An eigenvalue problem related to the critical Sobolev exponent: variable coefficient case

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Abstract

Let u_ε be a least energy solution to the nearly critical problem:

$$-\Delta u = c_0 K(x) u^{p_\varepsilon} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $c_0 = N(N-2)$, $p_\varepsilon = (N+2)/(N-2) - \varepsilon$ where $\varepsilon > 0$ is a small parameter and $K \in C^2(\overline{\Omega})$ is a positive function.

Under some assumption of K, we prove several asymptotic estimates of eigenvalues $\lambda_{i,\varepsilon}$ and corresponding eigenfunctions $v_{i,\varepsilon}$ to the eigenvalue problem:

$$\begin{cases} -\Delta v_{i,\varepsilon} = \lambda_{i,\varepsilon} \left(c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} \right) v_{i,\varepsilon} & \text{in } \Omega, \\ v_{i,\varepsilon} = 0 & \text{on } \partial\Omega, \\ \|v_{i,\varepsilon}\|_{L^\infty(\Omega)} = 1 \end{cases}$$

as $\varepsilon \to 0$, for $i = 2, \cdots, N+1, N+2$.

1 Introduction

We consider the problem

$$\begin{cases} -\Delta u = c_0 K(x) u^{p_\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $c_0 = N(N-2)$, $p_\varepsilon = p - \varepsilon$, $p = (N+2)/(N-2)$ is the critical Sobolev exponent with respect to the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and $\varepsilon > 0$ is a small parameter. Here, K is a positive function in $C^2(\overline{\Omega})$.

Throughout this paper, u_ε will denote a least energy solution of (1.1) with

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\left(\int_\Omega K(x)|u_\varepsilon|^{p_\varepsilon+1}dx\right)^{\frac{2}{p_\varepsilon+1}}} = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega K(x)|u|^{p_\varepsilon+1}dx\right)^{\frac{2}{p_\varepsilon+1}}},$$

unless otherwise stated. It is known that $\|u_\varepsilon\|_{L^\infty(\Omega)} \to +\infty$ as $\varepsilon \to 0$ and the maximum point x_ε of u_ε converges to a maximum point of K in $\overline{\Omega}$.

From now on, we assume that the coefficient function K satisfies the following (see [5]):

(K): $K \in C^2(\overline{\Omega}), 0 < K(x) \leq 1, K^{-1}(\max_{\overline{\Omega}} K) = \{x_0\} \subset \Omega$ with $K(x_0) = 1$, and x_0 is a nondegenerate critical point of K.

From the assumption, we see that $\mu_i < 0$ $(1 \leq \forall i \leq N)$ where $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$ denote eigenvalues of the Hessian matrix $\mathrm{Hess}K(x_0) = \left(\frac{\partial^2 K}{\partial x_i \partial x_j}(x_0)\right)_{1 \leq i,j \leq N}$. In addition to (K), we assume the following:

(A):

$$|\Delta K(x_0)| > \left(\frac{N-2}{2}\right)|\mu_1|. \tag{1.2}$$

We note that the assumption (A) is automatically satisfied when $N = 3, 4$.

Under the assumption (K), we can obtain a precise asymptotic behavior of least energy solutions as $\varepsilon \to 0$ along the argument by Hebey [4]; see Lemma 2.3 below.

Now, let us consider the eigenvalue problem:

$$\begin{cases} -\Delta v = \lambda \left(c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1}\right) v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \|v\|_{L^\infty(\Omega)} = 1. \end{cases} \tag{1.3}$$

It is well known that (1.3) admits a countable sequence of eigenvalues $\lambda_{1,\varepsilon} < \lambda_{2,\varepsilon} \leq \cdots \leq \lambda_{i,\varepsilon} \leq \cdots \to +\infty$ and corresponding eignefunctions $v_{1,\varepsilon}, v_{2,\varepsilon}, \cdots, v_{i,\varepsilon}, \cdots$,

$\|v_{i,\varepsilon}\|_{L^\infty(\Omega)} = 1$ $(i \in \mathbb{N})$, with the orthogonal property

$$\int_\Omega \left(c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1}\right) v_{i,\varepsilon} v_{j,\varepsilon} dx = 0, \quad (i \neq j). \tag{1.4}$$

In the following, $\| \ \|$ will denote $\| \ \|_{L^\infty(\Omega)}$. It is easy to check that $\lambda_{1,\varepsilon} = 1/p_\varepsilon$ and the corresponding eigenfunction is $v_{1,\varepsilon} = \frac{u_\varepsilon}{\|u_\varepsilon\|}$. In this paper, we prove several asymptotic estimates of eigenvalues $\lambda_{i,\varepsilon}$ and corresponding eigenfunctions $v_{i,\varepsilon}$ to the eigenvalue problem (1.3) for $i = 2, \cdots, N+1$ and $N+2$. To state our results, we introduce the scaled eigenfunctions

$$\tilde{v}_{i,\varepsilon}(y) = v_{i,\varepsilon}\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + x_\varepsilon\right), \quad y \in \Omega_\varepsilon = \|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}(\Omega - x_\varepsilon). \tag{1.5}$$

