TOPOLOGICAL CLASSIFICATION OF TORUS MANIFOLDS WHICH HAVE CODIMENSION ONE EXTENDED G-ACTIONS

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Abstract. The problem whether cohomology ring determines topological types of (quasi)toric manifolds (the cohomological rigidity problem) is still open problem and some affirmative evidences are known by the first authors. On the other hand, the second author classified torus manifolds (the generalization of (quasi)toric manifolds) with codimension one extended G-actions up to G-equivariantly homeomorphism. They are equivariantly homeomorphic to the sphere bundle or projective bundle over some torus manifold. The goal of this paper is to classify such torus manifolds up to homeomorphism (that is, we forget the G-action). As a result, we show that their topological types are completely determined by their cohomology rings and characteristic classes. Due to this result, we find the counterexample of the cohomological rigidity problem in the category of torus manifolds. Moreover, we find the class of manifolds in torus manifolds with codimension one extended G-actions which is not in the class of (quasi)toric manifolds but cohomologically rigid.

Contents

1. Introduction

A *toric variety* of dimension n is a normal algebraic variety on which an algebraic torus $(\mathbb{C}^*)^n$ acts with a dense orbit. In this paper, we call a compact non-singular toric variety

2000 *Mathematics Subject Classification.* Primary 57S25; Secondary 22F30.

Key words and phrases. Sphere bundle, Complex projective bundle, Torus manifold, Non-singular toric variety, Quasitoric manifold, Toric topology.

a *toric manifold*. We regard the compact torus T^n as the standard compact subgroup in $(\mathbb{C}^*)^n$. The orbit space of a toric manifold with T^n can be identified with the simple polytope and the action of T^n on a toric manifold is *locally standard*, that is, locally modelled by the standard action on $\mathbb{C}^n$. Davis and Januszkiewicz first introduced the notion of a toric manifold as a topological counterpart, which is now called a *quasitoric manifold*, by taking these two characteristic properties as the starting point (see [1], [5]). A quasitoric manifold is a smooth closed manifold of dimension $2n$ with a locally standard T^n-action whose orbit space is a simple polytope. Obviously not all quasitoric manifolds belong to the family of toric manifolds. For instance, a connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ of two $\mathbb{C}P^2$'s is a quasitoric manifold with an appropriate action of T^2 but not a toric manifold because it does not allow an almost complex structure. Moreover, the family of quasitoric manifolds does not contain the family of toric manifolds entirely, but the theory can be extended to a certain family of manifolds containing both toric manifolds and quasitoric manifolds. As an ultimate generalization of (quasi)toric manifolds, Hattori and Masuda introduced a *torus manifold* (or *unitary toric manifold* in the earlier terminology) in [7], [11], which is an oriented, closed smooth manifold of dimension $2n$ with an effective T^n-action with a non-empty fixed point set. Among the definition of torus manifolds, if a torus manifold M satisfies the following two conditions:

1. $H^{odd}(M) = 0$ (resp. $H^*(M)$ is generated by $H^2(M)$ as ring);
2. M, M_i's and connected components of any multiple intersection of M_i's are all simply connected,

where M_i's are characteristic submanifolds of M, then the orbit space is a homotopy cell (resp. homotopy polytope) and the converse also holds (see [13]). Thus Masuda and Suh believe in [15] that the toric theory can be developed to the family of torus manifolds satisfying the two conditions above in the topological category in a nice way.

On the other hand, the topological classification of these manifolds has recently attracted the attention of toric topologists. Masuda and Suh ask several problems in [15]. Of special interest is the following problem which is now called a *cohomological rigidity problem* for toric manifolds:

Problem 1.1. *Are toric manifolds diffeomorphic (or homeomorphic) if their cohomology rings are isomorphic as graded rings?*

They also ask the problem for quasitoric manifolds and torus manifolds satisfying the two conditions above. Because there are yet no negative answer to Problem 1.1 for toric or quasitoric manifolds but there are some affirmative evidences, Problem 1.1 has been still open in the toric or quasitoric categories (see [2], [13]). In the present paper, we find negative answers to Problem 1.1 for the family of torus manifolds each of whose orbit space is a homotopy cell. In order to do, we study a topological classification of torus manifolds which have extended G-actions, where G is a compact, connected Lie group with maximal torus T^n. Recently the second author has classified the torus manifolds which have extended G-actions with codimension 0 or 1 principal orbits in [8], [9], [10]. The (simply connected) torus manifold which has a codimension 0 extended G-action (i.e., transitive G-action) is a product of complex projective spaces and spheres. This is nothing but a product of projective spaces and spheres from the (non-equivariant) topological point of view. Thus, this paper will be restricted to consideration of the topological types of the torus manifolds which have extended G-actions with codimension 1 principal orbits. We denote such class as $\mathfrak{M}$. The aim of this paper is to prove the following theorem (see Corollary 4.2 in Section 4).

Theorem 1.2. *Homeomorphism types of $\mathfrak{M}$ are completely determined by their cohomology rings, Pontrjagin classes and Stiefel-Whitney classes.*

This paper is organized as follows. In Section 2, we recall the results of [9], [10] and prepare notations. In Section 3, we compute cohomology rings and characteristic classes of these manifolds. In Section 4, we present the main result of this paper. Finally, in Section 5, we prove the main result and exhibit several non-trivial examples.

2. Notations

We first recall the definition of torus manifolds (see [7], [11]). Let M be a $2n$-dimensional, oriented, closed manifold with an effective half dimensional torus T^n action. We call M a *torus manifold* if its fixed point set M^T is non empty. Remark that in [7], [11], we need to choice the *omniorientations*, namely, an orientation of the torus manifold and its characteristic submanifolds, on torus manifolds. However, in this paper we do not assume the omniorientations on torus manifolds because we focus only on the topological types of torus manifolds.

Next we recall the definition of quasitoric manifolds (see [1], [5]). If the torus manifold M^{2n} satisfies the following two properties:

(1) T^n-action is *locally standard*, that is, locally looks like the standard torus representation in $\mathbb{C}^n$;
(2) there is a projection map $M^{2n} \to P^n$ constant on T^n-orbits which maps every k-dimensional orbit to a point in the interior of k-dimensional face of P^n for $k = 0, \ldots, n$, where P^n is a convex, simple, n-dimensional polytope,

then we call M^{2n} a *quasitoric manifold.*

Let $\widetilde{\mathfrak{M}}$ be the set of simply connected torus manifolds M^{2n} which have extended G-actions with codimension 1 principal orbits, where G is a compact, connected Lie group with maximal torus T^n. Due to the main results in [9], [10], $\widetilde{\mathfrak{M}}$ consists of the following three types of manifolds:

- TYPE 1: $\prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} S^{2\ell_i+1} \times_{(S^1)^a} P(\mathbb{C}^{k_1}_{\rho} \oplus \mathbb{C}^{k_2})\right)$;
- TYPE 2: $\prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} S^{2\ell_i+1} \times_{(S^1)^a} S(\mathbb{C}^{k}_{\rho} \oplus \mathbb{R})\right)$;
- TYPE 3: $\prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} S^{2\ell_i+1} \times_{(S^1)^a} S(\mathbb{C}^{k_1}_{\rho} \oplus \mathbb{R}^{2k_2+1})\right)$,

where $P(\mathbb{C}^{k_1}_{\rho} \oplus \mathbb{C}^{k_2}) = (\mathbb{C}^{k_1}_{\rho} \oplus \mathbb{C}^{k_2} - \{0\})/\mathbb{C}^*$ is a complex projective space, $S(\mathbb{C}^{\ell} \oplus \mathbb{R}^m) \subset \mathbb{C}^{\ell} \oplus \mathbb{R}^m$ is a sphere, and $(S^1)^a$ acts on $\prod_{i=1}^{a} S^{2\ell_i+1}$ naturally and on $\mathbb{C}^{k_1}_{\rho}$ through the following representation $\rho : (S^1)^a \to S^1$:

$$\rho(t_1, \ldots, t_a) = t_1^{\alpha_1} \cdots t_a^{\alpha_a}$$

for $\alpha_i \in \mathbb{Z}$ $(i = 1, \ldots, a)$.

