Nondegeneracy of least energy solutions for an elliptic problem with nearly critical nonlinearity

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Abstract

Consider the problem $-\Delta u = c_0 K(x) u^{p_\varepsilon}$, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 3)$, $c_0 = N(N-2), p_\varepsilon = (N+2)/(N-2) - \varepsilon$ and K is a smooth positive function on $\overline{\Omega}$.

We prove that least energy solutions of the above problem are nondegenerate for $\varepsilon > 0$ small, under some assumptions on the coefficient function K. This is a generalization of the recent result by Grossi [6] for $K \equiv 1$, and needs precise estimates and a new argument.

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1 Introduction

We consider the problem

$$(P_{\varepsilon,K})\begin{cases}-\Delta u = c_0 K(x) u^{p_\varepsilon} & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega\end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $c_0 = N(N-2)$, $p_\varepsilon = p - \varepsilon$, $p = (N+2)/(N-2)$ is the critical Sobolev exponent with respect to the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and $\varepsilon > 0$ is a small parameter. Here, K is a positive function in $C^2(\overline{\Omega})$.

Since $(P_{\varepsilon,K})$ is a subcritical problem, there exists a least energy solution u_ε such that

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\left(\int_\Omega K(x)|u_\varepsilon|^{p_\varepsilon+1}dx\right)^{\frac{2}{p_\varepsilon+1}}} = \inf_{u\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega K(x)|u|^{p_\varepsilon+1}dx\right)^{\frac{2}{p_\varepsilon+1}}} =: S_{\varepsilon,K}$$

for any small $\varepsilon > 0$.

In the following, we put an assumption on the coefficient function K.

(K): $K \in C^2(\overline{\Omega}), 0 < K(x) \leq 1, K^{-1}(\max_{\overline{\Omega}} K) = \{x_0\} \subset \Omega$ with $K(x_0) = 1$, and x_0 is a nondegenerate critical point of K.

It is easy to see that

$$S_{\varepsilon,K} \to (\max_{\overline{\Omega}} K)^{-(N-2)/N} S_N = S_N$$

as $\varepsilon \to 0$, where S_N is the best Sobolev constant with respect to the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. Since S_N is never achieved on bounded domains, we have $\|u_\varepsilon\|_{L^\infty(\Omega)} = u_\varepsilon(x_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. It is also known that the maximum point x_ε of u_ε converges to a maximum point of K in $\overline{\Omega}$. Thus under the assumption (K), we have $x_\varepsilon \to x_0$ as $\varepsilon \to 0$ where x_0 is the unique interior maximum point of K.

In this paper, we prove a nondegeneracy property of least energy solutions to $(P_{\varepsilon,K})$ when $\varepsilon > 0$ is sufficiently small, under the assumption (K).

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a smooth bounded domain. Assume (K). Then the least energy solution u_ε to $(P_{\varepsilon,K})$ is nondegenerate for $\varepsilon > 0$ small, in the sense that the linearized problem around u_ε:*

$$(L_{\varepsilon,K}) \begin{cases} -\Delta v = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

admits only the trivial solution $v \equiv 0$.

When $K \equiv 1$, Grossi [6] obtained a nondegeneracy result for solutions $\{u_\varepsilon\}$ to $(P_{\varepsilon,K})$ satisfying

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\left(\int_\Omega |u_\varepsilon|^{p_\varepsilon+1} dx\right)^{\frac{2}{p_\varepsilon+1}}} \to S_N \quad \text{as } \varepsilon \to 0. \tag{1.1}$$

It is known ([8], [10]) that a solution sequence with the property (1.1) blows up at one point in the domain, and the blow up point is a critical point of the Robin function associated to the Dirichlet Green function. Under the assumption that the blow up point is a nondegenerate critical point of the Robin function, Grossi obtained the nondegeneracy of solutions satisfying (1.1) for $\varepsilon > 0$ small. This result was former obtained in [1] when $N \geq 4$ by another method. More recently, Grossi and Pacella [7] studied the linearized eigenvalue problem associated with the blow up solutions satisfying (1.1), and obtained the same nondegeneracy result, again when $K \equiv 1$.

When $K \not\equiv 1$, corresponding results to [1] or [7] are still not known. Main purpose of this paper is to generalize the result in [6] to the inhomogeneous case $K \not\equiv 1$. In this case, Hebey [9] obtained the precise asymptotic behavior of least energy solutions as $\varepsilon \to 0$ under a stronger assumption than (K). We remark here that the same asymptotic result as [9] can be obtained under the assumption (K) by using the local blow up analysis of YanYan Li [11]; see Remark after Theorem 2.3.

Based on Hebey's result, we prove Theorem 1.1 with a new argument. Note that, even in the case $K \equiv 1$, our argument simplifies the proof in [6].

2 Preliminaries

In this section, we prepare some facts which are needed in the sequel. Let $G = G(x, z)$ denote the Green function of $-\Delta$ under the Dirichlet boundary condition:

$$\begin{cases} -\Delta G(\cdot, z) = \delta_z & \text{in } \Omega, \\ G(\cdot, z) = 0 & \text{on } \partial\Omega. \end{cases}$$

Define the (positive) Robin function R associated to the Green function

$$R(z) = \lim_{x \to z} \left[\frac{1}{(N-2)\sigma_N} |x - z|^{2-N} - G(x, z) \right],$$

here σ_N denotes the volume of the unit sphere in $\mathbb{R}^N$.

Lemma 2.1 *The identities*

$$\int_{\partial\Omega} ((x - z) \cdot \nu_x) \left(\frac{\partial G(x, z)}{\partial \nu_x} \right)^2 ds_x = (N - 2)R(z) \tag{2.1}$$

and

$$\int_{\partial\Omega} \left(\frac{\partial G}{\partial x_i} \right) \frac{\partial}{\partial \nu_x} \left(\frac{\partial G}{\partial z_j} \right) (x, z) ds_x = \frac{1}{2} \frac{\partial^2 R}{\partial z_i \partial z_j}(z) \tag{2.2}$$

hold true for any $z \in \Omega$. Here, ν_x is the outer unit normal at $x \in \partial\Omega$.

