Characterization of homogeneous torus manifolds

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ABSTRACT. This is the first of a series of papers which will be devoted to the study of the extended G-actions on torus manifolds $(M^{2n},\ T^n)$, where G is a compact, connected Lie group whose maximal torus is T^n. The goal of this paper is to characterize codimension 0 extended G-actions up to essential isomorphism. For technical reasons, we do not assume that torus manifolds are omnioriented. The main result of this paper is as follows: a homogeneous torus manifold M^{2n} is (weak equivariantly) diffeomorphic to a product of complex projective spaces $\prod \mathbb{C}P(l)$ and quotient spaces of a product of spheres $(\prod S^{2m})/\mathcal{A}$ with standard torus actions, where $\mathcal{A}$ is a subgroup of $\prod \mathbb{Z}_2$ generated by the antipodal involutions on S^{2m}. In particular, if the homogeneous torus manifold M^{2n} is a compact (non-singular) toric variety or a quasitoric manifold, then M^{2n} is just a product of complex projective spaces $\prod \mathbb{C}P(l)$.

1. Introduction

A *torus manifold* is an even dimensional oriented manifold M^{2n} acted on by a half-dimensional torus T^n with non-empty fixed point set: typical examples are the complex projective space $\mathbb{C}P(n)$ and the even dimensional sphere S^{2n} equipped with the natural T^n-actions. As is well known, the natural T^n-action on $\mathbb{C}P(n)$ is induced from the transitive $U(n+1)$-action (or $PU(n+1)$-action) on $\mathbb{C}P(n)$, that is, this T^n-action extends to a $U(n+1)$-action or $PU(n+1)$-action (see Example 2.2). Moreover, there is a similar property for the T^n-action on S^{2n} (see Example 2.3). So we can naturally ask which torus manifolds possess such *extended actions* (the exact definition is in Section 2.1). In a series of papers we focus on this extension problem of torus actions on torus manifolds.

This problem is reminiscent of the study of automorphism groups of *toric varieties* by Demazure in [**5**], where here a toric variety is a normal algebraic variety V on which an algebraic torus $(\mathbb{C}^*)^n$ acts with a dense orbit (see [**6**]). We note that a compact non-singular toric variety is a torus manifold by restricting its $(\mathbb{C}^*)^n$-action to a T^n-action ($(\mathbb{C}^*)^n$ contains the topological torus $T^n = (S^1)^n$ as its maximal compact subgroup). The automorphism group $Aut(V)$ of V contains $(\mathbb{C}^*)^n$ and the action of $Aut(V)$ restricted to $(\mathbb{C}^*)^n$ coincides with the original $(\mathbb{C}^*)^n$-action on V. Hence, we can regard Demazure's study as the study of the extension problem of

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$(\mathbb{C}^*)^n$-actions on toric varieties. In fact the notion of torus manifold (or *unitary toric manifold* in the earlier terminology) was introduced by Hattori and Masuda in [**8, 11**] as a far-reaching topological generalization of compact non-singular toric varieties. Consequently, our extension problem may be interpreted as the topological version of Demazure's work. (From this point we assume our groups in this paper are always compact.)

In a series of papers, we will study extended G-actions. In particular, in the present paper and the next papers, we will characterize extended G-actions which have codimension 0 (i.e., G acts transitively) and 1 principal orbits up to *essential isomorphism* (i.e., the induced effective actions are same: see Section 2.4). We often call a torus manifold on which G acts transitively a *homogeneous torus manifold*. For technical reasons, in this paper, we do not assume that torus manifolds are omnioriented.

Let $\mathbb{Z}_2$ be defined as $\{I_{2m_j+1},\ -I_{2m_j+1}\} \subset O(2m_j+1)$, and let $PU(x)$ be a projective unitary group (see Example 2.2), $SO(x)$ a special orthogonal group, $O(x)$ an orthogonal group. Our main result (Theorem 3.4) is as follows:

Theorem 1. *Suppose a torus manifold $(M^{2n},\ T^n)$ extends to a transitive G-action, where G is a compact, connected Lie group whose maximal torus is T^n. Then $(M^{2n},\ G)$ is essentially isomorphic to*

$$\left(\prod_{i=1}^{a}\mathbb{C}P(l_i)\times\frac{\prod_{j=1}^{b}S^{2m_j}}{\mathcal{A}},\ \prod_{i=1}^{a}PU(l_i+1)\times\prod_{j=1}^{b}SO(2m_j+1)\right),$$

where $\mathcal{A}$ can be any subgroup of $\prod_{j=1}^{b}\mathbb{Z}_2$, and $\prod_{i=1}^{a}PU(l_i+1)\times\prod_{j=1}^{b}SO(2m_j+1)$ acts on $\prod_{i=1}^{a}\mathbb{C}P(l_i)\times\prod_{j=1}^{b}S^{2m_j}/\mathcal{A}$ in the natural way, and $\sum_{i=1}^{a}l_i+\sum_{j=1}^{b}m_j=n$.

Furthermore, M^{2n} is orientable if and only if $\mathcal{A}\subset SO(2m_1+\cdots+2m_b+b)$.

We also have the following result (see Corollary 3.9):

Corollary 2. *Suppose a compact, non-singular toric variety or a quasitoric manifold $(M^{2n},\ T^n)$ extends to a transitive G-action, where G is a compact, connected Lie group whose maximal torus is T^n. Then $(M^{2n},\ G)$ is essentially isomorphic to*

$$\left(\prod_{i=1}^{a}\mathbb{C}P(l_i),\ \prod_{i=1}^{a}PU(l_i+1)\right),$$

where $\sum_{i=1}^{a}l_i=n$.

