi_t(1/2)))^{-1}\\
=&q^{(1)}(t,0)\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv\right)(q^{21}(\varphi_t(1/2)))^{-1},
\end{aligned}
$$

we have

$$
\begin{aligned}
&q^{(2)}(t,1)\\
=&q^{(1)}(t,0)\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv\right)(q^{21}(\varphi_t(1/2)))^{-1}\\
&\quad\times\exp\left(-\int_{1/2}^{1}(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\right)\\
=&q^{(1)}(t,0)(q^{21}(\varphi_t(1/2)))^{-1}\\
&\times\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv\right)\exp\left(-\int_{1/2}^{1}(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\right)\\
=&q^{(1)}(t,0)(q^{21}(\varphi_t(1/2)))^{-1}\\
&\quad\times\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv-\int_{1/2}^{1}(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\right)\\
=&(q^{21}(\varphi_t(1/2)))^{-1}\\
&\quad\times\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv-\int_{1/2}^{1}(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\right).
\end{aligned}
\tag{6.14}
$$

Since we use

$$
\begin{aligned}
\xi(t,1)&=q^{(2)}(t,1)(s^{(2)}\circ\varphi_t)(\gamma(1))\\
&=q^{(2)}(t,1)(s^{(2)}\circ\varphi_t)(p)\\
&=q^{(2)}(t,1)q^{21}(\varphi_t(p))(s^{(1)}\circ\varphi_t)(p)\\
&=q^{(2)}(t,1)q^{21}(\varphi_t(p))\xi(t,0),\\
\xi(t,0)&=q^{(1)}(t,0)(s^{(1)}\circ\varphi_t)(p)\\
&=(s^{(1)}\circ\varphi_t)(p),
\end{aligned}
\tag{6.15}
$$

by (6.14) we obtain

$$
\begin{aligned}
&\xi(t,1)\\
(6.16)\qquad &=q^{21}(\varphi_t(\gamma(\frac{1}{2})))^{-1}q^{21}(\varphi_t(p))\\
&\quad\times\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv-\int_{1/2}^1(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\right)\xi(t,0)
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\rho_t([\gamma])\\
(6.17)\qquad &=q^{21}(\varphi_t(\gamma(1/2)))^{-1}q^{21}(\varphi_t(p))\\
&\quad\times\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv-\int_{1/2}^1(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\right).
\end{aligned}
$$

We shall compute the derivative of $\rho_t([\gamma])$ with respect to t :

$$\frac{d}{dt}\rho_t([\gamma]) = \ ? \ .$$

Define $\Phi : [-\varepsilon,\varepsilon]\times L\to M$ by $\Phi(t,x):=\varphi_t(x)$. Then we compute

$$
\begin{aligned}
&\frac{d}{dt}\int_0^{1/2}\varphi_t^*\theta^{(1)}(\dot\gamma(v))dv\\
&=\int_0^{1/2}(L_{\frac{\partial}{\partial t}}\Phi^*\theta^{(1)})(\dot\gamma(v))dv\\
&=\int_0^{1/2}(d\circ\iota_{\frac{\partial}{\partial t}}+\iota_{\frac{\partial}{\partial t}}\circ d)\Phi^*\theta^{(1)}(\dot\gamma(v))dv\\
(6.18)\qquad &=\int_0^{1/2}d(\theta^{(1)}(V_t))(\dot\gamma(v))dv+\int_0^{1/2}\sqrt{-1}(\Phi^*\omega)(\frac{\partial}{\partial t},\dot\gamma(v))dv\\
&=\int_0^{1/2}d(\theta^{(1)}(V_t))(\dot\gamma(v))dv+\int_0^{1/2}\sqrt{-1}\alpha_{V_t}(\dot\gamma(v))dv\\
&=\theta^{(1)}(V_t)(\gamma(1/2))-\theta^{(1)}(V_t)(\gamma(0))+\int_0^{1/2}\sqrt{-1}\alpha_{V_t}(\dot\gamma(v))dv\\
&=\theta^{(1)}(V_t)(\gamma(1/2))-\theta^{(1)}(V_t)(p)+\int_0^{1/2}\sqrt{-1}\alpha_{V_t}(\dot\gamma(v))dv.
\end{aligned}
$$