Proof: See [2]:Theorem 4.3 for (2.1) and [6]:Lemma 3.2 for (2.2). □

Lemma 2.2 *Let u_ε be a solution to $(P_{\varepsilon,K})$ and v_ε be a solution to $(L_{\varepsilon,K})$. Then the following identities hold true:*

$$\int_{\partial\Omega} ((x - z) \cdot \nu_x) \left(\frac{\partial u_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial v_\varepsilon}{\partial \nu_x} \right) ds_x = c_0 \int_\Omega u_\varepsilon^{p_\varepsilon} v_\varepsilon \left((x - z) \cdot \nabla K(x) \right) dx \tag{2.3}$$

for any $z \in \mathbb{R}^N$ and

$$\int_{\partial\Omega} \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) \left(\frac{\partial v_\varepsilon}{\partial \nu_x} \right) ds_x = c_0 \int_\Omega \left(\frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon} v_\varepsilon dx, \quad i = 1, 2, \cdots, N. \tag{2.4}$$

Proof: Set $w_\varepsilon(x) = (x - z) \cdot \nabla u_\varepsilon(x) + \frac{2}{p_\varepsilon - 1} u_\varepsilon(x)$. Direct computation yields that

$$-\Delta w_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} w_\varepsilon + c_0 u_\varepsilon^{p_\varepsilon} (x - z) \cdot \nabla K(x).$$

Since v_ε satisfies $-\Delta v_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} v_\varepsilon$, we have

$$(\Delta v_\varepsilon) w_\varepsilon - (\Delta w_\varepsilon) v_\varepsilon = c_0 u_\varepsilon^{p_\varepsilon} v_\varepsilon (x - z) \cdot \nabla K(x).$$

Integrating this identity on Ω, using integration by parts and noting $w_\varepsilon(x) = (x - z) \cdot \nu_x (\frac{\partial u_\varepsilon}{\partial \nu_x})$ for $x \in \partial\Omega$, we have (2.3).

On the other hand, differentiating the equation in $(P_{\varepsilon,K})$ with respect to x_i, we have

$$-\Delta \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) + c_0 \left(\frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon}.$$

From this equation and the equation in $(L_{\varepsilon,K})$, we obtain

$$(\Delta v_\varepsilon) \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) - \left(\Delta \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) \right) v_\varepsilon = c_0 \left(\frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon} v_\varepsilon.$$

Finally, integration by parts yields (2.4). □

Next is a part of the main theorem of [9], with a result of [11]. In what follows, we abbreviate $\|\cdot\| = \|\cdot\|_{L^\infty(\Omega)}$.

Theorem 2.3 *(Hebey [9]) Let $\Omega \subset \mathbb{R}^N, N \geq 3$ be a smooth bounded domain. Assume (K). Let u_ε be a least energy solution to $(P_{\varepsilon,K})$ and let $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|$.*

Then there exists a constant $C > 0$ independent of ε such that for any $R_\varepsilon \to \infty$ with $r_\varepsilon = R_\varepsilon \|u_\varepsilon\|^{-(\frac{p_\varepsilon - 1}{2})} \to 0$, the following estimates hold true:

$$u_\varepsilon(x) \leq \begin{cases} \dfrac{\|u_\varepsilon\|}{\left(1 + \|u_\varepsilon\|^{\frac{4}{N-2}} |x - x_\varepsilon|^2\right)^{\frac{N-2}{2}}}, & \text{for } |x - x_\varepsilon| \leq r_\varepsilon, \\ \dfrac{C}{\|u_\varepsilon\|} \dfrac{1}{|x - x_\varepsilon|^{N-2}}, & \text{for } \{|x - x_\varepsilon| > r_\varepsilon\} \cap \Omega. \end{cases} \tag{2.5}$$

Furthermore, after passing to a subsequence, we have

$$\begin{cases} |x_\varepsilon - x_0| = O\left(\|u_\varepsilon\|^{-2}\right) & N = 3, \\ |x_\varepsilon - x_0| = o\left(\|u_\varepsilon\|^{-2/(N-2)}\right) & N \geq 4, \end{cases} \tag{2.6}$$

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|^\varepsilon = 1 \tag{2.7}$$

and

$$\|u_\varepsilon\| u_\varepsilon \to (N-2)\sigma_N G(\cdot, x_0) \quad \text{in } C^1(\omega) \tag{2.8}$$

for any neighborhood ω of $\partial\Omega$ not containing x_0.

Remark: Hebey assumed in [9] that Ω is star-shaped with respect to a point x_0 in Ω, $K \in C^2(\overline{\Omega}), 0 < K(x) \leq 1, K(x_0) = 1$ and $(x - x_0) \cdot \nabla K(x) \leq 0$ for any $x \in \Omega$. However, crucial pointwise estimate (2.5) can be obtained by the theory of *isolated simple blow up point* due to YanYan Li [11], since in our case, the blow up point of least energy solutions has to be a unique interior maximum point of K, and thus to be an isolated simple blow up point in the sense of [11]. Once the crucial pointwise estimate (2.5) is obtained, the rest of the proof in Hebey [9] is still valid under the assumption (K).

Now, let us consider the scaled function

$$\tilde{u}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} u_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + x_\varepsilon \right), \quad y \in \Omega_\varepsilon := \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}} (\Omega - x_\varepsilon). \tag{2.9}$$

Then $0 < \tilde{u}_\varepsilon \leq 1, \tilde{u}_\varepsilon(0) = 1$, and $\tilde{u}_\varepsilon$ satisfies

$$\begin{cases} -\Delta \tilde{u}_\varepsilon = c_0 K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon} & \text{in } \Omega_\varepsilon, \\ \tilde{u}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

where $K_\varepsilon(y) = K \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon \right)$. Since $\|u_\varepsilon\| \to \infty$ and $x_\varepsilon \to x_0 \in \Omega$, we see $\Omega_\varepsilon \to \mathbb{R}^N$ and $K_\varepsilon \to K(x_0) = 1$ compact uniformly on $\mathbb{R}^N$ as $\varepsilon \to 0$. By standard elliptic estimates, we have a subsequence denoted also by $\tilde{u}_\varepsilon$ that

$$\tilde{u}_\varepsilon \to U \quad \text{compact uniformly in } \mathbb{R}^N \tag{2.10}$$

for some function U. Passing to the limit, we obtain that U is a solution of

$$\begin{cases} -\Delta U = c_0 U^p \quad \text{in } \mathbb{R}^N, \\ 0 < U \leq 1, \ U(0) = 1, \\ \lim_{|y| \to \infty} U(y) = 0. \end{cases}$$

According to the uniqueness theorem by Caffarelli, Gidas and Spruck [4], we obtain

$$U(y) = \left(\frac{1}{1 + |y|^2} \right)^{\frac{N-2}{2}}.$$

Note that by (2.7), the estimate (2.5) is written as

$$\tilde{u}_\varepsilon(y) \leq \begin{cases} CU(y) & \text{for } \{|y| \leq R_\varepsilon\} \cap \Omega_\varepsilon, \\ C|y|^{2-N} & \text{for } \{|y| > R_\varepsilon\} \cap \Omega_\varepsilon. \end{cases} \tag{2.11}$$

We recall here the classification theorem by Bianchi and Egnell [3].

Lemma 2.4 *Let v_0 be a solution to*

$$\begin{cases} -\Delta v_0 = c_0 p U^{p-1} v_0 & in\ \mathbb{R}^N, \\ v_0 \in D^{1,2}(\mathbb{R}^N) \end{cases}$$

where $D^{1,2}(\mathbb{R}^N) = \{v \in L^{2N/(N-2)}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |\nabla v|^2 dy < \infty\}$.