In the remainder of the paper, we assume that $a = 1$ and $b = 0$ and $\mathfrak{M}$ denotes the subset of $\widetilde{\mathfrak{M}}$ which satisfied $a = 1$ and $b = 0$. Let $\mathfrak{M}_i$ $(i = 1, 2, 3)$ be the subset of $\mathfrak{M}$ of TYPE i with $a = 1$ and $b = 0$. By the definition of $\mathfrak{M}_i$, the element $N_i \in \mathfrak{M}_i$ is as follows:

$$N_1 = S^{2\ell+1} \times_{S^1} P(\mathbb{C}^{k_1}_{\rho} \oplus \mathbb{C}^{k_2});$$
$$N_2 = S^{2\ell+1} \times_{S^1} S(\mathbb{C}^{k}_{\rho} \oplus \mathbb{R});$$
$$N_3 = S^{2\ell+1} \times_{S^1} S(\mathbb{C}^{k_1}_{\rho} \oplus \mathbb{R}^{2k_2+1}).$$

Defining $\rho : S^1 \to S^1$ as $t \mapsto t^\rho$, we may regard $\rho \in \mathbb{Z}$. First we compute the cohomology rings and characteristic classes of N_i's. In order to compute them, we use the following standard symbols: $H^*(X)$ is the cohomology ring of X over $\mathbb{Z}$-coefficients; $w(X)$ (resp. $w_i(X)$) is the total (resp. i-th) Stiefel-Whitney class of X; and $p(X)$ (resp. $p_i(X)$) is the total (resp. i-th) Pontrjagin class of X. Moreover, $\mathbb{Z}[x_1, \ldots, x_m]$ denotes the polynomial ring generated by x_j ($j = 1, \ldots, m$), and $< f_1(x_1, \ldots, x_m),\ \ldots,\ f_s(x_1, \ldots, x_m) >$ denotes the ideal in $\mathbb{Z}[x_1, \ldots, x_m]$ generated by the polynomials $f_j(x_1, \ldots, x_m)$ ($j = 1, \ldots, s$). The symbol $E(\eta)$ represents the total space for the fibre bundle η.

3. Topological invariants

In this section, we will compute the following three topological invariants of N_i ($i = 1, 2, 3$):

- cohomology rings $H^*(N_i)$;
- Stiefel-Whiteny classes $w(N_i)$;
- Pontrjagin classes $p(N_i)$.

3.1. **Topological invariants of N_1.** The goal of this subsection is to compute topological invariants of

$$N_1 = S^{2\ell+1} \times_{S^1} P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}).$$

In order to compute them, we first recall the torus action on N_1, where in this case the dimension of torus is $\ell + k_1 + k_2 - 1$. The torus action on N_1 is defined as follows ($k_1, k_2 \geq 1$):

$$\begin{aligned} & (a_1, \ldots, a_\ell, b_1, \ldots, b_{k_1}, c_1, \ldots, c_{k_2-1}) \cdot [(x_0, \ldots, x_\ell), [y_1; \ldots; y_{k_1}; y_1'; \ldots; y_{k_2}']] \\ = \ & [(x_0,\ a_1x_1, \ldots, a_\ell x_\ell), [b_1y_1; \cdots; b_{k_1}y_{k_1}; c_1y_1'; \cdots; c_{k_2-1}y_{k_2-1}'; y_{k_2}']], \end{aligned}$$

where $a_i,\ b_j,\ c_k \in S^1$ and $(x_0, \ldots, x_\ell) \in S^{2\ell+1} \subset \mathbb{C}^{\ell+1}$, $[y_1; \cdots; y_{k_1}; y_1'; \cdots; y_{k_2}'] \in P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2})$. By this torus action, we can easily check that this manifold N_1 is a quasitoric manifold over $\Delta^\ell \times \Delta^{k_1+k_2-1}$ (product of two simplices) whose dimension is $2\ell + 2k_1 + 2k_2 - 2$. Therefore, we can use the *Davis-Januszkiewicz formula* in [5, Theorem 4.14, Corollary 6.8] for computing topological invariants of quasitoric manifolds.

3.1.1. *The Davis-Januszkiewicz formula.* Next we quickly review the Davis-Januszkiewicz formula for topological invariants (see [1], [5] for details). The equivariant cohomology of quasitoric manifolds M^{2n} can be described as follows:

$$H^*(ET \times_T M;\ \mathbb{Z}) = H^*_{T^n}(M^{2n};\ \mathbb{Z}) \simeq \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I},$$

where v_j ($\deg v_j = 2,\ j = 1, \ldots, m$) is the equivariant Poincaré dual of codimension two invariant submanifold M_j in M^{2n} (*characteristic submanifolds*) and $\mathcal{I}$ is an ideal of the polynomial ring $\mathbb{Z}[v_1, \ldots, v_m]$ generated by $\{\prod_{j \in I} v_j \mid \bigcap_{j \in I} M_j = \emptyset\}$. We call $\mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}$ *the face ring* of $M/T = P$. Let $\pi : ET \times_T M \to BT$ be the natural projection. Then, we can define the induced homomorphism

$$\pi^* : H^*(BT;\ \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_n] \longrightarrow H^*(ET \times_T M;\ \mathbb{Z}) = H^*_{T^n}(M^{2n};\ \mathbb{Z}).$$

Moreover, the π^*-image of t_i, $\pi^*(t_i)$ ($i = 1, \ldots, n$), can be described by the information of the T-action on M. Such information is called *the characteristic matrix* $\Lambda = (\lambda_1 \cdots \lambda_m) \in M(n, m;\ \mathbb{Z})$ (the set of $n \times m$ integer matrices), where $\lambda_j \in \mathbb{Z}^n$ ($j = 1, \ldots, m$) corresponds with the generator of Lie algebra of isotropy subgroup of characteristic submanifold M_j. Put $\lambda_j = (\lambda_{1j} \ \cdots \ \lambda_{nj})^t \in \mathbb{Z}^n$. Then we can describe $\pi^*(t_i)$ $(i = 1, \ldots, n)$ as follows:

$$\pi^*(t_i) = \sum_{j=1}^{m} \lambda_{ij} v_j.$$

Let $\mathcal{J}$ be the ideal in $\mathbb{Z}[v_1, \ldots, v_m]$ generated by $\pi^*(t_i)$ for all $i = 1, \ldots, n$. Then the ordinary cohomology of quasitoric manifolds are described as follows:

$$H^*(M^{2n};\ \mathbb{Z}) \simeq \mathbb{Z}[v_1, \ldots, v_m]/(\mathcal{I} + \mathcal{J}). \tag{1}$$

Moreover, for an inclusion $\iota : M \to ET \times_T M$, the Pontrjagin class[1] and the Stiefel-Whitney class can be described as follows:

$$p(M) = \iota^* \prod_{i=1}^{m} (1 + v_i^2), \tag{2}$$

$$w(M) = \iota^* \prod_{i=1}^{m} (1 + v_i). \tag{3}$$

3.1.2. *Topological invariants of* N_1. Now we may compute the topological invariants of N_1. In order to use the Davis-Januszkiewicz formula, we need to compute the characteristic matrix of N_1. By the definition of the torus action on N_1, the characteristic matrix of N_1 is as follows:

$$\begin{pmatrix} I_\ell & 0 & 0 & \mathbf{1} & 0 \\ 0 & I_{k_1} & 0 & \rho\mathbf{1} & \mathbf{1} \\ 0 & 0 & I_{k_2-1} & 0 & \mathbf{1} \end{pmatrix} \in M(n,\ n+2;\ \mathbb{Z}), \tag{4}$$

where $n = \ell + k_1 + k_2 - 1$ and $\mathbf{1} = (1, \ldots, 1)^t \in M(r,\ 1;\ \mathbb{Z})$ for $r = \ell, k_1$ and k_2.