Here, a T^n-manifold M^{2n} is called a *quasitoric manifold* over a simple polytope P^n if the following two conditions are satisfied (see [**3, 4**][1]):

(1) the T^n-action is *locally standard*, that is, locally modelled by the standard action on $\mathbb{C}^n$;
(2) there is the orbit projection map $\pi: M^{2n}\to M^{2n}/T^n=P^n$ constant on T^n-orbits which maps every k-dimensional orbit to a point in the interior of k-dimensional face of P^n, $k=0,\ \cdots,\ n$.

[1]Davis and Januszkiewicz use the term "toric manifold" in [**4**], but in this paper we use the term "quasitoric manifold" in [**3**] because we would like to reserve the use of the term "toric manifold" to mean a "non-singular toric variety".

Quasitoric manifolds were defined by Davis and Januszkiewicz as a topological counterpart of projective, non-singular toric varieties in [**4**]. We note that every smooth projective toric variety is a quasitoric manifold (e.g. [**3**, Chapter 5]). From our main result Theorem 1, we may conclude that if a compact, non-singular toric variety has a transitive G-action, then this manifold is a product of $\mathbb{C}P(l)$'s (Corollary 2). Hence, such manifolds are projective toric varieties; thus, they are also quasitoric manifolds.

The organization of this paper is as follows. In Section 2, we first set up some notation and basic facts. Next we prove our main result Theorem 3.4, that is, we characterize homogeneous torus manifolds G/H in Section 3. A key lemma for this characterization is Lemma 3.2, as well as the classification result of simply connected, simple Lie groups and their maximal rank maximal connected subgroups in classical Lie theory proved in [**1**]. Finally, we remark that our methods, in particular Lemma 3.2, do not work for other T-manifolds in Section 4.

2. Preliminaries to the characterization

In this section, we recall some fundamental results. We start with recalling some basic notation associated to a torus manifold and an extended G-action.

2.1. Basic notations and examples. A *torus manifold* is a $2n$-dimensional, closed, connected, smooth manifold $M^{2n}(=M)$ with smooth, finite kernel action of an n-dimensional torus $T^n = (S^1)^n (= T)$ such that $M^T \neq \emptyset$ (see Section 2.4 for the kernel of an action). Let M^T denote the T-fixed point set. Automatically, every fixed point is isolated, because $\dim M = 2 \dim T$ and T acts on M with finite kernel.

Remark 2.1. In the paper [**8**], the definition of torus manifolds involves the choice of orientations of manifold M and its *characteristic submanifolds* called an *omniorientation* on M. Because we will classify extended actions up to essential isomorphism in this paper, we do not need to choose an omniorientation on M. Moreover, the T-action on M does not need to be effective.

Let $\varphi : T^n \times M^{2n} \to M^{2n}$ be a T^n-action on M^{2n}. Assume that a compact, connected Lie group G has T^n as its maximal torus subgroup. If there exists an action $\Phi : G \times M^{2n} \to M^{2n}$ such that the restricted T^n-action $\Phi|_{T^n \times M^{2n}}$ is the given φ, then we call Φ an *extended G-action* of $(M^{2n},\ T^n)$, and we also denote Φ as $(M^{2n},\ G)$. If a principal G-orbit is of codimension k, we call $(M^{2n},\ G)$ a *codimension k extended G-action* of $(M^{2n},\ T^n)$. Here, the integer k satisfies $0 \leq k \leq n$. Because M and G are compact, if $(M^{2n},\ G)$ is a codimension 0 extended G-action then the G-action on M^{2n} is transitive and M^{2n} is a homogeneous manifold. The following three examples are standard and important.

Example 2.2. If the T^n-action $\varphi : T^n \times \mathbb{C}P(n) \to \mathbb{C}P(n)$ on $\mathbb{C}P(n)$ is defined by

$$\varphi((t_1,\ \cdots,\ t_n),\ [z_0;\ z_1; \cdots;\ z_n]) = [z_0;\ t_1 z_1; \cdots;\ t_n z_n],$$

where $(t_1,\ \cdots,\ t_n) \in T^n$ and $[z_0;\ z_1; \cdots;\ z_n] \in \mathbb{C}P(n)$, then we can easily check that this is a torus manifold whose fixed points are $(n+1)$ points:

$$[1;\ 0;\ \cdots;\ 0],\ [0;\ 1;\ 0;\ \cdots;\ 0],\ \cdots,\ [0;\ \cdots;\ 0;\ 1].$$

Considering the above T^n as the diagonal subgroup of $U(n+1)$ with a unit in the $(1,1)$-entry, this action extends to the transitive $U(n+1)/Z(U(n+1))$-action, where $Z(U(n+1))$ is the center of $U(n+1)$. Therefore $(\mathbb{C}P(n),\ U(n+1)/Z(U(n+1))) = (\mathbb{C}P(n),\ PU(n+1))$ is a codimension 0 extended action of the torus manifold $(\mathbb{C}P(n),\ T^n)$. Remark that $PU(n+1) = U(n+1)/Z(U(n+1)) \simeq SU(n+1)/\mathbb{Z}_{n+1} \approx SU(n+1)$ has T^n as its maximal torus subgroup, where $G \simeq G'$ means G and G' are isomorphic, $G \approx G'$ means G and G' have a same Lie algebra, and $\mathbb{Z}_{n+1}$ is the center of $SU(n+1)$.

Example 2.3. Assume the T^n-action $\varphi : T^n \times S^{2n} \to S^{2n}$ on $S^{2n} \subset \mathbb{R}^{2n} \oplus \mathbb{R} = \mathbb{R}^{2n+1}$ is defined as follows: first we identify T^n with $SO(2)^n \subset SO(2n)$; and next T^n acts on $\mathbb{R}^{2n}$ by the restriction of the natural $SO(2n)$-action on $\mathbb{R}^{2n}$. Then we can easily check that this is a torus manifold whose fixed points are 2 points: the North pole $(0,\ \cdots,\ 0,\ 1)$ and the South pole $(0,\ \cdots,\ 0,\ -1)$ of S^{2n}. Moreover this action extends to the $SO(2n)$-action whose orbits are principal orbits S^{2n-1} (codimension 1 orbits) and two singular orbits which are the 2 fixed points of the T^n-action. Therefore $(S^{2n},\ SO(2n))$ is a codimension 1 extended action of the torus manifold $(S^{2n},\ T^n)$. Remark that $(S^{2n},\ T^n)$ also extends to a codimension 0 extended action $(S^{2n},\ SO(2n+1))$.