Similarly, we compute

$$
\begin{aligned}
&\frac{d}{dt}\int_{1/2}^1\varphi_t^*\theta^{(2)}(\dot\gamma(v))dv\\
(6.19)\qquad &=\theta^{(2)}(V_t)(\gamma(1))-\theta^{(2)}(V_t)(\gamma(1/2))+\int_{1/2}^1\sqrt{-1}\alpha_{V_t}(\dot\gamma(v))dv.
\end{aligned}
$$

Combining (6.18) and (6.19), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_0^{1/2}\varphi_t^*\theta^{(1)}(\dot\gamma(v))dv+\frac{d}{dt}\int_{1/2}^1\varphi_t^*\theta^{(2)}(\dot\gamma(v))dv\\
=&\theta^{(1)}(V_t)(\gamma(1/2))-\theta^{(1)}(V_t)(p)+\theta^{(2)}(V_t)(\gamma(1))-\theta^{(2)}(V_t)(\gamma(1/2))\\
&+\int_0^{1/2}\sqrt{-1}\alpha_{V_t}(\dot\gamma(v))dv+\int_{1/2}^1\sqrt{-1}\alpha_{V_t}(\dot\gamma(v))dv\\
=&\theta^{(1)}(V_t)(\gamma(1/2))-\theta^{(1)}(V_t)(p)+\theta^{(2)}(V_t)(\gamma(1))-\theta^{(2)}(V_t)(\gamma(1/2))\\
&+\int_0^1\sqrt{-1}\alpha_{V_t}(\dot\gamma(v))dv\\
=&\theta^{(1)}(V_t)(\gamma(1/2))-\theta^{(1)}(V_t)(p)+\theta^{(2)}(V_t)(p)-\theta^{(2)}(V_t)(\gamma(1/2))\\
&+\sqrt{-1}\int_\gamma\alpha_{V_t}.
\end{aligned}
\tag{6.20}
$$

Since

$$
\begin{aligned}
&\theta^{(2)}(V_t)(p)\\
=&(q^{21}(\varphi_t(p)))^{-1}(dq^{21})_{\varphi_t(p)}(V_t)+\theta^{(1)}(V_t)(p),\\
&\theta^{(2)}(V_t)(\gamma(1/2))\\
=&(q^{21}(\varphi_t(\gamma(1/2))))^{-1}(dq^{21})_{\varphi_t(\gamma(1/2))}(V_t)+\theta^{(1)}(V_t)(\gamma(1/2)),
\end{aligned}
$$

(6.20) becomes

$$
\begin{aligned}
&\frac{d}{dt}\int_0^{1/2}\varphi_t^*\theta^{(1)}(\dot\gamma(v))dv+\frac{d}{dt}\int_{1/2}^1\varphi_t^*\theta^{(2)}(\dot\gamma(v))dv\\
=&(q^{21}(\varphi_t(p)))^{-1}(dq^{21})_{\varphi_t(p)}(V_t)\\
&-(q^{21}(\varphi_t(\gamma(1/2))))^{-1}(dq^{21})_{\varphi_t(\gamma(1/2))}(V_t)\\
&+\sqrt{-1}\int_\gamma\alpha_{V_t}.
\end{aligned}
\tag{6.21}
$$

On the other hand we see

$$
\begin{aligned}
&\frac{d}{dt}q^{21}(\varphi_t(\gamma(1/2)))^{-1}\\
=&-q^{21}(\varphi_t(\gamma(1/2)))^{-1}(dq^{21})_{\varphi_t(\gamma(1/2))}(V_t)q^{21}(\varphi_t(\gamma(1/2)))^{-1},\\
&\frac{d}{dt}q^{21}(\varphi_t(p))=(dq^{21})_{\varphi_t(p)}(V_t).
\end{aligned}
\tag{6.22}
$$