Now we may compute the topological invariants of N_1. Because the equivariant cohomology ring $H^*_{T^n}(N_1)$ is the face ring of $\Delta^\ell \times \Delta^{k_1+k_2-1}$, we have that

$$H^*_{T^n}(N_1) \simeq \mathbb{Z}[v_1, \ldots, v_{\ell+1}, w_1, \ldots, w_{k_1+k_2}]/\mathcal{I}, \tag{5}$$

where $\deg v_i = \deg w_j = 2$. Moreover, by the definition of the torus action, we can easily check that the ideal $\mathcal{I}$ (see Section 3.1.1) is generated by

$$v_1 \cdots v_{\ell+1}$$

and

$$w_1 \cdots w_{k_1+k_2}.$$

Now the ideal $\mathcal{J}$ (see Section 3.1.1) is generated by the following elements by using (4):

$$\begin{aligned} & v_1 + v_{\ell+1},\ \ldots,\ v_\ell + v_{\ell+1}, \\ & w_1 + \rho v_{\ell+1} + w_{k_1+k_2},\ \ldots,\ w_{k_1} + \rho v_{\ell+1} + w_{k_1+k_2}, \\ & w_{k_1+1} + w_{k_1+k_2},\ \ldots,\ w_{k_1+k_2-1} + w_{k_1+k_2}. \end{aligned} \tag{6}$$

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[1] In [5, Corollary 6.8], the Pontrjagin class of quasitoric manifolds (*toric manifolds* in [5]) is $\iota^* \prod_{i=1}^m (1 - v_i^2)$. However, this formula coincides with $1 - p_1(M) + p_2(M) - \cdots = \sum_{i=0}^m (-1)^i p_i(M)$. Therefore, by [15], the Pontrjagin class of quasitoric manifolds must be $p(M) = 1 + p_1(M) + p_2(M) + \cdots = \iota^* \prod_{i=1}^m (1 + v_i^2)$

By using (1), (5) and (6), we have the following formula:

$$\begin{aligned} H^*(N_1) &\simeq \mathbb{Z}[v_1,\ldots,v_{\ell+1},w_1,\ldots,w_{k_1+k_2}]/(\mathcal{I}+\mathcal{J}) \\ &\simeq \mathbb{Z}[v_{\ell+1},w_{k_1+k_2}]/<(-1)^\ell(v_{\ell+1})^{\ell+1},(-w_{k_1+k_2}-\rho v_{\ell+1})^{k_1}(-w_{k_1+k_2})^{k_2}> \\ &\simeq \mathbb{Z}[x,y]/<x^{\ell+1},y^{k_2}(y+\rho x)^{k_1}>, \end{aligned}$$

where $x=v_{\ell+1}$, $y=w_{k_1+k_2}$.

Because of (2), (3) and (6), we have the characteristic classes of N_1 as follows:

$$\begin{aligned} p(N_1) &= (1+v_{\ell+1}^2)^{\ell+1}(1+(\rho v_{\ell+1}+w_{k_1+k_2})^2)^{k_1}(1+w_{k_1+k_2}^2)^{k_2} \\ &= (1+x^2)^{\ell+1}(1+(\rho x+y)^2)^{k_1}(1+y^2)^{k_2}; \\ w(N_1) &\equiv_2 (1+v_{\ell+1})^{\ell+1}(1+\rho v_{\ell+1}+w_{k_1+k_2})^{k_1}(1+w_{k_1+k_2})^{k_2} \\ &\equiv_2 (1+x)^{\ell+1}(1+\rho x+y)^{k_1}(1+y)^{k_2}. \end{aligned}$$

In summary, we have the following proposition.

Proposition 3.1. *Topological invariants of N_1 are as follows:*

$$\begin{aligned} H^*(N_1) &= \mathbb{Z}[x,y]/<x^{\ell+1},y^{k_2}(y+\rho x)^{k_1}>; \\ p(N_1) &= (1+x^2)^{\ell+1}(1+(\rho x+y)^2)^{k_1}(1+y^2)^{k_2}; \\ w(N_1) &\equiv_2 (1+x)^{\ell+1}(1+\rho x+y)^{k_1}(1+y)^{k_2}; \end{aligned}$$

where $\deg x=\deg y=2$ *and* $\ell,k_1,k_2\in\mathbb{N}$.

3.2. **Topological invariants of N_2 and N_3.** Recall the following two manifolds:

$$N_2 = S^{2\ell+1}\times_{S^1} S(\mathbb{C}_\rho^k\oplus\mathbb{R});$$
$$N_3 = S^{2\ell+1}\times_{S^1} S(\mathbb{C}_\rho^{k_1}\oplus\mathbb{R}^{2k_2+1}).$$

The goal of this subsection is to prove the following propositions.

Proposition 3.2. *Topological invariants of N_2 are as follows:*

$$\begin{aligned} H^*(N_2) &= \mathbb{Z}[x,z]/<x^{\ell+1},z(z+(\rho x)^k)>; \\ p(N_2) &= (1+x^2)^{\ell+1}(1+\rho^2x^2)^k; \\ w(N_2) &\equiv_2 (1+x)^{\ell+1}(1+\rho x)^k, \end{aligned}$$

where $\deg x=2$, $\deg z=2k$ *and* $\ell,k\in\mathbb{N}$.

Proposition 3.3. *Topological invariants of N_3 are as follows:*

$$\begin{aligned} H^*(N_3) &= \mathbb{Z}[x,z]/<x^{\ell+1},z^2>; \\ p(N_3) &= (1+x^2)^{\ell+1}(1+\rho^2x^2)^{k_1}; \\ w(N_3) &\equiv_2 (1+x)^{\ell+1}(1+\rho x)^{k_1}, \end{aligned}$$

where $\deg x=2$, $\deg z=2(k_1+k_2)$ *and* $\ell,k_1,k_2\in\mathbb{N}$.

Now N_2 has the following fibration:

$$S^{2k}=S(\mathbb{C}_\rho^k\oplus\mathbb{R})\longrightarrow N_2\xrightarrow{\pi} S^{2\ell+1}/S^1\cong \mathbb{C}P(\ell),$$

that is, N_2 is a sphere bundle over a complex projective space. Therefore, we can use the following lemma ([6, Lemma 4]):