Then there exist constants a_j $(j = 1, 2, \cdots, N)$ and b in $\mathbb{R}$ such that v_0 has the form

$$v_0 = \sum_{j=1}^{N} a_j \frac{y_j}{(1+|y|^2)^{N/2}} + b\frac{1-|y|^2}{(1+|y|^2)^{N/2}}. \tag{2.12}$$

Next lemma concerns a well-known unique solvability for the linear first order PDE's with the initial condition. The proof will be done by the standard method of characteristics, so we omit it.

Lemma 2.5 *Let $a = (a_1, a_2, \cdots, a_N) \neq 0$ is a constant vector and $f, g \in C^1(\mathbb{R}^N)$. Let $\Gamma_a = \{x \in \mathbb{R}^N | a \cdot x = 0\}$ be the hyperplane perpendicular to a through the origin. Then there exists a unique solution of the following initial value problem of the linear first order PDE*

$$\begin{aligned} a \cdot \nabla u &= f, \\ u|_{\Gamma_a} &= g. \end{aligned}$$

More precisely, this solution is obtained as

$$u(x) = \int_0^{\phi(x)} f(\tau a + \alpha(\psi(x)))d\tau + g(\alpha(\psi(x))), \quad x \in \mathbb{R}^N$$

where

$$\begin{aligned} \phi(x) &= \frac{a \cdot x}{|a|^2}, \quad \psi(x) = (\psi_1(x), \cdots, \psi_{N-1}(x)), \\ \psi_j(x) &= \frac{|a|^2 x_j - (a \cdot x)a_j}{|a|^2}, (j = 1, \cdots, N-1) \\ \alpha(s) &= (s, -\frac{1}{a_N}\sum_{j=1}^{N-1} a_j s_j) \in \mathbb{R}^N, \quad s = (s_1, \cdots, s_{N-1}) \in \mathbb{R}^{N-1}, \end{aligned}$$

if we assume (w.l.o.g) $a_N \neq 0$. Furthermore, if $f(x) = O(|x|^\beta), g(x) = O(|x|^\beta)$ as $|x| \to \infty$, then $u(x) = O(|x|^{\beta+1})$ as $|x| \to \infty$.

3 The nondegeneracy result

In this section, we will prove Theorem 1.1. In the course of proof, we need precise estimates and a new argument which are not in [6].

We argue by contradiction and assume that there exists a non-trivial solution v_ε to $(L_{\varepsilon,K})$ satisfying $\|v_\varepsilon\| = \|u_\varepsilon\|$ for any $\varepsilon > 0$, without loss of generality.

Let us consider the scaled function

$$\tilde{v}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} v_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon\right), \quad y \in \Omega_\varepsilon = \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}(\Omega - x_\varepsilon). \tag{3.1}$$

Then $0 < \tilde{v}_\varepsilon \leq 1$ and $\tilde{v}_\varepsilon$ satisfies

$$\begin{cases} -\Delta \tilde{v}_\varepsilon = c_\varepsilon(y)\tilde{v}_\varepsilon & \text{in } \Omega_\varepsilon, \\ \tilde{v}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \\ \|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1 \end{cases} \tag{3.2}$$

where $c_\varepsilon(y) := c_0 p_\varepsilon K_\varepsilon(y)\tilde{u}_\varepsilon^{p_\varepsilon-1}(y)$. By $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$ and the elliptic estimate, we see there exists v_0 such that

$$\tilde{v}_\varepsilon \to v_0 \quad \text{uniformly on compact subsets of } \mathbb{R}^N \tag{3.3}$$

and v_0 satisfies

$$-\Delta v_0 = c_0 p U^{p-1} v_0 \quad \text{in } \mathbb{R}^N.$$

Now, we claim that

$$\int_{\mathbb{R}^N} |\nabla v_0|^2 dy \leq C \tag{3.4}$$

for some $C > 0$.

Indeed, let $0 < \delta < \min(2, 4/(N-2) - 2\varepsilon)$. By (3.2) and the Sobolev inequality, we have

$$S_N \left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy\right)^{2/(p+1)} \leq \int_{\Omega_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2 dy = \int_{\Omega_\varepsilon} c_\varepsilon(y)\tilde{v}_\varepsilon^2 dy \leq \int_{\Omega_\varepsilon} |c_\varepsilon(y)||\tilde{v}_\varepsilon|^{2-\delta} dy,$$

here, the last inequality comes from the fact that $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq 1$. Now, by the Hölder inequality and (2.11), we have

$$\begin{aligned} \int_{\Omega_\varepsilon} |c_\varepsilon(y)||\tilde{v}_\varepsilon|^{2-\delta} dy &\leq \left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy\right)^{(2-\delta)/(p+1)} \left(\int_{\Omega_\varepsilon} |c_\varepsilon(y)|^{(p+1)/(p-1+\delta)} dy\right)^{(p-1+\delta)/(p+1)} \\ &\leq C \left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy\right)^{(2-\delta)/(p+1)} \left(\int_{\Omega_\varepsilon} U(y)^{(p_\varepsilon-1)(p+1)/(p-1+\delta)} dy\right)^{(p-1+\delta)/(p+1)}, \end{aligned}$$

thus we obtain

$$\left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy\right)^{\delta/(p+1)} \le C \left(\int_{\mathbb{R}^N} U(y)^{(p_\varepsilon-1)(p+1)/(p-1+\delta)} dy\right)^{(p-1+\delta)/(p+1)}.$$

Note that $(N-2)(p_\varepsilon - 1)(p+1)/(p-1+\delta) > N$ if $\delta < 4/(N-2) - 2\varepsilon$, so the last integral is bounded by a constant. Therefore, we have

$$\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \le C. \tag{3.5}$$

Finally, again by the Hölder inequality, (3.5) and (2.11), we have

$$\begin{aligned}
&\int_{\Omega_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2 dy \le \int_{\Omega_\varepsilon} c_\varepsilon(y)|\tilde{v}_\varepsilon|^{2-\delta} dy \\
&\le \left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy\right)^{(2-\delta)/(p+1)} \left(\int_{\Omega_\varepsilon} |c_\varepsilon(y)|^{(p+1)/(p-1+\delta)} dy\right)^{(p-1+\delta)/(p+1)} \\
&\le C.
\end{aligned}$$

Thus we have confirmed

$$\int_{\Omega_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2 dy \le C \tag{3.6}$$

for some $C > 0$ independent of $\varepsilon > 0$. (3.6) and Fatou's lemma implies (3.4).

Now, by (3.4) and Lemma 2.4, we have (2.12), i.e.

$$v_0 = \sum_{j=1}^{N} a_j \frac{y_j}{(1+|y|^2)^{N/2}} + b\frac{1-|y|^2}{(1+|y|^2)^{N/2}}. \tag{3.7}$$

In the following, we divide the proof into several steps.

Step 1. $b = 0$ in (3.7).

Step 2. $a_j = 0, j = 1, \cdots, N$ in (3.7).

Step 3. $v_0 = 0$ leads to a contradiction.

In the sequel, we need the following pointwise estimate for $\tilde{v}_\varepsilon$.

