# Morse indices and the number of maximum points of some solutions to a two-dimensional elliptic problem

Futoshi Takahashi
Department of Mathematics, Osaka City University
Sumiyoshi-ku, Osaka, 558-8585, Japan
Tel: (+81)(0)6-6605-2508
E-mail: futoshi@sci.osaka-cu.ac.jp

**Abstract.** In this note, we consider the problem

$$-\Delta u = u^p \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0$$

on a smooth bounded domain $\Omega$ in $\mathbb{R}^2$ for $p > 1$. Let $u_p$ be a positive solution of the above problem with Morse index less than or equal to $m \in \mathbb{N}$. We prove that if $u_p$ further satisfies the assumption $p \int_\Omega |\nabla u_p|^2 dx = O(1)$ as $p \to \infty$, then the number of maximum points of $u_p$ is less than or equal to $m$ for $p$ sufficiently large. If $\Omega$ is convex, we also show that a solution of Morse index one satisfying the above assumption has a unique critical point and the level sets are star-shaped for $p$ sufficiently large.

**Keywords**: Morse index, maximum point, semilinear elliptic equation.

**2000 Mathematics Subject Classifications**: 35B40, 35J20, 35J25.

## 1. Introduction.

In this note we consider the problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^2$ and $p > 1$. Since the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact for any $p > 1$, the existence of at least one solution is easy to obtain. In fact, if we define

$$S_p = \inf_{u \in H_0^1(\Omega)} \{ \int_\Omega |\nabla u|^2 dx \mid \int_\Omega |u|^{p+1} dx = 1 \},$$

then a standard variational method implies that $S_p$ is achieved by a positive function $\overline{u}_p \in H_0^1(\Omega)$ and $u_p = S_p^{1/(p-1)}\overline{u}_p$ is a solution of (1.1).

For the least energy solution $u_p$ obtained in this way, several studies on the asymptotic behavior have been done in [6], [7], [3] and [1]. In particular, in [6] and [7], along a suitable subsequence $p \to \infty$, least energy solution $u_p$ is shown to develop a single spiky pattern at an interior point of the domain. More precisely, $u_p$ satisfies a uniform $L^\infty$-norm estimate

$$C_1 \leq \|u_p\|_{L^\infty(\Omega)} \leq C_2$$

for some $0 < C_1 \leq C_2 < \infty$ independent of $p$, and to "concentrate" at an interior point of the domain, i.e.,

$$\frac{u_p^p}{\int_\Omega u_p^p dx} \to \delta(x_0) \quad \text{as } p \to \infty$$

for some $x_0 \in \Omega$ in the sense of Radon measures. Moreover, the estimate

$$p \int_\Omega |\nabla u_p|^2 dx \to 8\pi e \quad \text{as } p \to \infty$$

is proved for least energy solution $u_p$. Recently, for any $m \in \mathbb{N}$, a solution sequence $\{u_p\}$ which exhibits the asymptotic behavior

$$p \int_\Omega |\nabla u_p|^2 dx \to 8\pi m e \quad \text{as } p \to \infty$$

has been constructed in [5] under some topological assumption of the domain.

In the following, we restrict our attention to the solution $u_p$ of (1.1) which satisfies the assumption

$$p \int_\Omega |\nabla u_p|^2 dx = O(1) \quad \text{as } p \to \infty. \tag{1.2}$$

Before stating the results in this paper, we recall that the *Morse index* of a solution $u$ of (1.1) is the number of negative eigenvalues of the linearized operator $L_u = -\Delta - pu^{p-1}\cdot$ acting on $H_0^1(\Omega)$. In this paper, we prove the following theorems.