Lemma 3.4. *Let $\pi : E \to M$ be a smooth, oriented r-sphere bundle over an oriented manifold M which has a section $s : M \to E$. Let the normal bundle ν of the embedding s be oriented by π, and let $\chi(\nu) \in H^r(M)$ be the Euler class of ν with respect to this orientation. Then there exists a unique class $z \in H^r(E)$ such that*

$$s^*(z) = 0 \in H^*(M) \text{ and } < i^*(z),\ [S^r] >= 1.$$

Furthermore $H^*(E)$, as a module over $H^*(M)$, has the basis $\{1,\ z\}$ subject to the relation

$$z^2 + \pi^*(\chi(\nu))z = 0;$$

We define the section of $\pi : N_2 \to \mathbb{C}P(\ell)$ as follows:

$$s : \mathbb{C}P(\ell) \ni [z_0; \ldots ; z_\ell] \mapsto ([z_0; \ldots ; z_\ell],\ (0, \ldots , 0,\ 1)) \in N_2,$$

where $(0, \ldots , 0,\ 1) \in S(\mathbb{C}^k_\rho \oplus \mathbb{R})$. We can easily check this map is well-defined because $(0, \ldots , 0,\ 1) \in S(\mathbb{C}^k_\rho \oplus \mathbb{R})$ is one of the fixed points of the S^1-action on $S(\mathbb{C}^k_\rho \oplus \mathbb{R})$. Then the normal bundle of this section is isomorphic to the following bundle ξ_ρ:

$$\mathbb{C}^k \longrightarrow S^{2\ell+1} \times_{S^1} \mathbb{C}^k_\rho \longrightarrow \mathbb{C}P(\ell),$$

where $\rho : S^1 \to S^1$ defined by $t \to t^\rho$. Then, we have that

$$\xi_\rho \equiv \gamma_\rho^{\oplus k}$$

where $E(\gamma_\rho) = S^{2\ell+1} \times_{S^1} \mathbb{C}_\rho$ such that S^1 acts on $\mathbb{C}_\rho$ by the representation $t \mapsto t^\rho$. If $\rho = 1$, then γ_ρ is isomorphic to the orientation reversing, canonical line bundle over $\mathbb{C}P(\ell)$ as a complex line bundle. Hence, $\gamma_\rho = (\overline{\gamma})^{\otimes \rho}$ for the orientation reversing, canonical line bundle $\overline{\gamma}(= \gamma_1)$. Therefore, the Euler class of ξ_ρ is

$$\chi(\xi_\rho) = c_k(\xi_\rho) = c_k(\gamma_\rho^{\oplus k}) = c_1(\gamma_\rho)^k = c_1((\overline{\gamma})^{\otimes \rho})^k = (\rho c)^k = \rho^k c^k,$$

where $c \in H^2(\mathbb{C}P(\ell))$ is the generator (determined by $c_1(\overline{\gamma})$) of the cohomology ring $H^*(\mathbb{C}P(\ell))$. Now π^* is injective by using $H^{odd}(\mathbb{C}P(\ell)) = 0$ and $H^{odd}(S^{2k}) = 0$ (see [17]). Hence, by using $H^*(\mathbb{C}P(\ell)) = \mathbb{Z}[c]/ < c^{\ell+1} >$ and Lemma 3.4, there are the following relations in the cohomology ring $H^*(N_2)$:

$$x^{\ell+1} = 0;$$
$$z^2 + \rho^k x^k z = 0,$$

for $x = \pi^*(c) \in H^2(N_2)$ and some $z \in H^{2k}(N_2)$. Making use of the Serre spectral sequence for the bundle $\pi : N_2 \to \mathbb{C}P(\ell)$, there is an epimorphism $\mathbb{Z}[x,\ z] \to H^*(N_2)$, and additively the cohomology of $H^*(N_2)$ coincides with that of $\mathbb{C}P(\ell) \times S^{2k}$. Hence, there is no other relations except those mentioned in the above arguments. Thus, we have the cohomology formula in Proposition 3.2.

In order to compute characteristic classes, we regard $N_2 = S^{2\ell+1} \times_{S^1} S(\mathbb{C}^k_\rho \oplus \mathbb{R})$ as the unit sphere bundle of the following vector bundle over $\mathbb{C}P(\ell)$:

$$\xi = \xi_\rho \oplus \underline{\mathbb{R}} \equiv \gamma_\rho^{\oplus k} \oplus \underline{\mathbb{R}}, \tag{7}$$

where $\underline{\mathbb{R}}$ is the trivial line bundle. Note that $E(\xi) = S^{2\ell+1} \times_{S^1} (\mathbb{C}^k_\rho \oplus \mathbb{R})$. We often denote N_2 as $S(\xi)$, i.e., the unit sphere bundle of ξ.

Now $\mathcal{T}$ denotes the tangent bundle of $E(\xi)$. Then, there is the following pull-back diagram:

$$\begin{array}{ccc} \iota^*\mathcal{T} & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow \\ S(\xi) & \xrightarrow{\iota} & E(\xi) \end{array}$$

where $\iota : (N_2 =)S(\xi) \to E(\xi)$ is the natural inclusion, and the following relation holds:

$$\mathcal{T}|_{N_2} = \iota^*\mathcal{T} = \tau_2 \oplus \nu_2,$$

where τ_2 is the tangent bundle of $N_2 = S(\xi)$ and ν_2 is the normal bundle of the inclusion $\iota : N_2 = S(\xi) \to E(\xi)$. Note that ν_2 is a real 1-dimensional bundle by the equation $\dim E(\xi) - \dim S(\xi) = 1$. Because N_2 is simply connected, we have the following lemma for ν_2 (see [20]).

Lemma 3.5. *ν_2 is the trivial real line bundle over N_2, i.e., $E(\nu_2) = N_2 \times \mathbb{R}$.*

Hence, we have

$$\iota^*p(\mathcal{T}) = p(\iota^*\mathcal{T}) = p(\tau_2 \oplus \nu_2) = p(\tau_2) = p(N_2), \tag{8}$$

$$\iota^*w(\mathcal{T}) = w(\iota^*\mathcal{T}) = w(\tau_2 \oplus \nu_2) = w(\tau_2) = w(N_2). \tag{9}$$

We also remark $\iota^* : H^*(E(\xi)) \to H^*(S(\xi))$ is injective, because $\pi : S(\xi) \to \mathbb{C}P(\ell)$ is decomposed into $\pi = \widetilde{\pi} \circ \iota$ where $\widetilde{\pi} : E(\xi) \to \mathbb{C}P(\ell)$ and π^* is injective. In order to prove Proposition 3.2, we compute $p(\mathcal{T})$ and $w(\mathcal{T})$.

Let $\widetilde{s}$ be the zero section of $\widetilde{\pi} : E(\xi) \to \mathbb{C}P(\ell)$. Consider the following pull-back diagram:

$$\begin{array}{ccc} \widetilde{s}^*\mathcal{T} & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow \\ \mathbb{C}P(\ell) & \xrightarrow{\widetilde{s}} & E(\xi). \end{array}$$

Because the normal bundle $\nu(\mathbb{C}P(\ell))$ of the image of $\widetilde{s}$ is isomorphic to ξ, we have that

$$\widetilde{s}^*\mathcal{T} \equiv \tau(\mathbb{C}P(\ell)) \oplus \nu(\mathbb{C}P(\ell)) \equiv \tau(\mathbb{C}P(\ell)) \oplus \xi,$$

where $\tau(\mathbb{C}P(\ell))$ is the tangent bundle over $\mathbb{C}P(\ell)$. Therefore, by (7), we have

$$\widetilde{s}^*(p(\mathcal{T})) = p(\widetilde{s}^*\mathcal{T}) = p(\tau(\mathbb{C}P(\ell)) \oplus \xi) = p(\tau(\mathbb{C}P(\ell)))p(\xi) = (1 + c^2)^{\ell+1}(1 + \rho^2 c^2)^k,$$
$$\widetilde{s}^*(w(\mathcal{T})) = w(\widetilde{s}^*\mathcal{T}) = w(\tau(\mathbb{C}P(\ell)) \oplus \xi) = w(\tau(\mathbb{C}P(\ell)))w(\xi) = (1 + c)^{\ell+1}(1 + \rho c)^k.$$

Because $\widetilde{s}^* : H^*(E(\xi)) \to H^*(\mathbb{C}P(\ell)) \simeq \mathbb{Z}[c]/ < c^{\ell+1} >$ induces the isomorphism and $\widetilde{s}^* = (\widetilde{\pi}^*)^{-1}$, we have

$$p(\mathcal{T}) = (1 + \widetilde{\pi}^*(c)^2)^{\ell+1}(1 + \rho^2\widetilde{\pi}^*(c)^2)^k,$$
$$w(\mathcal{T}) = (1 + \widetilde{\pi}^*(c))^{\ell+1}(1 + \rho\widetilde{\pi}^*(c))^k.$$

Hence, we have that

$$p(N_2) = (1 + x^2)^{\ell+1}(1 + \rho^2 x^2)^k,$$
$$w(N_2) = (1 + x)^{\ell+1}(1 + \rho x)^k,$$

by using $\iota^* \circ \widetilde{\pi}^*(c) = \pi^*(c) = x$, (8) and (9). Thus, we have the characteristic classes in Proposition 3.2.