Example 2.4. In the above Example 2.3, $S^{2n} \subset \mathbb{R}^{2n+1}$ has a free involution by $-I_{2n+1} \in O(2n+1)$, where I_{2n+1} is the identity element in the orthogonal group $O(2n+1)$. Now we define the manifold $\mathbb{R}P(2n)$ by $S^{2n}/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{I_{2n+1},\ -I_{2n+1}\}$. Because the T^n-action on S^{2n} commutes with the $\mathbb{Z}_2$-action, we can define a T^n-action on $\mathbb{R}P(2n)$ induced by the T^n-action on S^{2n}. Moreover, the two T^n-fixed points on S^{2n}, the North and South poles, go to the same point in $\mathbb{R}P(2n)$ under the $\mathbb{Z}_2$-quotient, and this point is the unique fixed point of this T^n-action on $\mathbb{R}P(2n)$. We can easily check $(\mathbb{R}P(2n),\ T^n)$ is effective, because $(S^{2n},\ T^n)$ is effective and $T^n \cap \mathbb{Z}_2 = \{I_{2n+1}\}$. Therefore, $(\mathbb{R}P(2n),\ T^n)$ is a torus manifold; however, we remark that $\mathbb{R}P(2n)$ is a non-orientable manifold. Furthermore, we have that $(\mathbb{R}P(2n),\ T^n)$ has a codimension 1 extended action $(\mathbb{R}P(2n),\ SO(2n))$ and a codimension 0 extended action $(\mathbb{R}P(2n),\ SO(2n+1))$, because the $SO(2n)$-action and the $SO(2n+1)$-action on S^{2n} in Example 2.3 commute with this $\mathbb{Z}_2$-action. Remark that the orbits of the codimension 1 extended action $(\mathbb{R}P(2n),\ SO(2n))$ consist of: principal orbits S^{2n-1}, one singular orbit which coincides with the unique T^n-fixed point of $(\mathbb{R}P(2n),\ T^n)$, and one exceptional orbit $\mathbb{R}P(2n-1)$.

In order to characterize torus manifolds which have codimension 0 extended actions, i.e., homogeneous torus manifolds, we will also need the following basic facts, summarized in Section 2.2 to 2.4.

2.2. Homogeneous space G/H with finite T-fixed points. First we discuss homogeneous spaces with T-actions. Let T be a maximal torus in a compact Lie group G, and H a closed subgroup in G. Suppose that $(G/H,\ T)$ is a torus manifold, that is, it satisfies the following three properties:

(1) the T-action on G/H has finite kernel;
(2) $\dim G/H = 2 \dim T$;
(3) the T-fixed point set $(G/H)^T \neq \emptyset$.

Because of the third property, there is an element $gH \in G/H$ such that $TgH = gH$. It follows that $T \subset gHg^{-1}$ for some $g \in G$. Hence we can take a subgroup H as follows:

$$T \subset H^o \subset H \subset G,$$

where H^o is the identity component of H. Since T is a maximal torus in G, we have

$$\text{rank } G = \text{rank } H^o = \dim T = n,$$

where the *rank* of a compact connected Lie group is the dimension of a maximal torus subgroup. Consequently, we need to consider maximal rank subgroups of G.

2.3. Facts from classical Lie theory. In order to consider maximal rank subgroups, we recall some classical Lie theory (see [**13**, Chapter V]).

For any compact, connected Lie group G, there is a finite covering map:

$$p : \widetilde{G} = G_1 \times \cdots \times G_k \to G, \tag{2.1}$$

where G_i $(i = 1, \cdots, k)$ is a compact, simply connected, simple Lie group, or a compact, connected, commutative Lie group, i.e., a torus. Let the kernel of p be denoted by N. Then we have

$$G \simeq (G_1 \times \cdots \times G_k)/N,$$

where N is some finite central normal subgroup in $G_1 \times \cdots \times G_k$.

Now we have the following lemma for a product of Lie groups (see [**13**, Theorem 7.2]).

Lemma 2.5. *Let G_i $(i = 1, \cdots, k)$ be compact, connected Lie groups and let G be their product. Assume H^o is the identity connected maximal rank subgroup in G. Then $H^o = H_1 \times \cdots \times H_k$, where H_i is a maximal rank subgroup in G_i.*

2.4. Essential isomorphism and a remark for the characterization. In this subsection, we define an essential isomorphism.

We first need some notations. The *kernel* of (M, G) is defined as the intersection of all isotropy subgroups $\cap_{x \in M} G_x$. If the kernel N is the identity element, then this action is an *effective action*. The induced action $(M, G/N)$ is always effective, and we call it the *induced effective action*. We may now define an essential isomorphism.

Definition 2.6. Let N be the kernel of (M, G) and N' the kernel of (M', G'). We say that (M, G) and (M', G') are *essentially isomorphic* if their induced effective actions $(M, G/N)$ and $(M', G'/N')$ are weak equivariantly diffeomorphic, that is, there are an isomorphism $\rho : G/N \to G'/N'$ and a diffeomorphism $f : M \to M'$ such that $f(\varphi(g, x)) = \psi(\rho(g), f(x))$ for $(g, x) \in G/N \times M$, where $\varphi : G/N \times M \to M$ and $\psi : G'/N' \times M' \to M'$ are two induced effective actions.

Example 2.7. In Example 2.2, $(\mathbb{C}P(n), PU(n+1))$ is essentially isomorphic to the natural transitive action $(\mathbb{C}P(n), SU(n+1))$.