Theorem 1.1 *Assume (K) and (A). Then for $i = 2, 3, \cdots, N+1$, we have*

$$\tilde{v}_{i,\varepsilon}(y) \to \sum_{j=1}^N a_{i,j} \frac{y_j}{(1+|y|^2)^{\frac{N}{2}}} \quad in\ C^1_{loc}(\mathbb{R}^N), \tag{1.6}$$

$$\|u_\varepsilon\|^{2+\frac{2}{N-2}} v_{i,\varepsilon}(x) \to \sigma_N \sum_{j=1}^N a_{i,j} \left(\frac{\partial G}{\partial z_j}\right)(x,z)|_{z=x_0} \quad in\ C^1_{loc}(\overline{\Omega} \setminus \{x_0\}) \tag{1.7}$$

for some $\vec{a}_i = (a_{i,1}, a_{i,2}, \cdots, a_{i,N}) \neq \vec{0}$ as $\varepsilon \to 0$. Here $G(x,z)$ is Green's function of $-\Delta$ acting on $H^1_0(\Omega)$ and σ_N is the volume of the unit sphere in $\mathbb{R}^N$. In addition,

$$\|u_\varepsilon\|^{\frac{4}{N-2}}(\lambda_{i,\varepsilon} - 1) \to M|\mu_{N+2-i}|, \quad \varepsilon \to 0, \tag{1.8}$$

where $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N < 0$ are eigenvalues of $HessK(x_0) = \left(\frac{\partial^2 K}{\partial x_i \partial x_j}(x_0)\right)_{1 \leq i,j \leq N}$, and $M = \frac{2(N+1)}{N(N+2)}$. Furthermore, $\vec{a}_i$ is an eigenvector of $HessK(x_0)$ corresponding to the eigenvalue μ_{N+2-i} and $\vec{a}_i$ is perpendicular to $\vec{a}_j$ in $\mathbb{R}^N$ if $i \neq j$.

Note that the Morse index of u_ε is the number of eigenvalues of (1.3) such that $\lambda_{i,\varepsilon} < 1$, and the nullity of u_ε is the multiplicity of the eigenvalue 1 to (1.3), if it exists. Thus, as a corollary of Theorem 1.1, we see that any least energy solution u_ε to (1.1) is nondegenerate with Morse index 1 for $\varepsilon > 0$ small, under the assumptions (K) and (A). However, this fact has been proved in [5] under the assumption (K) only.

Further, we prove the following.

Theorem 1.2 *Assume (K) and (A). We have*

$$\tilde{v}_{N+2,\varepsilon}(y) \to b_{N+2} \frac{1-|y|^2}{(1+|y|^2)^{\frac{N}{2}}} \quad \text{in } C^1_{loc}(\mathbb{R}^N) \tag{1.9}$$

for some $b_{N+2} \neq 0$, *and*

$$\lambda_{N+2,\varepsilon} - 1 = \varepsilon\Gamma + o(\varepsilon) \tag{1.10}$$

as $\varepsilon \to 0$, *where*

$$\Gamma = \begin{cases} \frac{2(N-2)(N+1)}{N(N+2)}, & (N \geq 4), \\ \frac{4}{15}, & (N = 3). \end{cases}$$

In [2], Grossi and Pacella considered the eigenvalue problem

$$-\Delta v = \lambda c_0 p_\varepsilon u_\varepsilon^{p_\varepsilon - 1} v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \quad \|v\|_{L^\infty(\Omega)} = 1$$

on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$, where u_ε is a blowing-up solution of the slightly subcritical problem

$$-\Delta u = c_0 u^{p_\varepsilon}, \ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

with the property

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{\left(\int_\Omega |u_\varepsilon|^{p_\varepsilon + 1} dx\right)^{2/(p_\varepsilon+1)}} \to S_N,$$

where S_N is the best Sobolev constant. In addition to the qualitative properties of eigenfunctions, they obtained analogous results on the asymptotic behavior of eigenvalues and eigenfunctions as $\varepsilon \to 0$. In this case, the Robin function associated with Green's function

$$R(z) = \lim_{x \to z} \left[\frac{1}{(N-2)\sigma_N} |x-z|^{2-N} - G(x,z) \right]$$

plays a crucial role on the asymptotic behavior of eigenfunctions. Also note that u_ε need not be a least energy solution in [2].

The main purpose of this paper is to generalize the results of [2] to the inhomogeneous case, and to obtain precise effect of the coefficient function K on the spectral property of blowing-up solutions. Proofs of theorems in this paper will be done along the line in [2], however, the presence of the term $K(x)u^{p_\varepsilon}$ causes the difficulty and more efforts will be needed to obtain various estimates.

Once the precise asymptotic behavior is established, we have the same results on the qualitative properties of eigenfunctions as in [2]. We omit the proof of the next theorem since the proof in [2] with a trivial modification works well also in our inhomogeneous case.

Theorem 1.3 *Denote $N_{i,\varepsilon} = \{x \in \Omega \mid v_{i,\varepsilon}(x) = 0\}$ for $i \in \mathbb{N}$. Then for $\varepsilon > 0$ sufficiently small, we have the followings.*

(1) The eigenfunction $v_{i,\varepsilon}$ has only two nodal regions for $i = 2, \cdots, N+1$.