**Theorem 1.** *Let $\{u_p\}$ be a solution sequence of (1.1) satisfying the assumption (1.2) with the Morse index less than or equal to $m, m \in \mathbb{N}$. Then the* *number of maximum points of $u_p$ is less than or equal to $m$ for $p$ sufficiently large.*

**Theorem 2.** *Let $\{u_p\}$ be a solution sequence of (1.1) satisfying the assumption (1.2) with the Morse index one. If $\Omega$ is convex, then $u_p$ has only one critical point $x_p$ which is the global maximum point of $u_p$, and*

$$(x - x_p) \cdot \nabla u_p(x) < 0, \quad \forall x \in \Omega \setminus \{x_p\}$$

*holds for $p$ sufficiently large. In particular, the level sets of $u_p$ are strict star-shaped with respect to $x_p$.*

In [4], El Mehdi and Pacella treated the problem

$$\begin{cases} -\Delta u = N(N-2)u^{p-\varepsilon} - \lambda u & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega$ is a smooth bounded, star-shaped domain in $\mathbb{R}^N, N \geq 3$, $p = (N+2)/(N-2)$, $\varepsilon > 0$ and $\lambda \geq 0$. They proved similar results on the relation between the Morse index and the number of maximum points of blowing-up solutions $\{u_\varepsilon\}$ to this problem via a blow-up analysis. Note that in this case, it holds $\|u_\varepsilon\|_{L^\infty(\Omega)} \to +\infty$ as $\varepsilon \to 0$. Theorems in this paper are two-dimensional counterparts to the results in [4]. However, in our situation, solutions may not blow up in the $L^\infty$-norm sense, so the usual blow-up analysis as in the higher-dimensional case does not work. To overcome this difficulty, we combine the arguments in [4] and the two-dimensional blow-up technique by Adimurthi and Grossi [1].

# 2. Proof of Theorem 1.

Let $x_p^1 \in \Omega$ be a maximum point of $u_p$ for $p$ large, that is, $\|u_p\|_\infty = u_p(x_p^1)$. First, we recall a result in [6] that for any solution $u_p$ of (1.1), there holds an estimate

$$\|u_p\|_\infty \geq \lambda_1^{1/(p-1)}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ under the Dirichlet boundary condition. Proof of this fact in [6] is simple, so we recall it here for the readers' convenience. Let $\phi_1 > 0$ be the first eigenfunction associated to $\lambda_1$.

Then we see

$$0 = \int_\Omega (u_p \Delta \phi_1 - \phi_1 \Delta u_p) dx = \int_\Omega \phi_1 u_p (u_p^{p-1} - \lambda_1) dx,$$

thus we have $\|u_p\|_\infty^{p-1} \geq \lambda_1$.

From this fact, we see

$$\varepsilon_p := \frac{1}{\sqrt{p}\|u_p\|_\infty^{(p-1)/2}} \to 0 \quad (p \to \infty). \tag{2.1}$$

Next, we claim that $x_p^1$ is away from $\partial\Omega$ uniformly in $p$ sufficiently large. Indeed, (1.1) and (1.2) imply that $p \int_\Omega u_p^{p+1} dx = O(1)$ as $p \to \infty$. Then we have

$$p \int_\Omega u_p^p dx \leq \left( p \int_\Omega u_p^{p+1} dx \right)^{p/(p+1)} p^{1/(p+1)} |\Omega|^{1/(p+1)} \leq C$$

uniformly in $p$, thus $\int_\Omega u_p^p dx = O(1/p)$ as $p \to \infty$. Therefore, at any maximum point $x_p$ of $u_p$, $v_p(x) := u_p(x) / \int_\Omega u_p^p dx$ satisfies

$$v_p(x_p) = \frac{u_p(x_p)}{\int_\Omega u_p^p dx} \geq \frac{\lambda_1^{1/(p-1)}}{\int_\Omega u_p^p dx} \to +\infty$$

as $p \to \infty$. On the other hand, applying Lemma 4.1 in [6] and the elliptic $L^1$ estimate to $v_p$, we have, as in Lemma 4.2 in [6], that $v_p$ is bounded in $L^\infty$ near $\partial\Omega$ uniformly in $p$. Thus any maximum point $x_p$ of $u_p$ cannot approach to $\partial\Omega$ and for some neighborhood $\omega$ of $\partial\Omega$, we have $\{x_p \in \Omega | u_p(x_p) = \|u_p\|_\infty\} \subset \Omega \setminus \omega$ for large $p$.