Example 2.8. Let $Spin(m)$ be the universal (double) covering of $SO(m)$ $(m \geq 3)$. This group $Spin(m)$ acts on a sphere and a real projective space through the projection to $SO(m)$. In Example 2.3 (resp. Example 2.4), the codimension

1 extended action $(S^{2n},\ SO(2n))$ (resp. $(\mathbb{R}P(2n),\ SO(2n)))$ is essentially isomorphic to $(S^{2n},\ Spin(2n))$ (resp. $(\mathbb{R}P(2n),\ Spin(2n)))$ for $n \geq 2$, and the codimension 0 extended action $(S^{2n},\ SO(2n+1))$ (resp. $(\mathbb{R}P(2n),\ SO(2n+1)))$ is essentially isomorphic to the natural transitive action $(S^{2n},\ Spin(2n+1))$ (resp. $(\mathbb{R}P(2n),\ Spin(2n+1)))$ for $n \geq 1$.

Let $\widetilde{G} = G_1 \times \cdots \times G_k$ be a covering of G defined in (2.1) such that each G_i $(i = 1, \cdots, k)$ is a compact, simply connected, simple Lie group, or a torus group. Then $(M,\ G)$ is essentially isomorphic to

$$(M,\ \widetilde{G}) = (M,\ G_1 \times \cdots \times G_k). \tag{2.2}$$

Therefore, we only need to consider products of simply connected, simple Lie groups and tori as the transformation groups on a homogeneous torus manifold. In the next section we characterize homogeneous torus manifolds.

3. Characterization of homogeneous torus manifolds

Assume $(M,\ G)$ is a codimension 0 extended G-action of a torus manifold $(M^{2n},\ T^n)$. In this section, we will classify such $(M,\ G)$ up to essential isomorphism.

3.1. Structure of torus manifolds. Now we can put $M = G/H$ and T is a maximal torus subgroup of H and G by the argument in Section 2.2. Moreover, a T-action of $(G/H,\ T)$ is defined by a natural inclusion of T to G. By (2.2), $(G/H,\ G)$ is essentially isomorphic to

$$(G/H,\ \widetilde{G}) = (G/H,\ G_1 \times \cdots \times G_k).$$

Let $p : \widetilde{G} \to G$ be the projection of (2.1). Then we have

$$G/H \cong \widetilde{G}/p^{-1}(H), \tag{3.1}$$

where $X \cong Y$ means X and Y are diffeomorphic. Therefore it is sufficient to classify $\widetilde{G}$ and its subgroup $p^{-1}(H)$. To classify such $\widetilde{G}$ and $p^{-1}(H)$, we first consider the identity component of $p^{-1}(H)$.

Let $\widetilde{H}$ (resp. $\widetilde{T}$) be the identity component of $p^{-1}(H)$ (resp. $p^{-1}(T)$)[2]. Because of Lemma 2.5, $\widetilde{G}/\widetilde{H}$ is decomposed into a product as follows:

$$\widetilde{G}/\widetilde{H} = G_1/H_1 \times \cdots \times G_k/H_k,$$

where $H_i \subset G_i$ is a maximal rank, connected subgroup for all $i = 1, \cdots, k$. Because T is a maximal torus subgroup of G and H, we see that $\widetilde{T}$ is also a maximal torus subgroup of $\widetilde{G}$ and $\widetilde{H}$ such that $p(\widetilde{T}) = T$. Moreover, we have the following lemma.

Lemma 3.1. *If $(G/H,\ T)$ is a torus manifold, then $(\widetilde{G}/\widetilde{H},\ \widetilde{T})$ is also a torus manifold, and each G_i is a compact, simply connected, simple Lie group.*

Proof. We prove $(\widetilde{G}/\widetilde{H},\ \widetilde{T})$ satisfies the three properties in Section 2.2. Because $\widetilde{T}$ is a maximal torus subgroup of $\widetilde{G}$ and $\widetilde{H}$, we can easily check $(\widetilde{G}/\widetilde{H})^{\widetilde{T}} \neq \emptyset$, i.e., property (3) holds. Because $\dim G/H = 2\dim T$, we have $\dim \widetilde{G}/\widetilde{H} = 2\dim \widetilde{T}$, i.e., property (2) holds. Since $(G/H,\ T)$ is almost effective, $(\widetilde{G}/\widetilde{H},\ \widetilde{T})$ is also almost effective, i.e., property (1) holds. Moreover, we have each G_i is not a torus by property (1), i.e., each G_i is a compact, simply connected, simple Lie group. □

[2]We remark that $p^{-1}(T) = \widetilde{T}$ by [**13**, Theorem 4.9 in Chapter V].

Taking a maximal torus subgroup T_i in H_i, the maximal torus subgroup $\widetilde{T}$ is decomposed into

$$\widetilde{T} = T_1 \times \cdots \times T_k \subset H_1 \times \cdots \times H_k \subset G_1 \times \cdots \times G_k.$$

The following structure lemma holds.

Lemma 3.2. *Suppose that the torus manifold $(M,\ T)$ has a codimension 0 extended G-action and $M = G/H$ such that $T \subset H \subset G$. Put $\widetilde{G}/\widetilde{H} = G_1/H_1 \times \cdots \times G_k/H_k$, $\widetilde{T} = T_1 \times \cdots \times T_k$, and $G_i/H_i = M_i$ for all $i = 1,\ \cdots,\ k$, where $\widetilde{G}$ is a universal covering of G, its subgroup $\widetilde{H}$ (resp. $\widetilde{T}$) is the identity component of $p^{-1}(H)$ (resp. $p^{-1}(T)$), and T_i is a maximal torus subgroup in G_i and H_i. Then each factor $(M_i,\ T_i) = (G_i/H_i,\ T_i)$ is a torus manifold.*