(2) If Ω is convex and $i = 2, \cdots, N+1$, then $\overline{N_{i,\varepsilon}} \cap \partial\Omega \neq \phi$.

(3) $\lambda_{N+2,\varepsilon}$ is simple, $v_{N+2,\varepsilon}$ has only two nodal regions, and $\overline{N_{N+2,\varepsilon}} \cap \partial\Omega = \phi$.

2 Preliminaries

In this section, we collect lemmas which are needed in the proof.

Lemma 2.1 *For any $i \in \mathbb{N}$ and for any $y \in \mathbb{R}^N$, the following identities hold true.*

$$\int_{\partial\Omega} (x-y)\cdot\nu \left(\frac{\partial u_\varepsilon}{\partial \nu}\right)\left(\frac{\partial v_{i,\varepsilon}}{\partial \nu}\right) ds_x = c_0 p_\varepsilon (1-\lambda_{i,\varepsilon}) \int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1} w_\varepsilon v_{i,\varepsilon} dx + c_0 \int_\Omega (x-y)\cdot \nabla K(x) u_\varepsilon^{p_\varepsilon} v_{i,\varepsilon} dx, \tag{2.1}$$

where $w_\varepsilon(x) = (x-y)\cdot\nabla u_\varepsilon + \frac{2}{p_\varepsilon - 1} u_\varepsilon$, and

$$\int_{\partial\Omega} \left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial v_{i,\varepsilon}}{\partial \nu}\right) ds_x = (1-\lambda_{i,\varepsilon}) c_0 p_\varepsilon \int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1} \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) v_{i,\varepsilon} dx + c_0 \int_\Omega \left(\frac{\partial K}{\partial x_j}\right) u_\varepsilon^{p_\varepsilon} v_{i,\varepsilon} dx \quad \text{for } j = 1, \cdots, N, \tag{2.2}$$

where $\nu = \nu(x)$ is the unit outer normal at $x \in \partial\Omega$.

Proof. By an easy calculation, w_ε satisfies

$$-\Delta w_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} w_\varepsilon + c_0 (x-y)\cdot\nabla K(x) u_\varepsilon^{p_\varepsilon} \quad \text{in } \Omega. \tag{2.3}$$

Also by differentiating the equation of (1.1), we see

$$-\Delta\left(\frac{\partial u_\varepsilon}{\partial x_j}\right) = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right) + c_0\left(\frac{\partial K}{\partial x_j}\right) u_\varepsilon^{p_\varepsilon}. \tag{2.4}$$

Then, follow the proof of Lemma 4.3 and Lemma 5.1 in [2]. □

Lemma 2.2 *Let $G = G(x,z)$ be Green's function of $-\Delta$ under the Dirichlet boundary condition. Then we have for any $y \in \Omega$,*

$$\int_{\partial\Omega} ((x-y)\cdot\nu)\left(\frac{\partial G(x,y)}{\partial\nu}\right)^2 ds_x = (N-2)R(z)\Big|_{z=y}, \tag{2.5}$$

$$\int_{\partial\Omega} \left(\frac{\partial G(x,y)}{\partial\nu}\right)^2 \nu_i(x) ds_x = \frac{\partial R}{\partial z_i}(z)\Big|_{z=y}, \tag{2.6}$$

$$\int_{\partial\Omega} \left(\frac{\partial G(x,y)}{\partial x_i}\right)\frac{\partial}{\partial\nu_x}\left(\frac{\partial G}{\partial z_j}\right)(x,y) ds_x = \frac{1}{2}\frac{\partial^2 R}{\partial z_i \partial z_j}(z)\Big|_{z=y}. \tag{2.7}$$

Proof. See [3]; note that the sign of R is negative in [3]. □

The scaled eigenfunction $\tilde{v}_{i,\varepsilon}$ satisfies

$$\begin{cases} -\Delta\tilde{v}_{i,\varepsilon} = \lambda_{i,\varepsilon}\left(c_0 p_\varepsilon K_\varepsilon(y)\tilde{u}_\varepsilon^{p_\varepsilon - 1}\right)\tilde{v}_{i,\varepsilon} & \text{in } \Omega_\varepsilon, \\ \tilde{v}_{i,\varepsilon} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \|\tilde{v}_{i,\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} = 1 \quad (i \in \mathbb{N}), \end{cases} \tag{2.8}$$

where $K_\varepsilon(y) = K\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + x_\varepsilon\right)$ and

$$\tilde{u}_\varepsilon(y) = \frac{1}{\|u_\varepsilon\|} u_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + x_\varepsilon\right), \quad y \in \Omega_\varepsilon. \tag{2.9}$$

Note that $K_\varepsilon(y) \to K(x_0) = 1$ uniformly on compact sets on $\mathbb{R}^N$. Also by a result in [3] [4], we see

$$\tilde{u}_\varepsilon \to U(y) = \left(\frac{1}{1+|y|^2}\right)^{\frac{N-2}{2}} \quad \text{in } C^2_{loc}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N), \tag{2.10}$$

where U is the unique solution of

$$-\Delta U = c_0 U^p \text{ in } \mathbb{R}^N,\ 0 < U \le 1,\ U(0) = 1.$$

Lemma 2.3 *(Hebey [4]; see also [5]) Let $\Omega \subset \mathbb{R}^N, N \ge 3$ be a smooth bounded domain. Let u_ε be a least energy solution to (1.1) and $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|$. Assume (K). Then after passing to a subsequence, we have the followings:*