Now, we define the scaled function

$$\tilde{u}_p(y) = \frac{p}{\|u_p\|_\infty} \left\{ u_p \left( \varepsilon_p y + x_p^1 \right) - u_p(x_p^1) \right\}, \quad y \in \Omega_p^1 = \frac{\Omega - x_p^1}{\varepsilon_p} \tag{2.2}$$

as in [1], which satisfies

$$\begin{cases} -\Delta \tilde{u}_p(y) = \left( 1 + \frac{\tilde{u}_p}{p}(y) \right)^p & \text{in } \Omega_p^1, \\ 0 < 1 + \frac{\tilde{u}_p}{p}(y) \leq 1 & \text{in } \Omega_p^1, \\ \tilde{u}_p(y) = -p & y \in \partial\Omega_p^1. \end{cases} \tag{2.3}$$

By the above claim, the limit domain of $\Omega_p^1$ as $p \to \infty$ is $\mathbb{R}^2$. As in [1] p.1015, we can pass to the limit in (2.3) to obtain some function $U \in C^2(\mathbb{R}^2)$ such that $\tilde{u}_p \to U$ as $p \to \infty$ in $C^2_{loc}(\mathbb{R}^2)$. Passing to the limit in (2.3), we see that $U$ satisfies

$$-\Delta U = e^U \quad \text{in } \mathbb{R}^2, \quad \max_{y \in \mathbb{R}^2} U(y) = U(0) = 0.$$

Moreover, by the assumption (1.2) and Fatou's lemma, we can check that

$$\int_{\mathbb{R}^2} e^U dy < +\infty. \tag{2.4}$$

In fact, since $\tilde{u}_p(y) \to U(y)$ a.e. $y \in \mathbb{R}^2$, we see

$$p \log\left(1 + \frac{\tilde{u}_p}{p}(y)\right) \to U(y) \quad a.e. y \in \mathbb{R}^2$$

and

$$e^{p \log\left(1 + \frac{\tilde{u}_p}{p}(y)\right)} \to e^{U(y)} \quad a.e. y \in \mathbb{R}^2.$$

Thus Fatou's lemma and a simple change of variables using (2.2) imply that

$$\begin{aligned}\int_{\mathbb{R}^2} e^U dy &\leq \liminf_{p\to\infty} \int_{\Omega_p^1} \left(1 + \frac{\tilde{u}_p}{p}(y)\right)^p dy \\ &= \liminf_{p\to\infty} \int_\Omega \left(\frac{u_p(x)}{\|u_p\|_\infty}\right)^p p\|u_p\|_\infty^{p-1} dx \leq (1/C) \liminf_{p\to\infty} p \int_\Omega u_p^p dx,\end{aligned}$$

where we have used the fact $\|u_p\|_\infty \geq C > 0$ uniformly for $p$ large. The last term is bounded as before thanks to (1.2), so we get (2.4).

At this point, we see by a result of Chen and Li [2] that

$$U(y) = -2 \log\left(1 + \frac{|y|^2}{8}\right). \tag{2.5}$$

Next, we define two elliptic operators

$$L_p := -\Delta_x - p u_p^{p-1}(x) \cdot : H_0^1(\Omega) \to H^{-1}(\Omega), \tag{2.6}$$

$$\tilde{L}_p := -\Delta_y - \left(1 + \frac{\tilde{u}_p}{p}(y)\right)^{p-1} \cdot : H_0^1(\Omega_p^1) \to H^{-1}(\Omega_p^1). \tag{2.7}$$

Note that the operators (2.6) and (2.7) are related to each other by a simple scaling

$$\varepsilon_p^2 L_p \Big|_{u_p(x)=\|u_p\|_\infty \left(1+\frac{\tilde{u}_p}{p}(y)\right)} = \tilde{L}_p,$$

here, $x = \varepsilon_p y + x_p^1$ for $x \in \Omega$ and $y \in \Omega_p^1$. Further if we write the $j$-th eigenvalue of an elliptic operator $L$ acting on $H_0^1(D)$ for a bounded domain $D$ as $\lambda_j(L, D), j \in \mathbb{N}$, then

$$\varepsilon_p^2 \lambda_j(L_p, D) = \lambda_j(\tilde{L}_p, D_p), \quad D_p = \frac{D - x_p^1}{\varepsilon_p}. \tag{2.8}$$

Now, we claim that there exists $R > 0$ such that $\lambda_1(L_p, B(x_p^1, \varepsilon_p R)) < 0$ for $p$ sufficiently large.