Proof. By Lemma 3.1, $(\widetilde{G}/\widetilde{H},\ \widetilde{T})$ is a torus manifold and each G_i is a simply connected, simple Lie group. Because $(\widetilde{G}/\widetilde{H},\ \widetilde{T})$ is almost effective, we also have that $(G_i/H_i,\ T_i)$ is almost effective for all $i = 1,\ \cdots,\ k$. Since T_i is a maximal torus subgroup in G_i and H_i, we have $(G_i/H_i)^{T_i} \neq \emptyset$ for all $i = 1,\ \cdots,\ k$. Therefore, we have

$$2 \dim T_i \leq \dim(G_i/H_i)$$

for all $i = 1,\ \cdots,\ k$. Hence, the following equation holds:

$$2 \dim \widetilde{T} = \sum_{i=1}^{k} 2 \dim T_i \leq \sum_{i=1}^{k} \dim(G_i/H_i) = \dim \widetilde{G}/\widetilde{H}.$$

On the other hand $2 \dim \widetilde{T} = \dim \widetilde{G}/\widetilde{H}$. Consequently, we have $2 \dim T_i = \dim G_i/H_i$ for all $i = 1,\ \cdots,\ k$, and each factor $(M_i,\ T_i) = (G_i/H_i,\ T_i)$ is a torus manifold. □

From Lemma 3.2, in order to classify $\widetilde{G}/\widetilde{H}$, we need to consider each factor G_i/H_i constructed by a compact, simply connected, simple Lie group G_i and its maximal rank connected subgroup H_i, such that

$$\dim G_i/H_i = \dim G_i - \dim H_i = 2 \dim T_i = 2 \text{ rank } G_i = 2 \text{ rank } H_i.$$

In the next subsection we classify all codimension 0 extended G-actions of torus manifolds $(M,\ T^n)$ up to essential isomorphism.

3.2. Characterization of homogeneous torus manifolds. Let S be a compact, connected, simple Lie group, and S' a compact, connected, maximal rank, maximal subgroup of S. Here, a *maximal subgroup* means that if S'' is another compact, connected, maximal rank subgroup in S and there is an element $g \in S$ such that $S' \subset gS''g^{-1}$ then $S' = gS''g^{-1}$. For such S and S', the classification of these types, i.e., the Lie algebras of S and S', is known by classical Lie theory: see the Table 1 (see [**1**] or [**13**, Chapter V][3]).

Here, in the Table 1, $A_l \approx SU(l+1)$, $B_l \approx SO(2l+1)$, $C_l \approx Sp(l)$, $D_l \approx SO(2l)$ are the classical Lie groups and E_6, E_7, E_8, F_4, G_2 are the exceptional Lie groups

[3]The Table 1 is based on the list in [**13**]. In the list in [**1, 13**], the cases $i = 1$, l in A_l do not appear; however, we can easily check these cases should be included in their list by making use of [**13**, Theorem 7.16 in Chapter V]. Moreover, for the cases that i is not fixed in B_l, C_l and D_l in [**13**], we can easily check that indices of S' (G' in [**13**]) should be the indices in the above list by making use of [**13**, Theorem 7.16 in Chapter V].

Table 1. maximal rank subgroups

S	$A_l \quad (l \geq 1)$		$B_l \quad (l \geq 2)$
S'	$A_{i-1} \times A_{l-i} \times T^1 \quad (1 < i < l)$	$A_{l-1} \times T^1 \quad (i = 1,\ l)$	$B_{l-1} \times T^1 \quad (i = 1)$
S	$B_l \quad (l \geq 2)$		$C_l \quad (l \geq 3)$
S'	$D_i \times B_{l-i} \quad (1 < i < l)$	$D_l \quad (i = l)$	$C_i \times C_{l-i} \quad (1 \leq i < l)$
S	$C_l \quad (l \geq 3)$	$D_l \quad (l \geq 4)$	
S'	$A_{l-1} \times T^1 \quad (i = l)$	$D_{l-1} \times T^1 \quad (i = 1)$	$D_i \times D_{l-i} \quad (1 < i < l-1)$
S	$D_l \quad (l \geq 4)$	E_6	
S'	$A_{l-1} \times T^1 \quad (i = l-1,\ l)$	$D_5 \times T^1$	$A_1 \times A_5$
S	E_6	E_7	
S'	$A_2 \times A_2 \times A_2$	$D_6 \times A_1$	A_7
S	E_7		E_8
S'	$A_2 \times A_5$	$E_6 \times T^1$	D_8
S	E_8		
S'	A_8	$A_4 \times A_4$	$E_6 \times A_2$
S	E_8	F_4	
S'	$E_7 \times A_1$	$C_3 \times A_1$	$A_2 \times A_2$
S	F_4	G_2	
S'	B_4	A_2	$A_1 \times A_1$

(indices indicate their rank), and $S \approx X$ means S and X have the same Lie algebra. Note that each dimension of S is as follows:

$$\dim A_l = l^2 + 2l; \quad \dim B_l = \dim C_l = (2l+1)l; \quad \dim D_l = l(2l-1);$$
$$\dim E_6 = 78;\ \dim E_7 = 133;\ \dim E_8 = 248;\ \dim F_4 = 52;\ \dim G_2 = 14.$$

Therefore $\dim S/S'$ can be computed as in Table 2 below.

It follows easily from Table 2 that the following two cases are the only possible cases of $\dim S/S' = 2\ \mathrm{rank} S$:

$$A_l/(A_{l-1} \times T^1) \quad \text{and} \quad B_l/D_l.$$

If S'' is not a maximal subgroup but is a maximal rank compact connected subgroup, then S'' is a subgroup of a conjugation of one of the maximal subgroups S' in Table 1. Hence we have $\dim S/S'' > \dim S/S'$. By Table 2, we have $\dim S/S'' > \dim S/S' \geq 2\ \mathrm{rank} S$, hence such S'' does not occur. Moreover, if S'_1 and S'_2 are compact, connected, maximal rank, maximal compact subgroups in the compact, connected, simple Lie group S with same Lie algebra type, i.e., $S'_1 \approx S'_2$, then S'_1 and S'_2 are unique up to conjugation in S (see [**13**, Chapter V]). Therefore we have the following Lemma 3.3.