With a method similar to that demonstrated in the above proof of Proposition 3.2, for $N_3 = S^{2\ell+1} \times_{S^1} S(\mathbb{C}^{k_1}_\rho \oplus \mathbb{R}^{2k_2+1})$, we can prove Proposition 3.3.

4. Main theorem and Preliminary

In this section, we state the main theorem and prepare to prove it.

4.1. **Main theorem.** Before we state the main theorem, we prepare some notations (also see [19]). A manifold M in the given family is said to be *cohomologically rigid* if for any other manifold M' in the family the ring isomorphism $H^*(M;\mathbb{Z}) \simeq H^*(M';\mathbb{Z})$ implies a homeomorphism $M \cong M'$. A manifold M in the given family is said to be *rigid by the cohomology ring and the Pontrjagin class* (resp. *the Stiefel-Whitney class*) if for any other manifold M' in the family the ring isomorphism $\phi : H^*(M;\mathbb{Z}) \simeq H^*(M';\mathbb{Z})$ such that $\phi(p(M)) = p(M')$ (resp. $\phi(w(M)) = w(M')$) implies a homeomorphism $M \cong M'$. We remark that if M is cohomologically rigid in the given family, then M is automatically rigid by the cohomology ring and the Pontrjagin class (and the Stiefel-Whitney class).

Now we may state the main theorem.

Theorem 4.1. *All manifolds $M \in \mathfrak{M}$ satisfy one of the following three properties.*

(1) *If M is cohomologically rigid in $\mathfrak{M}$, then M is one of the followings:*

$$\begin{aligned}
&S^{2\ell+1} \times_{S^1} P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2});\\
&S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^{k} \oplus \mathbb{R}) \quad \ell > 1 \text{ and } k \leq \ell;\\
&S^{3} \times_{S^1} S(\mathbb{C}_\rho^{k} \oplus \mathbb{R}) \quad \text{for} \quad k \equiv_2 0 \ (k > 1);\\
&S^{3} \times_{S^1} S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1}) \quad \text{for} \quad k_1 \equiv_2 0.
\end{aligned}$$

(2) *If M is not cohomologically rigid but rigid by the cohomology ring and the Pontrjagin class in $\mathfrak{M}$, then M is one of the followings:*

$$\begin{aligned}
&S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^{k} \oplus \mathbb{R}) \quad \text{for} \quad k > \ell > 1;\\
&S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1}) \quad \text{for} \quad \ell > 1.
\end{aligned}$$

(3) *Otherwise, M is rigid by the cohomology ring and the Stiefel-Whitney class in $\mathfrak{M}$ and one of the followings:*

$$\begin{aligned}
&S^{3} \times_{S^1} S(\mathbb{C}_\rho^{k} \oplus \mathbb{R}) \quad \text{for} \quad k \equiv_2 1 \ (k > 1);\\
&S^{3} \times_{S^1} S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1}) \quad \text{for} \quad k_1 \equiv_2 1.
\end{aligned}$$

It is easy to show that the manifolds in $\mathfrak{M}$ satisfy the two conditions in Section 1, i.e., for all $M \in \mathfrak{M}$,

(1) $H^{odd}(M) = 0$;
(2) M, M_i's and connected components of any multiple intersection of M_i's are all simply connected,

where M_i's are characteristic submanifolds of M. Due to Theorem 4.1, we have that torus manifolds do not satisfy the cohomological rigidity even if the above two conditions hold. This gives the negative answer to the cohomological rigidity problem of torus manifolds (see [15, Problem 1 and Section 7]).

As a corollary of Theorem 4.1, we have the following result.

Corollary 4.2. *Homeomorphism types of $\mathfrak{M}$ are completely determined by their cohomology rings, Pontrjagin classes and Stiefel-Whitney classes.*

4.2. **Preliminary.** In this subsection, we prepare to prove Theorem 4.1.

Due to the definition of $N_1 \in \mathfrak{M}_1$, this manifold N_1 is the projectify of the vector bundle η_ρ:

$$\mathbb{C}^{k_1+k_2} \longrightarrow S^{2\ell+1} \times_{S^1} (\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}) \longrightarrow \mathbb{C}P(\ell).$$

Now we have

$$\eta_\rho \equiv (\gamma^{\otimes(-\rho)})^{\oplus k_1} \oplus \underline{\mathbb{C}}^{\oplus k_2}$$

over $\mathbb{C}P(\ell)$ where γ is the canonical line bundle ($E(\gamma) = S^{2\ell+1} \times_{S^1} \mathbb{C}_{(-1)}$) and $\underline{\mathbb{C}}$ is the trivial complex line bundle. Thus, $\mathfrak{M}_1$ consists of *2-stage generalized Bott towers*, i.e., projectivizations of Whitney sums of line bundles over complex projective spaces (see [2], [3]). Therefore, we may use the following theorem.

Theorem 4.3 ([2]). *Top manifolds of 2-stage generalized Bott towers are diffeomorphic if and only if their integral cohomology rings are isomorphic.*

Note that for all $M \in \mathfrak{M}$, $H^*(M) = \mathbb{Z}[x, w]/ < x^{\ell+1}, f(x, w) >$, where f is a homogeneous polynomial and $\deg x = 2$ and $w = y$ for Proposition 3.1 or $w = z$ for Proposition 3.2, 3.3. We list up for each case:

Proposition 3.1: $f(x,\ y) = y^{k_2}(y + \rho x)^{k_1}$, $\deg y = 2$ for $k_1,\ k_2 \in \mathbb{N}$;
Proposition 3.2: $f(x,\ z) = z(z + (\rho x)^k)$, $\deg z = 2k$ for $k \in \mathbb{N}$;
Proposition 3.3: $f(x,\ z) = z^2$, $\deg z = 2k_1 + 2k_2$ for $k_1,\ k_2 \in \mathbb{N}$.

This implies that the cohomology ring determines $\ell, \deg w$ and $\deg f$ (remark that $\ell \geq 1$, $\deg w \geq 2$ and $\deg f \geq 4$). The proof of Theorem 4.1 is divided into the following two cases corresponding with the degree of w:

CASE 1: $\deg w = 2$, i.e., 2-dimensional sphere bundle or complex projective bundle;
CASE 2: $\deg w > 2$, i.e., m-dimensional sphere bundle and $m = \deg w > 2$;

moreover, we will divide CASE 2 into the following three cases corresponding with ℓ:

CASE 2 (1): $\deg w > 2$ and $\ell \geq 4$;
CASE 2 (2): $\deg w > 2$ and $\ell = 2,\ 3$;
CASE 2 (3): $\deg w > 2$ and $\ell = 1$.

5. Proof of the main theorem

In this final section, we prove Theorem 4.1.