Lemma 3.1 *Let $\tilde{v}_\varepsilon$ be a solution of (3.2). Then we have the estimate*

$$|\tilde{v}_\varepsilon(y)| \leq C\left(\frac{1}{1+|y|^2}\right)^{(N-2)/2}, \quad \forall y \in \Omega_\varepsilon \tag{3.8}$$

for some $C > 0$.

Proof: Consider the Kelvin transformation of $\tilde{v}_\varepsilon$:

$$\tilde{v}^*_\varepsilon(z) = |z|^{2-N}\tilde{v}_\varepsilon(\frac{z}{|z|^2}), \quad z \in \Omega^*_\varepsilon := \{\frac{y}{|y|^2} : y \in \Omega_\varepsilon\}. \tag{3.9}$$

To prove (3.8), it will be enough to show that $|\tilde{v}^*_\varepsilon|$ is bounded in $B(0,R) \cap \Omega^*_\varepsilon$ for some $R > 0$, since $|\tilde{v}_\varepsilon(y)| \leq 1$ for $y \in \Omega_\varepsilon$, $|y|$ small. Direct calculation shows that

$$\Delta_z \tilde{v}^*_\varepsilon(z) = |z|^{-2-N}\Delta_y \tilde{v}_\varepsilon(y), \quad z \in \Omega^*_\varepsilon,$$
$$\int_{\Omega^*_\varepsilon} |\tilde{v}^*_\varepsilon|^{p+1} dz = \int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy.$$

Thus by (3.2), $\tilde{v}^*_\varepsilon$ satisfies the equation

$$\begin{cases} -\Delta \tilde{v}^*_\varepsilon &= |z|^{-4} c_\varepsilon(\frac{z}{|z|^2})\tilde{v}^*_\varepsilon \quad \text{in } \Omega^*_\varepsilon, \\ \tilde{v}^*_\varepsilon &= 0 \quad \text{on } \partial\Omega^*_\varepsilon. \end{cases} \tag{3.10}$$

We claim that

$$a_\varepsilon(z) := |z|^{-4} c_\varepsilon(\frac{z}{|z|^2}) \in L^\infty(\Omega^*_\varepsilon). \tag{3.11}$$

Indeed, since $\Omega_\varepsilon \subset B(0, \gamma\|u_\varepsilon\|^{(p_\varepsilon-1)/2})$ for some $\gamma > 0$, the domain Ω^*_ε satisfies $\Omega^*_\varepsilon \subset \mathbb{R}^N \setminus B(0, \frac{1}{\gamma\|u_\varepsilon\|^{(p_\varepsilon-1)/2}})$. By (2.11), we have

$$|c_\varepsilon(y)| \leq CU^{p_\varepsilon-1}(y) \quad \text{for } y \in \Omega_\varepsilon.$$

Therefore, we have

$$\begin{aligned} |z|^{-4}c_\varepsilon(\frac{z}{|z|^2}) &\leq C|z|^{-4}\left(\frac{|z|^2}{1+|z|^2}\right)^{(\frac{N-2}{2})(p_\varepsilon-1)} \\ &= C|z|^{-4+(N-2)(p_\varepsilon-1)}\frac{1}{(1+|z|^2)^{2-\varepsilon(\frac{N-2}{2})}} \\ &\leq C|z|^{-4+(N-2)(p_\varepsilon-1)} = C|z|^{-\varepsilon(N-2)} \end{aligned}$$

Since $|z| \geq \frac{1}{\gamma \|u_\varepsilon\|^{(p_\varepsilon-1)/2}}$ for $z \in \Omega_\varepsilon^*$, we have

$$|z|^{-\varepsilon(N-2)} \leq \gamma^{\varepsilon(N-2)} \|u_\varepsilon\|^{\varepsilon(N-2)(p_\varepsilon-1)/2} \to 1$$

as $\varepsilon \to 0$ by (2.7). From these, we confirm that the claim (3.11).

Now, for any $R > 0$, we have

$$\begin{aligned}\int_{\Omega_\varepsilon^* \cap B(0,2R)} |\tilde{v}_\varepsilon^*|^{p+1} dz &\leq \int_{\Omega_\varepsilon^*} |\tilde{v}_\varepsilon^*|^{p+1} dz = \int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \\ &\leq \left(S_N^{-1} \int_{\Omega_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2 dy \right)^{(p+1)/2} \leq C,\end{aligned}$$

here we have used the Sobolev inequality and (3.6). Then by a result of classical elliptic regularity ([5] Theorem 8.17), we obtain

$$\sup_{B(0,R)\cap\Omega_\varepsilon^*} |\tilde{v}_\varepsilon^*| \leq C \left[\frac{1}{R^N} \int_{B(0,2R)\cap\Omega} |\tilde{v}_\varepsilon^*|^{p+1} dz \right]^{1/(p+1)} \leq C$$

for some $R > 0$. □

By Lemma 3.1, we have the following convergence result.

Lemma 3.2 *Let $\omega \subset \Omega$ be any neighborhood of $\partial\Omega$ not containing x_0. Then we have*

$$\|u_\varepsilon\| v_\varepsilon \to -(N-2)\sigma_N b G(\cdot, x_0) \quad \textit{in } C^1(\omega). \tag{3.12}$$

Proof: We see

$$-\Delta \left(\|u_\varepsilon\| v_\varepsilon \right) = \|u_\varepsilon\| c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} v_\varepsilon =: f_\varepsilon(x) \tag{3.13}$$

for $x \in \Omega$ with the boundary condition $\|u_\varepsilon\| v_\varepsilon = 0$ on $\partial\Omega$. By using (2.11), (3.8), (2.7) and the dominated convergence theorem, we obtain

$$\begin{aligned}\int_\Omega f_\varepsilon(x) dx &= c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon + 1 - (\frac{p_\varepsilon - 1}{2})N} \int_{\Omega_\varepsilon} K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon - 1}(y) \tilde{v}_\varepsilon(y) dy \\ &\to c_0 p \int_{\mathbb{R}^N} U^{p-1} v_0 dy = c_0 p b \int_{\mathbb{R}^N} \frac{1 - |y|^2}{(1+|y|^2)^{N/2+2}} dy \\ &= c_0 p b \sigma_N \left(\int_0^\infty \frac{r^{N-1}}{(1+r^2)^{N/2+2}} dr - \int_0^\infty \frac{r^{N+1}}{(1+r^2)^{N/2+2}} dr \right) \\ &= -(N-2) b \sigma_N.\end{aligned}$$