There exists a constant $C > 0$ independent of ε such that for any $R_\varepsilon \to \infty$ with $r_\varepsilon = R_\varepsilon \|u_\varepsilon\|^{-(\frac{p_\varepsilon - 1}{2})} \to 0$, we have

$$u_\varepsilon(x) \le \begin{cases} \frac{\|u_\varepsilon\|}{\left(1+\|u_\varepsilon\|^{\frac{4}{N-2}}|x-x_\varepsilon|^2\right)^{\frac{N-2}{2}}}, & \text{for } |x - x_\varepsilon| \le r_\varepsilon, \\ \frac{C}{\|u_\varepsilon\|}\frac{1}{|x-x_\varepsilon|^{N-2}}, & \text{for } \{|x - x_\varepsilon| > r_\varepsilon\} \cap \Omega. \end{cases} \tag{2.11}$$

Furthermore, after passing to a subsequence,

$$\begin{cases} |x_\varepsilon - x_0| = O\left(\|u_\varepsilon\|^{-2}\right) & N = 3, \\ |x_\varepsilon - x_0| = o\left(\|u_\varepsilon\|^{-2/(N-2)}\right) & N \ge 4, \end{cases} \tag{2.12}$$

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|^\varepsilon = 1 \tag{2.13}$$

and

$$\|u_\varepsilon\| u_\varepsilon \to (N-2)\sigma_N G(\cdot, x_0) \quad in\ C^1(\omega) \tag{2.14}$$

for any neighborhood ω of $\partial\Omega$ not containing x_0.

Finally, we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \varepsilon \|u_\varepsilon\|^2 = 128 R(x_0) && N = 3, \\ &\lim_{\varepsilon \to 0} \varepsilon \|u_\varepsilon\|^2 = 48\pi^2 R(x_0) - \frac{1}{2}\Delta K(x_0) && N = 4, \\ &\lim_{\varepsilon \to 0} \varepsilon \|u_\varepsilon\|^{\frac{4}{N-2}} = -\frac{2}{(N-2)^2}\Delta K(x_0) && N \ge 5. \end{aligned} \tag{2.15}$$

Note that the pointwise estimate (2.11) is equivalent to

$$\tilde{u}_\varepsilon(y) \le C U(y), \quad \forall y \in \Omega_\varepsilon. \tag{2.16}$$

Theorem 2.4 *(Bianchi and Egnell [1]) The eigenvalue problem*

$$\begin{cases} -\Delta V_i = \lambda_i c_0 p U^{p-1} V_i \quad \text{in } \mathbb{R}^N, \\ V_i \in D^{1,2}(\mathbb{R}^N) \end{cases} \tag{2.17}$$

where $D^{1,2}(\mathbb{R}^N) = \{V \in L^{2N/(N-2)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla V|^2 dy < +\infty\}$*, has eigenvalues*

$$\lambda_1 = 1/p < \lambda_2 = \lambda_3 = \cdots = \lambda_{N+1} = \lambda_{N+2} = 1 < \lambda_{N+3} \leq \cdots$$

with eigenfunctions

$$V_1 = U = \left(\frac{1}{1+|y|^2}\right)^{\frac{N-2}{2}}, \quad V_i = \frac{\partial U}{\partial y_{i-1}}, (i = 2, \cdots, N+1),$$
$$V_{N+2} = \frac{d}{d\lambda}\Big|_{\lambda=1} \lambda^{(N-2)/2} U(\lambda y) = y \cdot \nabla U + \frac{N-2}{2} U.$$

Lemma 2.5 *For any* $i \in \mathbb{N}$*, there exists a constant* $C > 0$ *independent of* ε *such that*

$$|\tilde{v}_{i,\varepsilon}(y)| \leq CU(y) \tag{2.18}$$

holds true for all $y \in \Omega_\varepsilon$*.*

Proof. See Lemma 3.1 in [5]. □

Lemma 2.6 *Suppose* $\lambda_i = \lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} = 1$*. Then*

$$\tilde{v}_{i,\varepsilon} \to V_i = \sum_{j=1}^{N} a_{i,j} \frac{y_j}{(1+|y|^2)^{N/2}} + b_i \frac{1-|y|^2}{(1+|y|^2)^{N/2}} \quad \text{in } C^1_{loc}(\mathbb{R}^N) \tag{2.19}$$

as $\varepsilon \to 0$ *for some* $(a_{i,1}, a_{i,2}, \cdots, a_{i,N}, b_i) \neq (0, 0, \cdots, 0)$*.*

Proof. By elliptic estimates, (2.16) and (2.18), we can check that there exists some V_i such that

$$\tilde{v}_{i,\varepsilon} \to V_i \quad \text{in } C^1_{loc}(\mathbb{R}^N) \quad (i \in \mathbb{N}).$$