Indeed, define

$$w_p(y) = y \cdot \nabla \tilde{u}_p(y) + \frac{2p}{p-1}\left(1 + \frac{\tilde{u}_p}{p}(y)\right), \quad y \in \Omega_p^1. \tag{2.9}$$

Then we have $-\Delta w_p(y) = \left(1 + \frac{\tilde{u}_p(y)}{p}\right)^{p-1} w_p(y)$ for $y \in \Omega_p^1$. Note that $w_p(0) = 2p/(p-1) \to 2$. Also, since $\tilde{u}_p \to U$ in $C^2_{loc}(\mathbb{R}^2)$, we have $w_p(y) \to 2(8 - |y|^2)/(8 + |y|^2) < 0$ if $|y| = R > 2\sqrt{2}$ as $p \to \infty$. Now, set $A_p = \{y \in B(0, R) : w_p(y) > 0\}$, $A_p \neq \phi$, and

$$\overline{w}_p(y) = \begin{cases} w_p(y) & y \in A_p, \\ 0 & y \in B(0, R) \setminus \overline{A_p}. \end{cases}$$

Testing

$$\lambda_1(\tilde{L}_p, B(0, R)) = \inf_{v \in H_0^1(B(0,R))} \frac{\int_{B(0,R)} |\nabla v|^2 dy - \int_{B(0,R)} \left(1 + \frac{\tilde{u}_p}{p}(y)\right)^{p-1} v^2 dy}{\int_{B(0,R)} v^2 dy}$$

by $\overline{w}_p \in H_0^1(B(0, R))$, we see that $\lambda_1(\tilde{L}_p, B(0, R)) \le 0$. Strict inequality $\lambda_1(\tilde{L}_p, B(0, R)) < 0$ actually follows since if equality holds, $\overline{w}_p$ would be the first eigenfunction of $\tilde{L}_p$ on $B(0, R)$, so it must be strictly positive on $B(0, R)$, which contradicts to the fact that $\overline{w}_p$ is 0 near $\partial B(0, R)$. By scaling (2.8), we prove the claim.

If there is another maximum point $x_p^2 \neq x_p^1$, we repeat the same procedure as before to obtain the ball $B(x_p^2, \varepsilon_p R)$ such that $\lambda_1(L_p, B(x_p^2, \varepsilon_p R)) < 0$.

We claim that $B(x_p^1, \varepsilon_p R)$ and $B(x_p^2, \varepsilon_p R)$ are disjoint for $p$ large. Indeed, since $\tilde{u}_p \to U$ in $C^2_{loc}(\mathbb{R}^2)$, $U(y) = -2\log(1 + |y|^2/8)$ is strictly concave and $y \cdot \nabla\tilde{u}_p(y) \to y \cdot \nabla U(y) < 0$ on $B(0, R) \setminus \{0\}$, we see that $u_p$ is also strictly concave and $(x - x_p^1) \cdot \nabla u_p(x) < 0$ on $B(x_p^1, \varepsilon_p R) \setminus \{x_p^1\}$ for $p$ large. The same property holds for $u_p$ on $B(x_p^2, \varepsilon_p R) \setminus \{x_p^2\}$, and this concavity property means the claim.

Now, if there are $N$ maximum points $x_p^1, \cdots, x_p^N$ of $u_p$, we have $N$ open balls $B^1, \cdots, B^N$, $B^j = B(x_p^j, \varepsilon_p R)$, which are disjoint, and

$$\lambda_1(L_p, B^j) < 0 \quad \text{for } j = 1, \cdots, N. \tag{2.10}$$

By a variational characterization of $N$-th eigenvalue of $L_p$ and the well-known argument as in a proof of Courant's Nodal Domain Theorem, we see that

$$\lambda_N(L_p, \Omega) \leq \sum_{j=1}^{N} \lambda_1(L_p, B^j). \tag{2.11}$$

From (2.10) and (2.11), we have $\lambda_N(L_p, \Omega) < 0$. On the other hand, the Morse index of $u_p$ is less than or equal to $m$ by assumption, we have $\lambda_{m+1}(L_p, \Omega) \geq 0$. Therefore we must have $N \leq m$, and we have proved Theorem 1. □