Lemma 3.3. *Let $(G_i/H_i,\ T_i)$ be a pair such that $G_i \supset H_i \supset T_i$ and $\dim G_i/H_i = 2\dim T_i = 2l$. Assume G_i is a connected, simple Lie group, H_i is a connected, closed subgroup, and* $\mathrm{rank}\ G_i = \mathrm{rank}\ H_i = \dim T_i$. *Then there are just the following two cases:*

$$G_i/H_i = SU(l+1)/S(U(l) \times U(1)) \cong \mathbb{C}P(l);$$
$$G_i/H_i = Spin(2l+1)/Spin(2l) \cong SO(2l+1)/SO(2l) \cong S^{2l}.$$

Table 2. dimension of S/S'

S/S'	$A_l/(A_{i-1}\times A_{l-i}\times T^1)$	$A_l/(A_{l-1}\times T^1)$	$B_l/(B_{l-1}\times T^1)$
dim	$2i(l-i+1);\quad 1<i<l$	$2l;\quad 1\le l$	$2(2l-1);\quad 2\le l$
S/S'	$B_l/(D_i\times B_{l-i})$	B_l/D_l	$C_l/(C_i\times C_{l-i})$
dim	$2i(2l-2i+1);\quad 1<i<l$	$2l;\quad 2\le l$	$4i(l-i);\quad 1\le i<l,\ 3\le l$
S/S'	$C_l/(A_{l-1}\times T^1)$	$D_l/(D_{l-1}\times T^1)$	$D_l/(D_i\times D_{l-i})$
dim	$l(l+1);\quad 3\le l$	$4(l-1);\quad 4\le l$	$4i(l-i);\quad 1<i<l-1,\ 4\le l$
S/S'	$D_l/(A_{l-1}\times T^1)$	$E_6/(D_5\times T^1)$	$E_6/(A_1\times A_5)$
dim	$l(l-1);\quad 4\le l$	32	40
S/S'	$E_6/(A_2\times A_2\times A_2)$	$E_7/(D_6\times A_1)$	E_7/A_7
dim	54	64	70
S/S'	$E_7/(A_2\times A_5)$	$E_7/(E_6\times T^1)$	E_8/D_8
dim	90	54	128
S/S'	E_8/A_8	$E_8/(A_4\times A_4)$	$E_8/(E_6\times A_2)$
dim	168	200	162
S/S'	$E_8/(E_7\times A_1)$	$F_4/(C_3\times A_1)$	$F_4/(A_2\times A_2)$
dim	112	28	36
S/S'	F_4/B_4	G_2/A_2	$G_2/(A_1\times A_1)$
dim	16	6	8

From Lemma 3.3, we have $\widetilde{G}=\prod_{i=1}^{a}SU(l_i+1)\times\prod_{j=1}^{b}Spin(2m_j+1)$. Because an $Spin(m)$-action can be identified with an $SO(m)$-action up to essential isomorphism (see Section 2.3 and Example 2.8), we can assume that

$$\widetilde{G}=\prod_{i=1}^{a}SU(l_i+1)\times\prod_{j=1}^{b}SO(2m_j+1),$$

where $a+b=k$.

Because $\widetilde{H}=H_1\times\cdots\times H_k$ is the identity component of $p^{-1}(H)$, we have the following relation:

$$\begin{aligned}&\widetilde{H}\subset p^{-1}(H)\subset N(\widetilde{H})\subset\widetilde{G};\\ &N(\widetilde{H})=\prod_{i=1}^{a}S(U(l_i)\times U(1))\times\prod_{j=1}^{b}S(O(2m_j)\times O(1)),\end{aligned}\tag{3.2}$$

where $N(\widetilde{H})$ is a normalizer group of $\widetilde{H}$ in $\widetilde{G}$. Hence, we have the following fibration for the torus manifold $G/H \cong \widetilde{G}/p^{-1}(H)$:

$$\mathcal{A} = p^{-1}(H)/\widetilde{H} \longrightarrow \widetilde{G}/\widetilde{H} \longrightarrow \widetilde{G}/p^{-1}(H) \cong G/H, \tag{3.3}$$

where $p^{-1}(H)/\widetilde{H} = \mathcal{A}$ is a subgroup in $N(\widetilde{H})/\widetilde{H} \simeq \prod_{j=1}^{b} \mathbb{Z}_2$, because of $S(O(2m_j)\times O(1))/SO(2m_j) \simeq \mathbb{Z}_2$ and the above (3.2). Note that we can regard $\mathbb{Z}_2$ as the group generated by the antipodal involution on S^{2m_j}, i.e., $\mathbb{Z}_2 \simeq O(2m_j+1)/SO(2m_j+1) = \{I_{2m_j+1},\ -I_{2m_j+1}\}$. Therefore, $\mathcal{A}$ acts on $\prod_{j=1}^{b} S^{2m_j}$ freely. Moreover, we have that this $\mathcal{A}$-action on $\prod_{j=1}^{b} S^{2m_j}$ is orientation preserving if and only if $\mathcal{A} \subset SO(2m_1 + \cdots + 2m_b + b)$. Consequently, we have the following theorem.

Theorem 3.4. *Suppose a torus manifold $(M^{2n},\ T^n)$ extends to a codimension 0 extended G-action, where G is a compact, connected Lie group whose maximal torus is T^n. Then $(M^{2n},\ G)$ is essentially isomorphic to*

$$\left(\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \frac{\prod_{j=1}^{b} S^{2m_j}}{\mathcal{A}},\ \prod_{i=1}^{a} PU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1)\right),$$

where $\mathcal{A}$ can be any subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$ whose factor $\mathbb{Z}_2 = \{I_{2m_j+1},\ -I_{2m_j+1}\}$ for $j = 1,\ \cdots,\ b$, and $\prod_{i=1}^{a} PU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1)$ acts on $\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b} S^{2m_j}/\mathcal{A}$ in the natural way, and $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n$.