Also by the fact $\int_{\Omega_\varepsilon} |\nabla \tilde{v}_{i,\varepsilon}|^2 dy \leq C$, we confirm that $V_i \in D^{1,2}(\mathbb{R}^N)$. Put $\lambda_i = \lim_{\varepsilon \to 0} \lambda_{i,\varepsilon}$ $(i \in \mathbb{N})$. Then by using (2.10) in (2.8), we see

$$\begin{cases} -\Delta V_i = \lambda_i \left(c_0 p U^{p-1}\right) V_i \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\nabla V_i|^2 dy < \infty. \end{cases} \tag{2.20}$$

Thus if $\lambda_i = 1$, we have some $a_{i,1}, \cdots, a_{i,N}$ and b_i such that

$$V_i = \sum_{j=1}^{N} a_{i,j} \frac{y_j}{(1+|y|^2)^{N/2}} + b_i \frac{1-|y|^2}{(1+|y|^2)^{N/2}}$$

by Theorem 2.4. We see that $V_i \not\equiv 0$ by the estimate (2.18). Indeed, if $V_i \equiv 0$, then the maximum point y_ε^i of $\tilde{v}_{i,\varepsilon}$ would satisfy $|y_\varepsilon^i| \to +\infty$, since $\tilde{v}_{i,\varepsilon} \to V_i \equiv 0$ compact uniformly on $\mathbb{R}^N$. But this is impossible because of the estimate (2.18). □

From Lemma 2.6, we can obtain the following convergence result; see [5].

Lemma 2.7 *Suppose* $\lambda_i = \lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} = 1$ *and* $b_i \neq 0$ *in (2.19). Then we have*

$$\|u_\varepsilon\|^2 v_{i,\varepsilon} \to -(N-2) b_i \sigma_N G(\cdot, x_0) \text{ in } C^1_{loc}(\overline{\Omega} \setminus \{x_0\}) \quad \text{as } \varepsilon \to 0. \tag{2.21}$$

Now, since the blow-up point x_0 is an interior point of Ω, we may assume that there exists $\rho > 0$ such that $B(x_\varepsilon, 2\rho) \subset \Omega$ for any $\varepsilon > 0$ sufficiently small. We employ a cut-off function $\phi = \phi(x)$ such that $\phi \in C_0^\infty(B(x_\varepsilon, 2\rho))$, $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on $B(x_\varepsilon, \rho)$. Denote

$$\psi_{j,\varepsilon}(x) = \phi(x) \left(\frac{\partial u_\varepsilon}{\partial x_j} \right), \quad j = 1, \cdots, N, \tag{2.22}$$

$$\psi_{N+1,\varepsilon}(x) = \phi(x) \left((x - x_\varepsilon) \cdot \nabla u_\varepsilon + \frac{2}{p_\varepsilon - 1} u_\varepsilon \right). \tag{2.23}$$

Then, as Lemma 3.1 in [2], we have the following lemma.

Lemma 2.8 $u_\varepsilon, \{\psi_{j,\varepsilon}\}_{j=1,\cdots,N}, \psi_{N+1,\varepsilon}$ *are linearly independent in* $H_0^1(\Omega)$.

Proof. Assume the contrary that there exist $\alpha_{0,\varepsilon}, \alpha_{1,\varepsilon}, \cdots, \alpha_{N,\varepsilon}, \alpha_{N+1,\varepsilon}$ such that $\sum_{j=0}^{N+1} \alpha_{j,\varepsilon}^2 \neq 0$ and

$$\alpha_{0,\varepsilon} u_\varepsilon + \sum_{j=1}^{N} \alpha_{j,\varepsilon} \psi_{j,\varepsilon} + \alpha_{N+1,\varepsilon} \psi_{N+1,\varepsilon} \equiv 0$$

in Ω. Without loss of generality, we may assume that $\sum_{j=0}^{N+1} \alpha_{j,\varepsilon}^2 = 1$.

First we claim that $\alpha_{0,\varepsilon} = 0$. Indeed, if $\alpha_{0,\varepsilon} \neq 0$, then we have

$$u_\varepsilon = \sum_{j=1}^{N+1} \beta_{j,\varepsilon} \psi_{j,\varepsilon} \tag{2.24}$$

where $\beta_{j,\varepsilon} = -\alpha_{j,\varepsilon}/\alpha_{0,\varepsilon}$. On the other hand, by (2.4) and $\phi \equiv 1$ on $B(x_\varepsilon, \rho)$, we see for $j = 1, \cdots, N$,

$$-\Delta \psi_{j,\varepsilon} = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} \psi_{j,\varepsilon} + c_0 \left(\frac{\partial K}{\partial x_j} \right) u_\varepsilon^{p_\varepsilon} \quad \text{on } B(x_\varepsilon, \rho). \tag{2.25}$$