Now, using (6.21) and (6.22), we can compute the derivative of (6.17) with respect to t as follows :

$$
\begin{aligned}
&\frac{d}{dt}\rho_t([\gamma])\\
=&q^{21}(\varphi_t(\gamma(1/2)))^{-1}\cdot q^{21}(\varphi_t(p))\\
&\quad\times\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv-\int_{1/2}^1(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\right)\\
&\quad\times(-q^{21}(\varphi_t(p))^{-1}(dq^{21})_{\varphi_t(p)}(V_t)+q^{21}(\varphi_t(\gamma(1/2))^{-1}(dq^{21})_{\varphi_t(\gamma(1/2))}(V_t)\\
&\qquad-\sqrt{-1}\int_\gamma\alpha_{V_t})\\
&+\frac{d}{dt}q^{21}(\varphi_t(\gamma(1/2))^{-1})\cdot q^{21}(\varphi_t(p))\\
&\quad\times\exp(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv-\int_{1/2}^1(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\\
&+q^{21}(\varphi_t(\gamma(1/2))^{-1}\cdot\frac{d}{dt}q^{21}(\varphi_t(p))\\
&\quad\times\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv-\int_{1/2}^1(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\right)\\
=&q^{21}(\varphi_t(\gamma(1/2)))^{-1}\cdot q^{21}(\varphi_t(p))\\
&\quad\times\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv-\int_{1/2}^1(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\right)\\
&\quad\times(-q^{21}(\varphi_t(p))^{-1}\cdot(dq^{21})_{\varphi_t(p)}(V_t)+q^{21}(\varphi_t(\gamma(1/2)))^{-1}\cdot(dq^{21})_{\varphi_t(\gamma(1/2))}(V_t)\\
&\qquad-\sqrt{-1}\int_\gamma\alpha_{V_t})\\
&-q^{21}(\varphi_t(\gamma(1/2)))^{-1}\cdot(dq^{21})_{\varphi_t(\gamma(1/2))}(V_t)\cdot q^{21}(\varphi_t(\gamma(1/2)))^{-1}\cdot q^{21}(\varphi_t(p))\\
&\quad\times\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv-\int_{1/2}^1(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\right)\\
&+q^{21}(\varphi_t(\gamma(1/2)))^{-1}\cdot(dq^{21})_{\varphi_t(p)}(V_t)\\
&\quad\times\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv-\int_{1/2}^1(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\right)\\
=&q^{21}(\varphi_t(\gamma(1/2)))^{-1}\cdot q^{21}(\varphi_t(p))\\
&\cdot\exp\left(-\int_0^{1/2}(\varphi_t^*\theta^{(1)})(\dot\gamma(v))dv-\int_{1/2}^1(\varphi_t^*\theta^{(2)})(\dot\gamma(v))dv\right)\cdot\left(-\sqrt{-1}\int_\gamma\alpha_{V_t}\right).
\end{aligned}
$$

Hence we obtain the formula

$$
\rho_t([\gamma])^{-1}\frac{d}{dt}\rho_t([\gamma])=-\sqrt{-1}\int_\gamma\alpha_{V_t}. \tag{6.23}
$$

Therefore we conclude

Theorem 6.1. *The following conditions are equivalent each other :*

(1) *The holonomy homomorphisms* ρ_t ($|t| \leq \varepsilon$) *are same, that is,*

$$\frac{d}{dt}\rho_t([\gamma]) \equiv 0 \textit{ for each } [\gamma] \in \pi_1(L).$$

(2) *For each* t *with* $|t| \leq \varepsilon$ *and each* $[\gamma] \in \pi_1(L)$,

$$\int_\gamma \alpha_t = 0.$$

(3) $\{\varphi_t \mid |t| \leq \varepsilon\}$ *is a Hamiltonian deformation.*

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