Note that the integral involving the a_j terms of v_0 must vanish by the oddness of the integrand. Last integral can be computed by the formula

$$\int_0^\infty \frac{r^\alpha}{(1+r^2)^\beta} dr = \frac{\Gamma((\alpha+1)/2)\Gamma(\beta-(\alpha+1)/2)}{2\Gamma(\beta)}$$

where $\alpha > 0, \beta > 0$ with $\beta - (\alpha+1)/2 > 0$. Furthermore, for any $x \neq x_0$, we have by (2.5) and (3.8),

$$\begin{aligned} f_\varepsilon(x) &\leq C\|u_\varepsilon\| \frac{\|u_\varepsilon\|^{p_\varepsilon}}{\left(1 + \|u_\varepsilon\|^{\frac{4}{N-2}} |x - x_\varepsilon|^2\right)^{\frac{N-2}{2}p_\varepsilon}} \\ &\leq C \frac{\|u_\varepsilon\|^{-(p_\varepsilon - 1)}}{|x - x_\varepsilon|^{(N-2)p_\varepsilon}} \to 0 \end{aligned}$$

since $-(p_\varepsilon - 1) = -4/(N-2) + \varepsilon < 0$ for $\varepsilon > 0$ small. In conclusion, we confirm that

$$f_\varepsilon \to -(N-2)\sigma_N b \delta_{x_0} \tag{3.14}$$

in the sense of distributions. On the other hand, from the equation (3.13) with the boundary condition, we have the uniform boundary $C^{1,\alpha}$-estimate ([8] Lemma 2)

$$\|\|u_\varepsilon\| v_\varepsilon\|_{C^{1,\alpha}(\omega)} \leq C(\omega) \left(\|f_\varepsilon\|_{L^1(\Omega)} + \|f_\varepsilon\|_{L^\infty(\omega')}\right),$$

here $\omega \subset\subset \omega'$ is a neighborhood of $\partial\Omega$ not containing 0. Since the RHS of the above estimate is bounded by a constant independent of ε, Ascoli-Arzelà theorem implies that the function $\|u_\varepsilon\| v_\varepsilon$ converges to some function in $C^{1,\alpha}$-topology. Finally, (3.14) implies that this limit function is $-(N-2)\sigma_N b G(x, x_0)$. □

Assume for the moment that the proof of Step 1 and Step 2 is finished. Then the proof of Step 3 is as follows. By Step 1 and Step 2, we deduce that $\lim_{\varepsilon\to 0} \tilde{v}_\varepsilon = v_0 \equiv 0$. Since $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, there exists $y_\varepsilon \in \Omega_\varepsilon$ such that $\tilde{v}_\varepsilon(y_\varepsilon) = 1$. Since $\tilde{v}_\varepsilon \to v_0 \equiv 0$ uniformly on compact sets of $\mathbb{R}^N$, we must have $|y_\varepsilon| \to \infty$. But this is not possible because of Lemma 3.1. □

Proof of Step 1.

First, we treat the case $N = 3$ or $N = 4$. Putting $z = x_0$ in (2.3) and multiplying $\|u_\varepsilon\|^2$, we have

$$\int_{\partial\Omega}((x - x_0)\cdot\nu_x)\left(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial\nu_x}\right)\left(\frac{\partial\|u_\varepsilon\|v_\varepsilon}{\partial\nu_x}\right)ds_x$$
$$= c_0\|u_\varepsilon\|^2\int_\Omega u_\varepsilon^{p_\varepsilon}v_\varepsilon(x)(x - x_0)\cdot\nabla K(x)dx. \qquad (3.15)$$

By Theorem 2.3 (2.8) and (3.12), we see

$$\text{(LHS) of (3.15)} \to -(N-2)^2\sigma_N^2 b\int_{\partial\Omega}((x - x_0)\cdot\nu_x)\left(\frac{\partial G(x, x_0)}{\partial\nu_x}\right)^2 ds_x$$
$$= -(N-2)^3\sigma_N^2 bR(x_0) \qquad (3.16)$$

as $\varepsilon \to 0$. Here we have used (2.1) in Lemma 2.1.

Also by Taylor's theorem, we have

$$K(x) = 1 + \frac{1}{2}\sum_{i,j=1}^N b_{ij}(x_i - x_{0i})(x_j - x_{0j}) + O(|x - x_0|^3)$$

and

$$\frac{\partial K}{\partial x_j}(x) = \sum_{i=1}^N b_{ij}(x_i - x_{0i}) + O(|x - x_0|^2)$$

where $b_{ij} = \frac{\partial^2 K}{\partial x_i\partial x_j}(x_0)$. Thus

$$(x - x_0)\cdot\nabla K(x) = \sum_{i,j=1}^N b_{ij}(x_i - x_{0i})(x_j - x_{0j}) + O(|x - x_0|^3).$$

When $N = 3$, we write

$$\begin{aligned}&\text{(RHS) of (3.15)}\\ &= \|u_\varepsilon\|^2\int_\Omega u_\varepsilon^{p_\varepsilon}v_\varepsilon O(|x - x_\varepsilon|^2)dx + \|u_\varepsilon\|^2\int_\Omega u_\varepsilon^{p_\varepsilon}v_\varepsilon O(|x_\varepsilon - x_0|^2)dx\\ &=: A_1 + A_2.\end{aligned}$$

By a change of variables, we see

$$|A_1| \le \|u_\varepsilon\|^2\int_\Omega u_\varepsilon^{p_\varepsilon}|v_\varepsilon|O(|x - x_\varepsilon|^2)dx$$
$$= \|u_\varepsilon\|^{3+p_\varepsilon-(\frac{p_\varepsilon-1}{2})N-(p_\varepsilon-1)}\int_{\Omega_\varepsilon}O(|y|^2)\tilde{u}_\varepsilon^{p_\varepsilon}(y)|\tilde{v}_\varepsilon(y)|dy.$$

By (2.11), (3.8), (2.7) and the dominated convergence theorem, we have

$$\int_{\Omega_\varepsilon} O(|y|^2)\tilde{u}_\varepsilon^{p_\varepsilon}(y)|\tilde{v}_\varepsilon(y)|dy \to \int_{\mathbb{R}^N} O(|y|^2)U^p(y)|v_0(y)|dy$$

which is finite if $N \geq 3$. On the other hand, the exponent of $\|u_\varepsilon\|$ is $3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - (p_\varepsilon - 1) = (2N-8)/(N-2) + \varepsilon N/2 < 0$ when $N = 3$ and $\varepsilon > 0$ small, thus $A_1 \to 0$ as $\varepsilon \to 0$. Similarly by Theorem 2.3 (2.6) and the dominated convergence theorem, we see

$$\begin{aligned} A_2 &= \|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^{p_\varepsilon} v_\varepsilon O(|x_\varepsilon - x_0|^2)dx \\ &= \|u_\varepsilon\|^2 \times \left(\int_{\mathbb{R}^N} U^p v_0(y)dy + o(1) \right) \times O\left(\|u_\varepsilon\|^{-4}\right) \\ &= o(1) \end{aligned}$$

as $\varepsilon \to 0$. Thus together with (3.16), we have

$$-\sigma_3^2 bR(x_0) = 0$$

when $N = 3$, which leads to $b = 0$.