Recall $w_\varepsilon(x) = (x - x_\varepsilon) \cdot \nabla u_\varepsilon(x) + \frac{2}{p_\varepsilon - 1} u_\varepsilon$ satisfies (2.3) (with $y = x_\varepsilon$), thus

$$-\Delta \psi_{N+1,\varepsilon} = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} \psi_{N+1,\varepsilon} + c_0 (x - x_\varepsilon) \cdot \nabla K(x) u_\varepsilon^{p_\varepsilon} \tag{2.26}$$

on $B(x_\varepsilon, \rho)$. Multiplying $\beta_{j,\varepsilon}$ to (2.25) and $\beta_{N+1,\varepsilon}$ to (2.26), and summing up, we have

$$\begin{aligned} - \Delta \left(\sum_{j=1}^{N+1} \beta_{j,\varepsilon} \psi_{j,\varepsilon} \right) &= c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} \left(\sum_{j=1}^{N+1} \beta_{j,\varepsilon} \psi_{j,\varepsilon} \right) \\ &+ c_0 u_\varepsilon^{p_\varepsilon} \left(\sum_{j=1}^{N} \beta_{j,\varepsilon} \left(\frac{\partial K}{\partial x_j} \right) + \beta_{N+1,\varepsilon} (x - x_\varepsilon) \cdot \nabla K(x) \right) \end{aligned} \tag{2.27}$$

on $B(x_\varepsilon, \rho)$. Furthermore, since $u_\varepsilon = \sum_{j=1}^{N+1} \beta_{j,\varepsilon} \psi_{j,\varepsilon}$ is a solution to (1.1), we have

$$-\Delta \left(\sum_{j=1}^{N+1} \beta_{j,\varepsilon} \psi_{j,\varepsilon} \right) = c_0 K(x) u_\varepsilon^{p_\varepsilon}. \tag{2.28}$$

From (2.27) and (2.28), we see

$$c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon} + c_0 u_\varepsilon^{p_\varepsilon} \left(\sum_{j=1}^{N} \beta_{j,\varepsilon} \left(\frac{\partial K}{\partial x_j} \right) + \beta_{N+1,\varepsilon} (x - x_\varepsilon) \cdot \nabla K(x) \right) = c_0 K(x) u_\varepsilon^{p_\varepsilon},$$

that is,

$$K(x)(1 - p_\varepsilon) \equiv \left(\sum_{j=1}^{N} \beta_{j,\varepsilon} \left(\frac{\partial K}{\partial x_j} \right) + \beta_{N+1,\varepsilon} (x - x_\varepsilon) \cdot \nabla K(x) \right) \tag{2.29}$$

on $B(x_\varepsilon,\rho)$. Note that for any small ε, the blow up point $x_0 \in B(x_\varepsilon,\rho)$. Inserting $x = x_0$ into (2.29) and noting that $K(x_0) = 1, \nabla K(x_0) = 0$, we must have $1 - p_\varepsilon = 0$, which is a contradiction. Thus we conclude that $\alpha_{0,\varepsilon} = 0$.

Next, we claim that $\alpha_{N+1,\varepsilon} = 0$. Indeed, putting $x = x_\varepsilon$ into $\sum_{j=1}^N \alpha_{j,\varepsilon}\psi_{j,\varepsilon} + \alpha_{N+1,\varepsilon}\psi_{N+1,\varepsilon} \equiv 0$, we see $\alpha_{N+1,\varepsilon}(\frac{2}{p_\varepsilon-1})u_\varepsilon(x_\varepsilon) = 0$. Thus we obtain $\alpha_{N+1,\varepsilon} = 0$.

Now, we obtain $\sum_{j=1}^N \alpha_{j,\varepsilon}\psi_{j,\varepsilon} \equiv 0$ on Ω. By scaling, this leads to

$$\sum_{j=1}^N \alpha_{j,\varepsilon}\phi_\varepsilon(y)\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}(y) \equiv 0$$

for $y \in \Omega_\varepsilon$, where $\phi_\varepsilon(y) = \phi(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} + x_\varepsilon)$. Using (2.10), we get that

$$\sum_{j=1}^N \alpha_j \frac{\partial U}{\partial y_j} \equiv 0 \quad \text{on } \mathbb{R}^N,$$

where $\alpha_j = \lim_{\varepsilon\to 0}\alpha_{j,\varepsilon}$. Since $\frac{\partial U}{\partial y_j}$ are linearly independent, we have that $\alpha_j = 0$ for all $j = 1, 2, \cdots, N$. But this is impossible since $\sum_{j=1}^N \alpha_j^2 = \lim_{\varepsilon\to 0}(\sum_{j=1}^N \alpha_{j,\varepsilon}^2) = 1$. Thus we have proved Lemma 2.8. □

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 along the line of [2].

Proposition 3.1 *For $i = 2, \cdots, N+1$, we have*

$$\lambda_{i,\varepsilon} \le 1 + \frac{C_1 + o(1)}{\|u_\varepsilon\|^{\frac{4}{N-2}}}, \tag{3.1}$$

where $C_1 = \frac{2(N+1)}{N(N+2)}|\mu_1|$. Moreover,

$$\lim_{\varepsilon\to 0}\lambda_{i,\varepsilon} = 1. \tag{3.2}$$

Proof. By the variational characterization, $\lambda_{i,\varepsilon}$ can be expressed as

$$\lambda_{i,\varepsilon} = \inf_{W\subset H_0^1(\Omega),\dim(W)=i}\ \max_{v\in W}\frac{\int_\Omega |\nabla v|^2 dx}{c_0 p_\varepsilon \int_\Omega K(x)u_\varepsilon^{p_\varepsilon-1}v^2 dx}.$$