5.1. **CASE 1 : $\deg w = 2$.** Assume $\deg w = 2$. Then this case is a 2-dimensional sphere bundle or a complex projective bundle over $\mathbb{C}P(\ell)$, i.e.,

$$N_1 = S^{2\ell+1} \times_{S^1} P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}) \quad \text{or}$$
$$N_2 = S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho \oplus \mathbb{R}).$$

First, we prove this case is equivalent to a toric manifold. By using $H^{odd}(N_i) = 0$ $(i = 1, 2, 3)$ and [13, Theorem 4.1], we see that the torus action on this manifold is locally standard. Moreover, the orbit space of this torus action is a product of two simplicies. Therefore, we have that this case is a quasitoric manifold. Consider the standard torus action on the following toric manifolds:

$$(\mathbb{C}^{\ell+1} \backslash \{0\}) \times_{\mathbb{C}^*} P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}).$$

Then its orbit space and the characteristic matrix are same as N_1 for all $k_1, k_2 \geq 1$ or N_2 for $k_1 = k_2 = 1$. Therefore, N_1 and N_2 are equivalent to the above toric manifolds.

Moreover, this case has the same cohomology rings as a 2-stage generalized Bott tower. Hence, by using [2, Theorem 6.4] and [4, Theorem 1.6], we also see that this case is a 2-stage generalized Bott tower. Hence, by using Theorem 4.3, we have that this case ($\deg w = 2$) satisfies the cohomological rigidity.

In summary, the following proposition holds.

Proposition 5.1. *If the CASE 1 holds, i.e.,* $\deg w = 2$, *then the manifold in this case is cohomologically rigid in* $\mathfrak{M}$.

5.2. **CASE 2 (1) :** $\deg w > 2$ **and** $\ell \geq 4$**.** Assume $\deg w > 2$ and $\ell \geq 4$. In this case, M must be in $\mathfrak{M}_2$ or $\mathfrak{M}_3$.

Let M_1 and M_2 be in $\mathfrak{M}$ (we will denote $w = z$). Without loss of generality,

$$M_1 = S((\gamma^{\otimes(-\rho_1)})^{\oplus k_{11}} \oplus \mathbb{R}^{2k_{12}+1})$$

and

$$M_2 = S((\gamma^{\otimes(-\rho_2)})^{\oplus k_{21}} \oplus \mathbb{R}^{2k_{22}+1})$$

for some $\rho_1,\ \rho_2 \in \mathbb{Z}$, $k_{11},\ k_{21} \in \mathbb{N}$ and $k_{12},\ k_{22} \geq 0$, where $(k_{11},\ k_{12}),\ (k_{21},\ k_{22}) \neq (1,\ 0)$ because $\deg z (= \deg w) > 2$.

Suppose $H^*(M_1) \simeq H^*(M_2)$. Because M_1 and M_2 are sphere bundles over complex projective spaces, we can assume their base spaces are same dimension 2ℓ and

$$\begin{aligned} & H^*(M_1) = \mathbb{Z}[x_1, z_1]/ < x_1^{\ell+1}, f_1(x_1, z_1) > \\ \simeq\ \ & H^*(M_2) = \mathbb{Z}[x_2, z_2]/ < x_2^{\ell+1}, f_2(x_2, z_2) >, \end{aligned}$$

where $\deg x_i = 2$, $\deg z_i = \deg w = 2k_{i1} + 2k_{i2}$ $(i = 1,\ 2)$ and

$$\begin{aligned} f_i(x_i,\ z_i) &= z_i(z_i + (\rho_i x_i)^{k_{i1}}) \quad \text{for} \quad k_{i2} = 0, \text{ i.e., } M_i \in \mathfrak{M}_2; \\ f_i(x_i,\ z_i) &= z_i^2 \quad \text{for} \quad k_{i2} > 0, \text{ i.e., } M_i \in \mathfrak{M}_3. \end{aligned}$$

Let $\phi : H^*(M_1) \to H^*(M_2)$ be an isomorphism. Because $\deg x_i = 2$, $\deg z_i > 2$ and ϕ preserves generators, we have

$$\phi(x_1) = \pm x_2. \tag{10}$$

Therefore, we have that the cohomology isomorphism ϕ induces the identity of Pontrjagin classes, i.e., by using Propositions 3.2 and 3.3,

$$\begin{aligned} & \phi(p(M_1)) = (1 + x_2^2)^{\ell+1}(1 + \rho_1^2 x_2^2)^{k_{11}} \\ =\ \ & p(M_2) = (1 + x_2^2)^{\ell+1}(1 + \rho_2^2 x_2^2)^{k_{21}}, \end{aligned}$$

if and only if each coefficient of $p_j(M_1)$ and $p_j(M_2)$ $(j = 1, \ldots, \ell + k_{i1} + k_{i2})$ are same.

5.2.1. *This case is rigid by the cohomology ring and the Pontrjagin class.* Assume coefficients of $p_j(M_1)$ and $p_j(M_2)$ are same for all $j = 1, \ldots, \ell + k_{i1} + k_{i2}$. Note that $x_2^4 \neq 0$ in $H^*(M_2)$ since $\ell \geq 4$. Then, by the above arguments, we have every isomorphism ϕ preserves their Pontrjagin classes and in particular we have

$$\begin{aligned} \phi(p_1(M_1)) = p_1(M_2) \quad &\Leftrightarrow \quad k_{11}\rho_1^2 = k_{21}\rho_2^2, \\ \phi(p_2(M_1)) = p_2(M_2) \quad &\Leftrightarrow \quad \binom{k_{11}}{2}\rho_1^4 = \binom{k_{21}}{2}\rho_2^4. \end{aligned}$$

Therefore, we can easily show that there are two cases; one is $\rho_1 = \rho_2 = 0$ and the other is $\rho_1 = \pm\rho_2$ and $k_{11} = k_{21}$. In both cases, the vector bundle $\left(\gamma^{\otimes(-\rho_1)}\right)^{\oplus k_{11}} \oplus \mathbb{R}^{2k_{12}+1}$ and $\left(\gamma^{\otimes(-\rho_2)}\right)^{\oplus k_{21}} \oplus \mathbb{R}^{2k_{22}+1}$ are isomorphic as a real vector bundle. This implies that M_1 and M_2 (unit sphere bundles of these vector bundles) are homeomorphic. Hence, in this case, we have that if cohomology rings and coefficients of $p_j(M_1)$ and $p_j(M_2)$ are same then M_1 and M_2 are homeomorphic. In other words, if there is a graded ring isomorphism $\phi : H^*(M_1) \to H^*(M_2)$ such that $\phi(p(M_1)) = p(M_2)$ then M_1 and M_2 are homeomorphic, i.e., this case is rigid by the cohomology ring and the Pontrjagin class.

In order to check whether this case satisfies the cohomological rigidity or not, we divide this case into the following three cases.

5.2.2. *The case $k_{12},\ k_{22} > 0$.* Assume $k_{12},\ k_{22} > 0$, i.e., $M_i \in \mathfrak{M}_3$ for $i = 1, 2$. Using their cohomology ring structures, we can easily show that

$$\phi(z_1) = \pm z_2.$$

Because of (10), ρ_1 and ρ_2 are independent of ϕ. Hence, by taking M_1 and M_2 with different coefficients of $p_j(M_1)$ and $p_j(M_2)$ for some j, we can easily construct examples which do not satisfy the cohomological rigidity. Therefore, this case does not satisfy the cohomological rigidity.

5.2.3. *The case $k_{12} > 0$ and $k_{22} = 0$.* Assume $k_{12} > 0$ and $k_{22} = 0$, i.e., $M_1 \in \mathfrak{M}_3$ and $M_2 \in \mathfrak{M}_2$. Then $k_{21} = k_{11} + k_{12}$. Put

$$\phi(z_1) = a x_2^{k_{21}} + b z_2.$$

We can easily show $b = \pm 1$ because ϕ preserves generators. Using $\phi(z_1)^2 = 0$, we have the following three cases:

(1) $k_{21} \geq \ell + 1$;
(2) $k_{21} < \ell + 1 \leq 2k_{21}$, $\rho_2 \equiv 0 \pmod 2$ and $a = \frac{\rho_2^{k_{21}}}{2} b$ $(b = \pm 1)$;
(3) $2k_{21} < \ell + 1$ and $\rho_2 = 0$.