Furthermore, M^{2n} is orientable if and only if $\mathcal{A} \subset SO(2m_1 + \cdots + 2m_b + b)$.

Here, we give two examples.

Example 3.5. Let $I = I_{2m+b}$ be the identity element in $O(2m+b)$. We have $\{I,\ -I\} \subset \prod_{j=1}^{b} \mathbb{Z}_2 \subset O(2m+b)$, and the following manifold is one of the homogeneous torus manifolds in Theorem 3.4:

$$\mathbb{R}P_b(2m) = (\prod_{j=1}^{b} S^{2m_j})/\{I,\ -I\},$$

where $\{I,\ -I\}$ acts on $\prod_{j=1}^{b} S^{2m_j} \subset \prod_{j=1}^{b} \mathbb{R}^{2m_j+1} = \mathbb{R}^{2m+b}$ canonically. Note that if $b = 1$, this manifold $\mathbb{R}P_1(2m)$ is an even dimensional real projective space $\mathbb{R}P(2m)$. If b is even (resp. odd), then $\{I,\ -I\} \subset SO(2m+b)$ (resp. $\{I,\ -I\} \not\subset SO(2m+b)$). Therefore the following two conditions are equivalent by Theorem 3.4:

(1) $\mathbb{R}P_b(2m)$ is orientable;
(2) b is even.

Example 3.6. Let us consider the product of three spheres $S^{2m_1} \times S^{2m_2} \times S^{2m_3} \subset \mathbb{R}^{2m_1+1} \times \mathbb{R}^{2m_2+1} \times \mathbb{R}^{2m_3+1}$. We define the group $\mathcal{A}$ as follows:

$$\begin{aligned}
\mathcal{A} &= \langle(-I_{2m_1+1},\ -I_{2m_2+1},\ I_{2m_3+1}),\ (I_{2m_1+1},\ -I_{2m_2+1},\ -I_{2m_3+1})\rangle \\
&= \{(I_{2m_1+1},\ I_{2m_2+1},\ I_{2m_3+1}),\ (-I_{2m_1+1},\ -I_{2m_2+1},\ I_{2m_3+1}), \\
&\quad\ (I_{2m_1+1},\ -I_{2m_2+1},\ -I_{2m_3+1}),\ (-I_{2m_1+1},\ I_{2m_2+1},\ -I_{2m_3+1})\} \\
&\simeq \mathbb{Z}_2 \times \mathbb{Z}_2,
\end{aligned}$$

where $I_{2m_j+1} \in O(2m_j+1)$ is the identity element. Because $\mathcal{A} \subset SO(2m_1+2m_2+2m_3+3)$, we see that $(S^{2m_1} \times S^{2m_2} \times S^{2m_3})/\mathcal{A}$ is a homogeneous torus manifold and orientable by Theorem 3.4.

We can easily show the following corollaries.

Corollary 3.7. *If a simply connected torus manifold has a codimension* 0 *extended* G*-action,* $(M^{2n},\ G)$ *is essentially isomorphic to*

$$\left(\prod_{i=1}^{a}\mathbb{C}P(l_i)\times\prod_{j=1}^{b}S^{2m_j},\ \prod_{i=1}^{a}PU(l_i+1)\times\prod_{j=1}^{b}SO(2m_j+1)\right),$$

where $\sum_{i=1}^{a} l_i+\sum_{j=1}^{b} m_j=n$.

Remark 3.8. We can assume $m_j\geq 2$ in the above Corollary 3.7, because $(S^2,\ SO(3))$ and $(\mathbb{C}P(1),\ PU(2))$ are essentially isomorphic. Therefore, diffeomorphism types of simply connected, homogeneous torus manifolds can be completely determined by sequences $(l_1,\ \cdots,\ l_a)$ and $(m_1,\ \cdots,\ m_a)$ such that $0<l_1\leq\cdots\leq l_a,\ 2\leq m_1\leq\cdots\leq m_b,\ \sum_{i=1}^{a} l_i+\sum_{j=1}^{b} m_j=n$. We also remark that these sequences do not determine the omniorientation on M.

The set of torus manifolds is a topological generalization of *compact non-singular toric varieties*, and this set also contains all *quasitoric manifolds*. As is well known, every quasitoric manifold is simply connected and their cohomology rings are generated by second degree cohomology classes (see [**6**] for the case of toric varieties, and [**3**, Theorem 5.18], or [**4**, Theorem 4.14] for the case of quasitoric manifolds). Consequently, we have the following corollary by Corollary 3.7.

Corollary 3.9. *If a compact non-singular toric variety or a quasitoric manifold* $(M^{2n},\ T^n)$ *has a codimension* 0 *extended* G*-action, then* $(M^{2n},\ G)$ *is essentially isomorphic to*

$$\left(\prod_{i=1}^{a}\mathbb{C}P(l_i),\ \prod_{i=1}^{a}PU(l_i+1)\right),$$

where $\sum_{i=1}^{a} l_i=n$.

Remark 3.10. If a compact algebraic variety has a codimension 0 extended compact G-action, then this variety is non-singular. Hence, in this case we may omit the assumption of non-singularity in the above corollary.

4. On other T-manifolds

Finally, in this section, we give an application of the above argument for other T-manifolds $(M,\ T)$.

Suppose a T^m-manifold $(M^{2n},\ T^m)$ extends to a transitive G-action, where G has T^m as its maximal torus. Assume that this T^m-action is almost effective and has finitely many fixed points. Then we have $n\geq m$ (if $n=m$ then M is a torus manifold). From the same argument as in Section 2.2, we also have $M\cong G/H$ such that $T^m\subset H^o\subset G$ and $\operatorname{rank} G=\operatorname{rank} H^o=m$. Therefore we can apply the same argument as for torus manifolds, and we obtain the diffeomorphism type of M^{2n}.