We take

$$W = W_i = \text{span}\{u_\varepsilon, \psi_{1,\varepsilon}, \cdots, \psi_{i-1,\varepsilon}\},$$

where $\psi_{j,\varepsilon}$ are defined in (2.22). For $a_0, a_1, \cdots, a_{i-1} \in \mathbb{R}$, we put

$$v = a_0 u_\varepsilon + \sum_{j=1}^{i-1} a_j \psi_{j,\varepsilon} = a_0 u_\varepsilon + \phi z_\varepsilon \in W_i,$$

where $z_\varepsilon = \sum_{j=1}^{i-1} a_j (\frac{\partial u_\varepsilon}{\partial x_j})$. By (2.4), we see z_ε satisfies

$$-\Delta z_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} z_\varepsilon + c_0 u_\varepsilon^{p_\varepsilon} \left(\sum_{j=1}^{i-1} a_j \frac{\partial K}{\partial x_j} \right) \quad \text{in } \Omega,$$

and

$$\int_\Omega \nabla z_\varepsilon \cdot \nabla(\phi^2 z_\varepsilon) dx = c_0 p_\varepsilon \int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1} \phi^2 z_\varepsilon^2 dx + c_0 \int_\Omega \sum_{j=1}^{i-1} a_j \left(\frac{\partial K}{\partial x_j} \right) u_\varepsilon^{p_\varepsilon} \phi^2 z_\varepsilon dx.$$

Thus,

$$\int_\Omega |\nabla v|^2 dx = \int_\Omega |\nabla(a_0 u_\varepsilon + \phi z_\varepsilon)|^2 dx$$
$$= a_0^2 \int_\Omega |\nabla u_\varepsilon|^2 dx + 2a_0 \int_\Omega \nabla u_\varepsilon \cdot \nabla(\phi z_\varepsilon) dx + \int_\Omega |\nabla(\phi z_\varepsilon)|^2 dx,$$
$$\int_\Omega |\nabla(\phi z_\varepsilon)|^2 dx = \int_\Omega |\nabla \phi|^2 z_\varepsilon^2 dx + \int_\Omega \nabla z_\varepsilon \cdot \nabla(\phi^2 z_\varepsilon) dx$$
$$= \int_\Omega |\nabla \phi|^2 z_\varepsilon^2 dx + c_0 p_\varepsilon \int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1} \phi^2 z_\varepsilon^2 dx + c_0 \int_\Omega \sum_{j=1}^{i-1} a_j \left(\frac{\partial K}{\partial x_j} \right) u_\varepsilon^{p_\varepsilon} \phi^2 z_\varepsilon dx,$$
$$\int_\Omega \nabla u_\varepsilon \cdot \nabla(\phi z_\varepsilon) dx = \int_\Omega (-\Delta u_\varepsilon) \phi z_\varepsilon dx = c_0 \int_\Omega K(x) u_\varepsilon^{p_\varepsilon} \phi z_\varepsilon dx.$$

Using these, we have

$$\max_{v \in W_i} \frac{\int_\Omega |\nabla v|^2 dx}{c_0 p_\varepsilon \int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1} v^2 dx} = \max_{a_0, a_1, \cdots, a_{i-1}} \left\{ 1 + \frac{N_\varepsilon}{D_\varepsilon} \right\}$$

where $N_\varepsilon = N_\varepsilon^1 + N_\varepsilon^2 + N_\varepsilon^3 + N_\varepsilon^4$,

$$N_\varepsilon^1 = a_0^2 c_0 (1 - p_\varepsilon) \int_\Omega K(x) u_\varepsilon^{p_\varepsilon+1} dx,$$

$$N_\varepsilon^2 = 2a_0 c_0 (1 - p_\varepsilon) \sum_{j=1}^{i-1} a_j \int_\Omega K(x) u_\varepsilon^{p_\varepsilon} \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \phi dx,$$

$$N_\varepsilon^3 = \sum_{j,l=1}^{i-1} a_j a_l \int_\Omega |\nabla \phi|^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) dx,$$

$$N_\varepsilon^4 = c_0 \sum_{j,l=1}^{i-1} a_j a_l \int_\Omega u_\varepsilon^{p_\varepsilon} \left(\frac{\partial K}{\partial x_j}\right) \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) \phi^2 dx$$

and $D_\varepsilon = D_\varepsilon^1 + D_\varepsilon^2 + D_\varepsilon^3$,

$$D_\varepsilon^1 = c_0 p_\varepsilon a_0^2 \int_\Omega K(x) u_\varepsilon^{p_\varepsilon+1} dx,$$

$$D_\varepsilon^2 = 2c_0 p_\varepsilon a_0 \sum_{j=1}^{i-1} a_j \int_\Omega K(x) u_\varepsilon^{p_\varepsilon} \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \phi dx,$$

$$D_\varepsilon^3 = c_0 p_\varepsilon \sum_{j,l=1}^{i-1} a_j a_l \int_\Omega K(x) u_\varepsilon^{p_\varepsilon-1} \phi^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) dx.$$