If $k_{21} \geq \ell+1$, then we have $\phi(z_1) = \pm z_2$ because $x_2^{\ell+1} = 0$. If $2k_{21} < \ell+1$ and $\rho_2 = 0$, then we can conclude $\phi(z_1) = \pm z_2$ by the easy computation. These cases are the same situation as the above case (Section 5.2.2). Hence, with the method similar to that demonstrated in Section 5.2.2, these cases do not satisfy the cohomological rigidity.

Therefore, it is sufficient to consider the case $k_{21} < \ell + 1 \leq 2k_{21}$ only. In this case, we have

$$\phi(z_1) = \pm\left(\frac{\rho_2^{k_{21}}}{2} x_2^{k_{21}} + z_2\right).$$

However, also in this case, by taking M_1 and M_2 with different coefficients of $p_j(M_1)$ and $p_j(M_2)$ for some j, we can easily construct examples which do not satisfy the cohomological rigidity. Therefore, this case does not satisfy the cohomological rigidity.

5.2.4. *The case $k_{12} = k_{22} = 0$.* Assume $k_{12} = 0 = k_{22}$. Then $k_{11} = k_{21}$ by $2k_{11} = \deg z_1 = \deg z_2 = 2k_{21}$.

If $k_{11} = k_{21} > \ell$, then we can easily show that $\phi(z_1) = \pm z_2$. Therefore, this case does not satisfy the cohomological rigidity by using the same argument in Section 5.2.2.

Assume $k_{11} = k_{21} \leq \ell$. Let $\phi(z_1) = ax_2^{k_{21}} + bz_2$, where $b = \pm 1$. Using the formula $z_i^2 + z_i(\rho_i x_i)^{k_{i1}} = 0$ $(i = 1, 2)$, we have

$$\begin{aligned}
& \phi(z_1(z_1 + (\rho_1 x_1)^{k_{11}})) \\
= \;& (ax_2^{k_{21}} + bz_2)^2 + (ax_2^{k_{21}} + bz_2)(\pm\rho_1 x_2)^{k_{11}} \\
= \;& a^2 x_2^{2k_{21}} + 2abx_2^{k_{21}} z_2 + b^2 z_2^2 + a(\pm\rho_1)^{k_{21}} x_2^{2k_{21}} + b(\pm\rho_1)^{k_{21}} z_2 x_2^{k_{21}} \\
= \;& a^2 x_2^{2k_{21}} + 2abx_2^{k_{21}} z_2 - b^2 z_2(\rho_2 x_2)^{k_{21}} + a(\pm\rho_1)^{k_{21}} x_2^{2k_{21}} + b(\pm\rho_1)^{k_{21}} z_2 x_2^{k_{21}} \\
= \;& ax_2^{2k_{21}}(a + (\pm\rho_1)^{k_{21}}) + bx_2^{k_{21}} z_2(2a - b\rho_2^{k_{21}} + (\pm\rho_1)^{k_{21}}) = 0.
\end{aligned}$$

If $a = 0$, then $|\rho_2| = |\rho_1|$ by using the above equation, $b = \pm 1$ and $\rho_1, \rho_2 \in \mathbb{Z}$. This implies that the vector bundles $(\gamma^{\otimes(-\rho_1)})^{\oplus k_{11}} \oplus \mathbb{R}$ and $(\gamma^{\otimes(-\rho_2)})^{\oplus k_{21}} \oplus \mathbb{R}$ are same as a real vector bundle. It follows that unit sphere bundles of these vector bundles are homeomorphic, i.e., the manifold in this case is cohomologically rigid. If $a \neq 0$, then we have $a = -(\pm\rho_1)^{k_{21}}$ and

$$-(\pm\rho_1)^{k_{21}} = b\rho_2^{k_{21}},$$

by using the above equation. Using $\rho_1,\ \rho_2 \in \mathbb{Z}$ and $b = \pm 1$, we have $|\rho_1| = |\rho_2|$. Hence, this case is also cohomologically rigid with the method similar to that demonstrated in the case $a = 0$.

In summary, the following proposition holds.

Proposition 5.2. *If the CASE 2-(1) holds, i.e.,* $\deg w > 2$ *(if and only if* $2k_1 + 2k_2 > 2$*) and* $\ell \geq 4$*, then we can put* $M = S^{2\ell+1} \times_{S^1} S((\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1})$ *for* $k_1 > 0$, $k_2 \geq 0$ *and there are the following two cases:*

1. *M is cohomologically rigid in $\mathfrak{M} \Leftrightarrow k_2 = 0$ and $1 < k_1 \leq \ell$;*
2. *M is not cohomologically rigid but rigid by the cohomology ring and the Pontrjagin class in $\mathfrak{M} \Leftrightarrow$ otherwise, i.e., $k_2 > 0$, or $k_2 = 0$ and $k_1 > \ell$.*

5.3. **CASE 2 (2) :** $\deg w > 2$ **and** $\ell = 2,\ 3$**.** Assume $\deg w > 2$ and $\ell = 2,\ 3$. With a method similar to that demonstrated in CASE 2 (1), we can put

$$M_1 = S((\gamma^{\otimes(-\rho_1)})^{\oplus k_{11}} \oplus \mathbb{R}^{2k_{12}+1}) = S^{2\ell+1} \times_{S^1} S(\mathbb{C}_{\rho_1}^{k_{11}} \oplus \mathbb{R}^{2k_{12}+1})$$

and

$$M_2 = S((\gamma^{\otimes(-\rho_2)})^{\oplus k_{21}} \oplus \mathbb{R}^{2k_{22}+1}) = S^{2\ell+1} \times_{S^1} S(\mathbb{C}_{\rho_2}^{k_{21}} \oplus \mathbb{R}^{2k_{22}+1})$$

for some $k_{11},\ k_{21} \in \mathbb{N}$ and $k_{12},\ k_{22} \geq 0$. Let $\phi : H^*(M_1) \to H^*(M_2)$ be an isomorphism. Then we similarly have that an isomorphic map ϕ satisfies

$$\phi(x_1) = \pm x_2.$$

Similarly to CASE 2 (1), we divide this case into the following three cases: the case $k_{12},\ k_{22} > 0$; the case $k_{12} > 0$, $k_{22} = 0$; and the case $k_{12} = k_{22} = 0$.

Assume $k_{12},\ k_{22} > 0$. Then we have $\phi(z_1) = \pm z_2$. Using the similar method demonstrated in Section 5.2.2, this case is not cohomologically rigid. Assume $\phi(p_1(M_1)) = p_1(M_2)$. Then, by the method similar to that demonstrated in Section 5.2, we have

$$\phi(p_1(M_1)) = p_1(M_2) \Leftrightarrow k_{11}\rho_1^2 = k_{21}\rho_2^2.$$

Note that in this case the second Pontrjagin class p_2 does not appear by its cohomology ring structure (see Propositions 3.2 and 3.3). So we need to use a KO-theoretical argument.