For example, applying the above argument for $(M^{4n},\ T^{n+1})$, we have the following proposition, where a *decomposable manifold* M means that the manifold M is diffeomorphic to $M_1\times M_2$ such that $\dim M_1,\ \dim M_2\neq 0$.

Proposition 4.1. *Assume that* $(M^{4n},\ T^{n+1})$ *has finitely many fixed points,* M *is simply connected, and that the* T^{n+1}*-action is almost effective. If* $(M^{4n},\ T^{n+1})$

has a codimension 0 *extended* G*-action (where rank* $G = n+1$*) and* M *is not a decomposable manifold, then* M^{4n} *is diffeomorphic to one of the followings:*

$$\begin{aligned} G_2(\mathbb{C}^{n+2}) &= SU(n+2)/S(U(n)\times U(2)); \\ \mathbb{H}P(n) &= Sp(n+1)/(Sp(n)\times Sp(1)); \\ Q_{2n} &= SO(2n+2)/(SO(2n)\times SO(2)), \end{aligned}$$

where $G_2(\mathbb{C}^{n+2})$ *is the complex Grassmannian of* 2*-planes in* $\mathbb{C}^{n+2}$*,* $\mathbb{H}P(n)$ *is the quaternionic projective space, and* Q_{2n} *is the complex quadric.*

Proof. Because M is not decomposable, we can assume that G is a simply connected, compact, simple Lie group. First we assume $M = G/H$. With a method similar to that demonstrated in Section 2.2, we can easily show that rank $G =$ rank H. Because M is simply connected, we have $H = H^o$. Assume H is a maximal compact subgroup. In this case, we will see from Table 2 that the pair $(G,\ H)$ such that rank $G =$ rank $H = n+1$ and $\dim G = \dim H = 4n$ is one of the three pairs in the statement of this proposition (remark that $(B_2,\ D_2) \approx (C_2,\ C_1\times C_1)$ and $(D_4,\ A_3\times T^1) \approx (D_4,\ D_3\times T^1)$).

Next we assume $M = G/K$ and $K = K^o$ is not maximal. Then we have $\dim G/K > \dim G/H$ where H is maximal and rank $G =$ rank $H =$ rank $K = n+1$. If G is one of the next Lie groups: $B_{n+1},\ E_6,\ E_7,\ E_8,\ F_4,\ G_2$, then we always have $\dim G/H > 4n$ from the list in Table 2. Therefore we can assume that G is one of the next three Lie groups: $A_{n+1},\ C_{n+1},\ D_{n+1}$.

If G is A_{n+1}, then an inequality $\dim G/H < 4n$ holds only when $A_{n+1}/(A_n\times T^1)$ from the list in Table 2. Thus, we have $G \supset H \supset K$ where $H \approx A_n\times T^1$. If K is maximal in such subgroup, we also have $K \approx (A_{j-1}\times A_{n-j}\times T^1)\times T^1$ for $1 \le j \le n$. Then we can easily check that $\dim G/K > 4n$. Hence this case does not occur.

If G is C_{n+1} or D_{n+1}, then we always have $\dim G/H \ge 4n$ from the list in Table 2. Hence only the maximal case occurs. $\square$

Remark 4.2. In general $(M^{4n},\ T^{n+1})$ can be decomposed into a product $(M_1\times \cdots \times M_k,\ T_1\times\cdots\times T_k)$, but we can easily prove that the type decomposition of Lemma 3.2 does not hold (except $k=1$). For example $(S^6\times S^2,\ T^2\times T^1)$ is one of the elements in the class $(M^{4n},\ T^{n+1})$ for $n=2$, and it has the codimension 0 extended $G_2\times SO(3)$-action, where $S^6 = G_2/SU(3)$ and $S^2 = SO(3)/SO(2)$. However, the two factors $(S^6,\ T^2)$ and $(S^2,\ T^1)$ are not in the class $(M^{4n},\ T^{n+1})$.

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References

[1] A. Borel and J. de Siebenthal: *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Comment. Helv., **23** (1949), 200–221.

[2] G.E. Bredon: Introduction to compact transformation groups, Academic Press, 1972.

[3] V.M. Buchstaber and T.E. Panov: Torus actions and their applications in topology and combinatorics, Amer. Math. Soc., 2002.

[4] M. Davis and T. Januszkiewicz: *Convex polytopes, Coxeter orbifolds and torus action*, Duke. Math. J., **62** (1991), no. 2, 417–451.

[5] M. Demazure: *Sous-groups algebriques de rang maximum du group de Cremona*, Ann. Sci. École Norm. Sup. (4) 3(1970), 507–588.
[6] W. Fulton: An Introduction to Toric Varieties, Ann. of Math. Studies **113**, Princeton Univ. Press, Princeton, N.J., 1993.
[7] V. Guillemin, T. S. Holm and C. Zara: *A GKM description of the equivariant cohomology ring of a homogeneous space*, J. Algebraic Combin., 23 (2006), no. 1, 21–41.
[8] A. Hattori and M. Masuda: *Theory of multi-fans*, Osaka. J. Math., **40** (2003), 1–68.
[9] K. Kawakubo: The theory of transformation groups, Oxford Univ. Press, London, 1991.
[10] S. Kuroki: *On transformation groups which act on torus manifolds*, Proceedings of 34th Symposium on Transformation Groups, 10–26, Wing Co., Wakayama, 2007.
[11] M. Masuda: *Unitary toric manifolds, multi-fans and equivariant index*, Tôhoku Math. J., **51** (1999), 237–265.
[12] M. Masuda and T. E. Panov: *On the cohomology of torus manifolds*, Osaka. J. Math., **43** (2006), 711–746.
[13] M. Mimura and H. Toda: Topology of Lie Groups, I and II, Amer. Math. Soc., 1991.

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