# 3. Proof of Theorem 2.

Assume $u_p$ is a solution of (1.1) of Morse index one with the property (1.2) as $p \to \infty$. By Theorem 1, we know there exists only one maximum point $x_p = x_p^1$ of $u_p$ and on the ball $B = B(x_p, \varepsilon_p R) \subset \Omega$,

$$\lambda_1(L_p, B) < 0 \quad \text{and } (x - x_p) \cdot \nabla u_p(x) < 0 \quad (\forall x \in B \setminus \{x_p\}).$$

Now, we claim that $(x - x_p) \cdot \nabla u_p(x) < 0$ for all $x \in \Omega \setminus \overline{B}$. Indeed, assume there exists some $\overline{x} \in \Omega \setminus B$ such that $(\overline{x} - x_p) \cdot \nabla u_p(\overline{x}) \geq 0$. By the variational characterization of the second eigenvalue, we have

$$\lambda_2(L_p, \Omega) \leq \lambda_1(L_p, B) + \lambda_1(L_p, \Omega \setminus \overline{B}).$$

Since $u_p$ is a solution of Morse index one, we have $\lambda_2(L_p, \Omega) \geq 0$, and thus $\lambda_1(L_p, \Omega \setminus \overline{B}) > 0$ for $p$ large. On the other hand, by a scaling $\overline{x} = \varepsilon_p \overline{y} + x_p$ and $\nabla_y = \varepsilon_p \nabla_x$, we have $\overline{y} \cdot \nabla\tilde{u}_p(\overline{y}) \geq 0$. By the convexity of $\Omega$, the scaled domain

$\Omega_p = (\Omega - x_p)/\varepsilon_p$ is star-shaped with respect to 0. Hence the Hopf lemma implies that $w_p$ in (2.9) satisfies $w_p(y) < 0$ for $y \in \partial\Omega_p$. Since $w_p(\overline{y}) > 0$, there would exist a connected component $C_p \subset \Omega_p \setminus \overline{B}(0, R), C_p \cap \partial\Omega_p = \phi$ such that $w_p > 0$ on $C_p$. Then we would have that $\lambda_1(\tilde{L}_p, C_p) \leq 0$, and by (2.8), $\lambda_1(L_p, \varepsilon_p(C_p + \{x_p\})) \leq 0$. Note that $\varepsilon_p(C_p + \{x_p\}) \subset \Omega \setminus \overline{B}$. This contradicts to the monotonicity $0 < \lambda_1(L_p, \Omega \setminus \overline{B}) \leq \lambda_1(L_p, \varepsilon_p(C_p + \{x_p\}))$. Thus we have proved $(x - x_p) \cdot \nabla u_p(x) < 0$ for all $x \in \Omega \setminus \{x_p\}$. The rest of the statement in Theorem 2 is a simple consequence of this inequality. □

# References


[1] Adimurthi, and M. Grossi, *Asymptotic estimates for a two-dimensional problem with polynomial nonlinearity,* Proc. A.M.S. **132**(4) (2003) 1013-1019.

[2] E. Chen, and C. Li, *Classification of solutions of some nonlinear elliptic equations,* Duke Math. J. **63**(3) (1991) 615-622.

[3] K. El Mehdi, and M. Grossi. *Asymptotic estimates and qualitative properties of an elliptic problem in dimension two,* Advances in Nonlinear Stud. **4**(1) (2004) 15-36.

[4] K. El Mehdi, and F. Pacella. *Morse index and blow-up points of solutions of some nonlinear problems,* Atti. Accad. Naz. Lincei Mat. Appl. **13** (2002) 101-105.

[5] P. Esposito, M. Musso and A. Pistoia. *Concentrating solutions for a planar elliptic problem involving nonlinearities with large exponent,* Journal of Differential Eq. **227**(1) (2006) 29-68.

[6] X. Ren, and J. Wei, *On a two-dimensional elliptic problem with large exponent in nonlinearlity,* Trans. A.M.S. **343**(2) (1994) 749-763.

[7] X. Ren, and J. Wei, *Single-point condensation and least-energy solutions,* Proc. A.M.S. **124**(1) (1996) 111-120.