Because of [18], in this case ($\ell = 2,\ 3$) we have $KO(\mathbb{C}P^{\ell}) \simeq \mathbb{Z}[y_{\ell}]/ < y_{\ell}^2 >$, where $y_{\ell} = r(\gamma) - 2$ for the canonical line bundle γ and the realification map $r : K(\mathbb{C}P^{\ell}) \to KO(\mathbb{C}P^{\ell})$. Moreover, we have $r(\gamma^{\otimes n}) = n^2 y_{\ell} + 2$ by [18]. So, we have that

$$r(\gamma^{\otimes(-\rho_1)}) = \rho_1^2 r(\gamma) - 2\rho_1^2 + 2, \quad r(\gamma^{\otimes(-\rho_2)}) = \rho_2^2 r(\gamma) - 2\rho_2^2 + 2.$$

Because $k_{11}\rho_1^2 = k_{21}\rho_2^2$ and $k_{11} + k_{12} = k_{21} + k_{22}$, we have the following equation:

$$k_{11} r(\gamma^{\otimes(-\rho_1)}) + 2k_{12} + 1 = k_{11}(\rho_1^2 r(\gamma) - 2\rho_1^2) + 2k_{11} + 2k_{12} + 1$$
$$= k_{21}(\rho_2^2 r(\gamma) - 2\rho_2^2) + 2k_{21} + 2k_{22} + 1 = k_{21} r(\gamma^{\otimes(-\rho_2)}) + 2k_{22} + 1.$$

Therefore, we have that

$$(\gamma^{\otimes(-\rho_1)})^{\oplus k_{11}} \oplus \mathbb{R}^{2k_{12}+1} \equiv_s (\gamma^{\otimes(-\rho_2)})^{\oplus k_{21}} \oplus \mathbb{R}^{2k_{22}+1}$$
$$\Leftrightarrow \quad k_{11}\rho_1^2 = k_{21}\rho_2^2 \quad \text{and} \quad k_{11} + k_{12} = k_{21} + k_{22},$$

where $\eta \equiv_s \xi$ means η and ξ are stably isomorphic. If $2k_{i1} + 2k_{i2} + 1 > 2\ell$ ($i = 1, 2$, and $\ell = 2, 3$), these bundles are in the stable range; therefore, two bundles are isomorphic $(\gamma^{\otimes(-\rho_1)})^{\oplus k_{11}} \oplus \mathbb{R}^{2k_{12}+1} \equiv (\gamma^{\otimes(-\rho_2)})^{\oplus k_{21}} \oplus \mathbb{R}^{2k_{22}+1}$. Otherwise, i.e., $\ell = 3$, $k_{i1} = k_{i2} = 1$ ($i = 1, 2$), we can easily show that $|\rho_1| = |\rho_2|$ by using $k_{11}\rho_1^2 = k_{21}\rho_2^2$; therefore, this case also satisfies that $\gamma^{\otimes(-\rho_1)} \oplus \mathbb{R}^3 \equiv \gamma^{\otimes(-\rho_2)} \oplus \mathbb{R}^3$. This implies that the case $k_{12},\ k_{22} > 0$ is rigid by the cohomology ring and the Pontrjagin class.

Here we exhibit some non-trivial examples.

Example 5.3. The following two manifolds are homeomorphic because $H^*(M_1) \simeq H^*(M_2)$ and $p_1(M_1) = 4x_1^2$ and $p_1(M_2) = 4x_2^2$ ($x_i \in H^2(M_i)$):

$$M_1 = S^7 \times_{S^1} S(\mathbb{C}_2^1 \oplus \mathbb{R}^9); \quad M_2 = S^7 \times_{S^1} S(\mathbb{C}_1^4 \oplus \mathbb{R}^1).$$

The following manifold has the same cohomology ring as the above two manifolds, but this manifold is not homeomorphic to the above manifolds because $p_1(M) = 16x$ for $x \in H^2(M)$.

$$M = S^7 \times_{S^1} S(\mathbb{C}_2^4 \oplus \mathbb{R}^1).$$

Using the similar argument in this section and Section 5.2.3 and 5.2.4, we have the same results for the cases $k_{12} > 0$, $k_{22} = 0$ and $k_{12} = k_{22} = 0$ as that in Section 5.2.3 and 5.2.4, respectively.

In summary, the following proposition holds.

Proposition 5.4. *If the CASE 2 (2) holds, i.e., $\ell = 2, 3$ and $2k_1 + 2k_2 > 2$, then we can put $M = S^{2\ell+1} \times_{S^1} S((\mathbb{C}_{\rho}^{k_1} \oplus \mathbb{R}^{2k_2+1})$ for $k_1 > 0$, $k_2 \geq 0$ and there are the following two cases:*

1. *M is cohomologically rigid in $\mathfrak{M} \Leftrightarrow k_2 = 0$ and $1 < k_1 \leq \ell$;*
2. *M is rigid by the cohomology ring and the Pontrjagin class in $\mathfrak{M} \Leftrightarrow$ otherwise, i.e., $k_2 > 0$, or $k_2 = 0$ and $k_1 > \ell$.*

5.4. **CASE 2 (3) : $\deg w > 2$ and $\ell = 1$.** Assume $\deg w > 2$ and $\ell = 1$. In this case, M must be in $\mathfrak{M}_2$ or $\mathfrak{M}_3$ (we will denote $w = z$). Thus we can put

$$M = S^3 \times_{S^1} S(\mathbb{C}_{\rho}^{k_1} \oplus \mathbb{R}^{2k_2+1})$$

for $\rho \in \mathbb{Z}$, $k_1 > 0$ and $k_2 \geq 0$, where $(k_1, k_2) \neq (1, 0)$ and $\deg z = 2k_1 + 2k_2$. Moreover, by using Propositions 3.2 and 3.3 and $\mathbb{C}P(1) \cong S^2$, we have that

$$H^*(M) \simeq H^*(S^{\deg z} \times S^2)$$

and

$$p(M) = 1, \quad w(M) = 1 + k_1\rho x.$$

It follows that the Pontrjagin class does not distinguish homeomorphism types of this case.

Recall that S^{n-1}-bundles over S^2 are classified by continuous maps from S^2 to $G_n = BO(n)$ up to homotopy and $\pi_2(G_n) \simeq \mathbb{Z}_2$ for $n > 2$ (see e.g. [20]). We can easily show that this $\mathbb{Z}_2$ is generated by $w_2(M)$. Therefore, if $k_1 \equiv_2 1$, the homeomorphism type of M is determined by ρ (mod 2). If $k_1 \equiv_2 0$, then $S^3 \times_{S^1} (\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1})$ is trivial bundle because its Stiefel-Whitney class is trivial. Hence, if $k_1 \equiv_2 0$ then $M \cong S^{\deg z} \times S^2$.

The above argument implies that homeomorphism types of CASE 2 (3) are determined as follows: if $k_1 \equiv_2 0$, then this case is cohomologically rigid; if $k_1 \equiv_2 1$, then this case is determined by the cohomology ring and the Stiefel-Whitney class. Moreover, in the case $k_1 \equiv_2 1$, we can easy to show that every cohomology graded ring isomorphism preserves Stiefel-Whitney classes. This implies that the case $k_1 \equiv_2 1$ is rigid by the cohomology ring and the Stiefel-Whitney class.

In summary, the following proposition holds.

Proposition 5.5. *If the CASE 2 (3) holds, i.e., $\ell = 1$ and $2k_1 + 2k_2 > 2$, then $M = S^3 \times_{S^1} S((\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1})$ for $k_1 > 0$, $k_2 \geq 0$ and there are the following two cases:*

(1) *M is cohomologically rigid in $\mathfrak{M} \Leftrightarrow k_1 \equiv_2 0$;*
(2) *M is rigid by the cohomology ring and the Stiefel-Whitney class in $\mathfrak{M} \Leftrightarrow k_1 \equiv_2 1$.*

By using Proposition 5.1, 5.2, 5.4, 5.5, we have Theorem 4.1.

Acknowledgements Finally the authors would like to thank Professors Zhi Lü and DongYoup Suh for providing excellent circumstances to do research. He also would like to thank for Professor Mikiya Masuda for his invaluable advices and comments.

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