When $N = 4$, we write

$$\begin{aligned} \text{(RHS) of (3.15)} = c_0\|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) \sum_{i,j=1}^N b_{ij}(x_i - x_{0i})(x_j - x_{0j})dx \\ + \|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) O(|x - x_0|^3)dx. \end{aligned}$$

As before, by Theorem 2.3 (2.6), we see

$$\begin{aligned} &\|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) O(|x - x_0|^3)dx \\ &= \|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) O(|x - x_\varepsilon|^3)dx + \|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) O(|x_\varepsilon - x_0|^3)dx \\ &= \|u_\varepsilon\|^{3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - 3(\frac{p_\varepsilon - 1}{2})} \left(\int_{\mathbb{R}^N} O(|y|^3)U^p(y)|v_0(y)|dy + o(1) \right) \\ &+ \|u_\varepsilon\|^2 \times \left(\int_{\mathbb{R}^N} U^p(y)|v_0(y)|dy + o(1) \right) \times o(\|u_\varepsilon\|^{-3}) \\ &= o(1) \end{aligned}$$

as $\varepsilon \to 0$, since $3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - 3(\frac{p_\varepsilon - 1}{2}) = \frac{2(N-5)}{N-2} + \varepsilon \frac{N+1}{2} < 0$ for $N = 4$ and $\varepsilon > 0$ small.

On the other hand, by a change of variables, we have

$$
\begin{aligned}
&c_0 \|u_\varepsilon\|^2 \int_\Omega u_\varepsilon^{p_\varepsilon} v_\varepsilon(x) \sum_{i,j=1}^N b_{ij}(x_i - x_{0i})(x_j - x_{0j}) dx \\
&= c_0 \|u_\varepsilon\|^{3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \times \\
&\sum_{i,j=1}^N b_{ij} \left(\frac{y_i}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + x_{\varepsilon i} - x_{0i} \right) \left(\frac{y_j}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + x_{\varepsilon j} - x_{0j} \right) dy \\
&=: B_1 + B_2 + B_3
\end{aligned}
$$

where

$$
B_1 = c_0 \|u_\varepsilon\|^{3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - (p_\varepsilon - 1)} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy,
$$

$$
B_2 = 2c_0 \|u_\varepsilon\|^{3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - (\frac{p_\varepsilon - 1}{2})} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i (x_{\varepsilon j} - x_{0j}) dy,
$$

$$
B_3 = c_0 \|u_\varepsilon\|^{3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} (x_{\varepsilon i} - x_{0i})(x_{\varepsilon j} - x_{0j}) dy.
$$

The exponents are

$$
\begin{aligned}
3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - (p_\varepsilon - 1) &= 2\varepsilon, \\
3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - (\frac{p_\varepsilon - 1}{2}) &= 1 + (3/2)\varepsilon, \\
3 + p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N &= 2 + \varepsilon
\end{aligned}
$$

when $N = 4$. Thus by Theorem 2.3 (2.6), (2.7) and the dominated convergence theorem as before, we see

$$
\begin{aligned}
|B_2| &\le 2c_0 \|u_\varepsilon\|^{1 + \frac{3}{2}\varepsilon} \left| \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i dy \right| o(\|u_\varepsilon\|^{-1}) \\
&= o(\|u_\varepsilon\|^{(3/2)\varepsilon}) = o(1) \quad \text{as } \varepsilon \to 0
\end{aligned}
$$

and

$$
\begin{aligned}
|B_3| &\le c_0\|u_\varepsilon\|^{2+\varepsilon}\left|\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_\varepsilon(y)dy\right| o(\|u_\varepsilon\|^{-2})\\
&= o(\|u_\varepsilon\|^{\varepsilon}) = o(1) \quad \text{as } \varepsilon \to 0.
\end{aligned}
$$

Now, we treat the term B_1. By (2.11), (3.8), (2.7) and the dominated convergence theorem as before, we have

$$
\begin{aligned}
B_1 &= c_0\|u_\varepsilon\|^{2\varepsilon}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_\varepsilon(y)\sum_{i,j=1}^{N} b_{ij}y_iy_j dy\\
&\to c_0\int_{\mathbb{R}^N} U^p(y)v_0(y)\sum_{i,j=1}^{N} b_{ij}y_iy_j dy\\
&= c_0\int_{\mathbb{R}^N} U^p(y)b\frac{1-|y|^2}{(1+|y|^2)^{N/2}}\sum_{i,j=1}^{N} b_{ij}y_iy_j dy\\
&= c_0\int_{\mathbb{R}^N} U^p(y)b\frac{1-|y|^2}{(1+|y|^2)^{N/2}}\sum_{i=1}^{N} b_{ii}y_i^2 dy\\
&= \frac{c_0}{N}\int_{\mathbb{R}^N} U^p(y)b\frac{1-|y|^2}{(1+|y|^2)^{N/2}}\Delta K(x_0)|y|^2 dy\\
&= (N-2)b\Delta K(x_0)\left(\int_{\mathbb{R}^N}\frac{|y|^2}{(1+|y|^2)^{N+1}}dy - \int_{\mathbb{R}^N}\frac{|y|^4}{(1+|y|^2)^{N+1}}dy\right)\\
&= (N-2)b\Delta K(x_0)\times -\sigma_N\frac{\Gamma(\frac{N}{2}+1)\Gamma(\frac{N}{2}-1)}{\Gamma(N+1)}\\
&= -\frac{b\Delta K(x_0)\sigma_4}{6} \quad \text{when } N=4.
\end{aligned}
$$

Note that by the oddness of the integrand,

$$
\int_{\mathbb{R}^N} U^p(y)\frac{1}{(1+|y|^2)^{N/2}}y_iy_jy_k dy = 0
$$

for any $i,j,k \in \{1,\cdots,N\}$, so the integral involving a_j terms in v_0 must vanish.

Returning to (3.16), we obtain

$$
-8\sigma_4^2 bR(x_0) = -\frac{b\Delta K(x_0)\sigma_4}{6}.
$$

Since $\Delta K(x_0) \le 0$ and $R(x_0) > 0$, we conclude that $b = 0$ when $N = 4$.

Next, we treat the case $N \ge 5$. In this case, we multiply (2.3) by $\|u_\varepsilon\|^{\frac{4}{N-2}}$ to get

$$\|u_\varepsilon\|^{\frac{4}{N-2}-2}\int_{\partial\Omega}((x-x_0)\cdot\nu_x)\left(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial\nu_x}\right)\left(\frac{\partial\|u_\varepsilon\|v_\varepsilon}{\partial\nu_x}\right)ds_x$$
$$= c_0\|u_\varepsilon\|^{\frac{4}{N-2}}\int_\Omega u_\varepsilon^{p_\varepsilon}v_\varepsilon(x)(x-x_0)\cdot\nabla K(x)dx. \tag{3.17}$$

Since $\frac{4}{N-2} < 2$ if $N \ge 5$, the LHS of (3.17) converges to 0 as $\varepsilon \to 0$. On the other hand, by Taylor's formula and the change of variables, we write

$$\text{(RHS) of (3.17)} =: C_1 + C_2 + C_3 + C_4$$

where

$$C_1 = c_0\|u_\varepsilon\|^{\frac{4}{N-2}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{2})N-(p_\varepsilon-1)}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_\varepsilon(y)\sum_{i,j=1}^N b_{ij}y_iy_jdy,$$

$$C_2 = 2c_0\|u_\varepsilon\|^{\frac{4}{N-2}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{2})N-(\frac{p_\varepsilon-1}{2})}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_\varepsilon(y)\sum_{i,j=1}^N b_{ij}y_i(x_{\varepsilon j}-x_{0j})dy,$$

$$C_3 = c_0\|u_\varepsilon\|^{\frac{4}{N-2}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{2})N}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_\varepsilon(y)\sum_{i,j=1}^N b_{ij}(x_{\varepsilon i}-x_{0i})(x_{\varepsilon j}-x_{0j})dy,$$

$$C_4 = c_0\|u_\varepsilon\|^{\frac{4}{N-2}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{2})N}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_\varepsilon(y)\left(O\left(\left|\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon-x_0\right|^3\right)\right)dy.$$

Again by (2.11), (3.8), (2.7), (2.6) and the dominated convergence theorem, we see

$$C_2 = O(\|u_\varepsilon\|^{\frac{2}{N-2}+\frac{N-1}{2}\varepsilon})\times O\left(\int_{\mathbb{R}^N}U^pv_0(y)|y|dy+o(1)\right)\times o(\|u_\varepsilon\|^{-\frac{2}{N-2}}) = o(1),$$

$$C_3 = O(\|u_\varepsilon\|^{\frac{4}{N-2}+\frac{N-2}{2}\varepsilon})\times O\left(\int_{\mathbb{R}^N}U^pv_0(y)dy+o(1)\right)\times o(\|u_\varepsilon\|^{-\frac{4}{N-2}}) = o(1),$$

$$C_4 = O(\|u_\varepsilon\|^{\frac{4}{N-2}+\frac{N-2}{2}\varepsilon})\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_\varepsilon(y)\left(O\left(\left|\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}\right|^3\right)+O(|x_\varepsilon-x_0|^3)\right)$$
$$= O(\|u_\varepsilon\|^{\frac{4}{N-2}})\times O(\|u_\varepsilon\|^{-\frac{6}{N-2}})\times O\left(\int_{\mathbb{R}^N}U^pv_0(y)(|y|^3+1)dy+o(1)\right)$$
$$= O(\|u_\varepsilon\|^{-\frac{2}{N-2}})$$

as $\varepsilon \to 0$. As for C_1, we see just as in the estimate of B_1,

$$C_1 = c_0\|u_\varepsilon\|^{(N/2)\varepsilon} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^{N} b_{ij} y_i y_j dy$$
$$\to = -(N-2) b \Delta K(x_0) \sigma_N \frac{\Gamma(\frac{N}{2}+1)\Gamma(\frac{N}{2}-1)}{\Gamma(N+1)}.$$

Thus letting $\varepsilon \to 0$ in (3.17), we have

$$0 = \Delta K(x_0) \times b.$$

Hence we obtain $b = 0$, because our nondegeneracy assumption of x_0 assures that $\Delta K(x_0) < 0$ strictly. This proves step 1 for all cases.

Proof of Step 2.

In this step, we prove $a_j = 0, j = 1, 2, \cdots, N$ in (3.7). For this purpose, we need a lemma, which is not in [6].

Lemma 3.3 *Assume $b = 0$ and $a = (a_1, \cdots, a_N) \neq 0$ in (3.7). Then we have*

$$\|u_\varepsilon\|^{N/(N-2)} v_\varepsilon \to \sigma_N \sum_{j=1}^{N} a_j \left(\frac{\partial G}{\partial z_j}(x, z) \right) \Big|_{z=x_0}$$

in $C^1_{loc}(\overline{\Omega} \setminus \{x_0\})$.

Proof. For any $x \in \overline{\Omega} \setminus \{x_0\}$, the Green representation formula to $(L_{\varepsilon,K})$ and a change of variables imply that

$$v_\varepsilon(x) = c_0 p_\varepsilon \int_\Omega G(x,z) K(z) u_\varepsilon^{p_\varepsilon - 1}(z) v_\varepsilon(z) dz$$
$$= c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N} \int_{\Omega_\varepsilon} G_\varepsilon(x,y) K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon - 1} \tilde{v}_\varepsilon(y) dy$$

where $G_\varepsilon(x,y) = G(x, \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + x_\varepsilon)$ and $K_\varepsilon(y) = K(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + x_\varepsilon)$ for $y \in \Omega_\varepsilon$. By (2.10) and (3.3) with $b = 0$, we see

$$\tilde{u}_\varepsilon^{p_\varepsilon - 1}(y) \to U^{p-1}(y),$$
$$\tilde{v}_\varepsilon(y) \to v_0 = \sum_{j=1}^{N} a_j \frac{y_j}{(1+|y|^2)^{N/2}} = \frac{-1}{(N-2)} \sum_{j=1}^{N} a_j \frac{\partial U}{\partial y_j}(y)$$

uniformly on compact subsets of $\mathbb{R}^N$, thus

$$\tilde{u}_\varepsilon^{p_\varepsilon-1}\tilde{v}_\varepsilon(y) \to \sum_{j=1}^N a_j \left(\frac{\partial}{\partial y_j} \frac{-1}{(N+2)} U^p(y) \right)$$

uniformly on compact subsets of $\mathbb{R}^N$.

Now, let us consider the following linear first order PDE

$$\sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon-1}\tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N \tag{3.18}$$

with the initial condition $w_\varepsilon|_{\Gamma_a} = \frac{-1}{(N+2)}U^p(y)$, where $\Gamma_a = \{x \in \mathbb{R}^N | x \cdot a = 0\}$. The RHS of (3.18) should be understood as 0 outside of Ω_ε. By Lemma 2.5, the solution w_ε satisfies the estimate $w_\varepsilon(y) = O(|y|^{-(N+1)})$ as $|y| \to \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon-1}\tilde{v}_\varepsilon(y) = O(U^{p_\varepsilon}(y)) = O(|y|^{-(N+2)})$ by (2.11) and (3.8). Also we have

$$w_\varepsilon \to \frac{-1}{(N+2)} U^p \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\Omega_\varepsilon} w_\varepsilon(y)dy \to \frac{-1}{(N+2)} \int_{\mathbb{R}^N} U^p dy = \frac{-1}{N(N+2)}\sigma_N$$

by the dominated convergence theorem.

Using integration by parts, we have

$$\begin{aligned}
v_\varepsilon(x) &= c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{2})N} \int_{\Omega_\varepsilon} G_\varepsilon(x,y) K_\varepsilon(y) \sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} dy \\
&= -c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{2})N} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j} \{G_\varepsilon(x,y)K_\varepsilon(y)\} w_\varepsilon(y) dy \\
&= -c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{2})N - (\frac{p_\varepsilon-1}{2})} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial z_j} \{G(x,z)K(z)\} \Big|_{z = \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon} w_\varepsilon(y) dy.
\end{aligned}$$

Now, we see

$$
\begin{aligned}
&\frac{\partial}{\partial z_j}\left\{G(x,z)K(z)\right\}\Big|_{z=\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon} \\
&\to \left(\frac{\partial G}{\partial z_j}(x,x_0)\right)K(x_0)+G(x,x_0)\left(\frac{\partial K}{\partial z_j}(x_0)\right) \\
&= \frac{\partial G}{\partial z_j}(x,x_0)
\end{aligned}
$$

uniformly on compact subsets of $\mathbb{R}^N$ as $\varepsilon \to 0$, since x_0 is a critical point of K with $K(x_0) = 1$. Also we note that $\frac{N}{N-2} + p_\varepsilon - (\frac{p_\varepsilon-1}{2})N - (\frac{p_\varepsilon-1}{2}) = (\frac{N-1}{2})\varepsilon$. Therefore, we have the convergence

$$
\begin{aligned}
&\|u_\varepsilon\|^{N/(N-2)}v_\varepsilon(x) \\
&= -c_0 p_\varepsilon \|u_\varepsilon\|^{\frac{N}{N-2}+p_\varepsilon-(\frac{p_\varepsilon-1}{2})N-(\frac{p_\varepsilon-1}{2})}\sum_{j=1}^N a_j\int_{\Omega_\varepsilon}\frac{\partial}{\partial z_j}\left\{G(x,z)K(z)\right\}\Big|_{z=\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon} w_\varepsilon(y)dy \\
&\to -c_0 p\sum_{j=1}^N a_j\frac{\partial G}{\partial z_j}(x,x_0)\times\left(\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} w_\varepsilon(y)dy\right) \\
&= \sigma_N\sum_{j=1}^N a_j\frac{\partial G}{\partial z_j}(x,x_0)
\end{aligned}
$$

for any $x \in \overline{\Omega}\setminus\{x_0\}$. Elliptic estimates implies this convergence holds true in $C^1_{loc}(\overline{\Omega}\setminus\{x_0\})$. This proves Lemma. □

Now, multiply both sides of (2.4) by $\|u_\varepsilon\|^{N/(N-2)}\times\|u_\varepsilon\|^{-1}$. Letting $\varepsilon \to 0$, we see the LHS is

$$
\begin{aligned}
&\|u_\varepsilon\|^{-2}\int_{\partial\Omega}\left(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial x_i}\right)\left(\frac{\partial\|u_\varepsilon\|^{N/(N-2)}v_\varepsilon}{\partial\nu_x}\right)ds_x \\
&= \|u_\varepsilon\|^{-2}\left((N-2)\sigma_N^2\int_{\partial\Omega}\sum_{j=1}^N a_j\left(\frac{\partial G}{\partial x_i}\right)(x,x_0)\frac{\partial}{\partial\nu_x}\left(\frac{\partial G}{\partial z_j}\right)(x,z)\Big|_{z=x_0}ds_x+o(1)\right) \\
&= \|u_\varepsilon\|^{-2}\left(\frac{N-2}{2}\sigma_N^2\sum_{j=1}^N a_j\frac{\partial^2 R}{\partial z_i\partial z_j}(x_0)+o(1)\right)\to 0,
\end{aligned}
$$

here we have used (2.8), Lemma 3.3 and (2.2).

On the other hand, using Lemma 2.5 again, we solve the linear PDE

$$\sum_{j=1}^{N} a_j \frac{\partial w_\varepsilon}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N \tag{3.19}$$

with the initial condition $w_\varepsilon|_{\Gamma_a} = \frac{-1}{2N} U^{p+1}(y)$, where $\Gamma_a = \{x \in \mathbb{R}^N | x \cdot a = 0\}$. Here as before, the RHS of (3.19) is understood as 0 outside of Ω_ε.

Lemma 2.5 implies that the solution w_ε satisfies the estimate $w_\varepsilon(y) = O(|y|^{-2N+1})$ as $|y| \to \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) = O(U^{p_\varepsilon+1}(y)) = O(|y|^{-2N})$ by (2.11) and (3.8). As before, we have

$$w_\varepsilon \to \frac{-1}{2N} U^{p+1} \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\Omega_\varepsilon} w_\varepsilon(y) dy \to \frac{-1}{2N} \int_{\mathbb{R}^N} U^{p+1} dy = \frac{-1}{2N} \sigma_N C_N$$

by the dominated convergence theorem, where $C_N = \int_0^\infty \frac{r^{N-1}}{(1+r^2)^N} dr = \frac{\Gamma(N/2)^2}{2\Gamma(N)}$. Thus, (RHS of (2.4)) $\times \|u_\varepsilon\|^{\frac{N}{N-2}-1}$ is

$$
\begin{aligned}
& c_0 \|u_\varepsilon\|^{-1+\frac{N}{N-2}} \int_\Omega \left(\frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon} v_\varepsilon dx \\
&= c_0 \|u_\varepsilon\|^{\frac{N}{N-2}+p_\varepsilon-(\frac{p_\varepsilon-1}{2})N} \int_{\Omega_\varepsilon} \left(\frac{\partial K}{\partial x_i} \right) \Big(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon \Big) \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon dy \\
&= c_0 \|u_\varepsilon\|^{\frac{N}{N-2}+p_\varepsilon-(\frac{p_\varepsilon-1}{2})N} \int_{\Omega_\varepsilon} \left(\frac{\partial K}{\partial x_i} \right) \Big(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon \Big) \sum_{j=1}^{N} a_j \frac{\partial w_\varepsilon}{\partial y_j} dy \\
&= -c_0 \|u_\varepsilon\|^{\frac{N}{N-2}+p_\varepsilon-(\frac{p_\varepsilon-1}{2})N} \sum_{j=1}^{N} a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j} \left\{ \left(\frac{\partial K}{\partial x_i} \right) \Big(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon \Big) \right\} w_\varepsilon(y) dy \\
&= -c_0 \|u_\varepsilon\|^{\frac{N}{N-2}+p_\varepsilon-(\frac{p_\varepsilon-1}{2})N-(\frac{p_\varepsilon-1}{2})} \sum_{j=1}^{N} a_j \int_{\Omega_\varepsilon} \left(\frac{\partial^2 K}{\partial x_i \partial x_j} \right) (x) \Big|_{x=\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon} w_\varepsilon(y) dy \\
&\to -c_0 \left(\lim_{\varepsilon \to 0} \|u_\varepsilon\|^{(\frac{N-1}{2})\varepsilon} \right) \sum_{j=1}^{N} a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0) \left(\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} w_\varepsilon(y) dy \right) \\
&= \frac{N-2}{2} \sigma_N C_N \sum_{j=1}^{N} a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0).
\end{aligned}
$$

Thus we have

$$\sum_{j=1}^{N} a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0) = 0.$$

By our assumption of the nondegeneracy of x_0, the matrix $\left(\frac{\partial^2 K}{\partial x_i \partial x_j}\right)(x_0)$ is invertible. Therefore we obtain that $a_j = 0$ for all $j = 1, \cdots, N$. Thus we have proved Step 2, and consequently, Theorem 1.1. □

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