As for the estimate of N_ε^2, we calculate

$$\int_\Omega K(x) u_\varepsilon^{p_\varepsilon} \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \phi dx = \frac{1}{p_\varepsilon + 1} \int_\Omega K(x) \phi \left(\frac{\partial u_\varepsilon^{p_\varepsilon+1}}{\partial x_j}\right) dx$$
$$= -\frac{1}{p_\varepsilon + 1} \int_\Omega u_\varepsilon^{p_\varepsilon+1} \left(\frac{\partial K(x)\phi}{\partial x_j}\right) dx = -\frac{1}{p_\varepsilon + 1} (I_1 + I_2)$$

where

$$I_1 = \int_\Omega K(x) u_\varepsilon^{p_\varepsilon+1} \left(\frac{\partial \phi}{\partial x_j}\right) dx, \quad I_2 = \int_\Omega \phi(x) u_\varepsilon^{p_\varepsilon+1} \left(\frac{\partial K}{\partial x_j}\right) dx.$$

Now,

$$I_1 = \frac{1}{\|u_\varepsilon\|^{p_\varepsilon+1}} \int_{\Omega \setminus B(x_\varepsilon, \rho)} (\|u_\varepsilon\| u_\varepsilon)^{p_\varepsilon+1} \left(\frac{\partial \phi}{\partial x_j}\right) dx = O\left(\frac{1}{\|u_\varepsilon\|^{p+1}}\right),$$

where we have used $\phi \equiv 1$ on $B(x_\varepsilon, \rho)$ and (2.13), (2.14).

Next, we treat I_2. By Taylor's theorem, we have

$$K(x) = 1 + \frac{1}{2}\sum_{j,k=1}^{N} b_{jk}(x_j - x_{0j})(x_k - x_{0k}) + O(|x - x_0|^3) \tag{3.3}$$

and

$$\frac{\partial K}{\partial x_j}(x) = \sum_{k=1}^{N} b_{jk}(x_k - x_{0k}) + O(|x - x_0|^2) \tag{3.4}$$

where $b_{jk} = \frac{\partial^2 K}{\partial x_j \partial x_k}(x_0)$. By change of variables $x = \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + x_\varepsilon$, we see

$$\begin{aligned}
&\int_\Omega \phi(x) u_\varepsilon^{p_\varepsilon+1}(x_k - x_{0k})dx \\
&= \|u_\varepsilon\|^{p_\varepsilon+1-(p_\varepsilon-1)N/2-(p_\varepsilon-1)/2} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon+1}\phi_\varepsilon(y) y_k dy \\
&+ \|u_\varepsilon\|^{p_\varepsilon+1-(p_\varepsilon-1)N/2} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon+1}\phi_\varepsilon(y) O(|x_{\varepsilon k} - x_{0k}|) dy \\
&= \|u_\varepsilon\|^{-2/(N-2)} \left[\int_{\mathbb{R}^N} U^{p+1} y_k dy + o(1)\right] \\
&+ \left[\int_{\mathbb{R}^N} U^{p+1} dy \times \begin{cases} o(\|u_\varepsilon\|^{-2/(N-2)}), & (N \geq 4) \\ O(\|u_\varepsilon\|^{-2}), & (N = 3) \end{cases} + o(1)\right] \\
&= \begin{cases} o(\|u_\varepsilon\|^{-2/(N-2)}), & (N \geq 4), \\ O(\|u_\varepsilon\|^{-2}), & (N = 3), \end{cases}
\end{aligned}$$

where we have used (2.12) and (2.13) in Lemma 2.3. Thus

$$\begin{aligned}
I_2 &= \int_\Omega \phi(x) u_\varepsilon^{p_\varepsilon+1}\left(\frac{\partial K}{\partial x_j}\right) dx \qquad (3.5)\\
&= \sum_{k=1}^{N} b_{jk} \int_\Omega u_\varepsilon^{p_\varepsilon+1}\phi(x)(x_k - x_{0k})dx + \int_\Omega u_\varepsilon^{p_\varepsilon+1}\phi(x) O(|x - x_0|^2)dx \\
&= \begin{cases} o(\|u_\varepsilon\|^{-2/(N-2)}), & (N \geq 4), \\ O(\|u_\varepsilon\|^{-2}). & (N = 3). \end{cases}
\end{aligned}$$

In conclusion, we have

$$\begin{aligned}&\int_\Omega K(x)u_\varepsilon^{p_\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\phi dx=-\frac{1}{p_\varepsilon+1}(I_1+I_2)\\&=O\left(\frac{1}{\|u_\varepsilon\|^{p+1}}\right)+O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right)=O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right).\end{aligned} \tag{3.6}$$

Also, as (3.25) in [2],

$$\begin{aligned}&\int_\Omega|\nabla\phi|^2\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial u_\varepsilon}{\partial x_l}\right)dx\\&=\frac{1}{\|u_\varepsilon\|^2}\int_{\Omega\setminus B(x_\varepsilon,\rho)}|\nabla\phi|^2\left(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial x_l}\right)dx=O\left(\frac{1}{\|u_\varepsilon\|^2}\right).\end{aligned} \tag{3.7}$$

Thus we have

$$N_\varepsilon^2=a_0O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right),\quad N_\varepsilon^3=O\left(\frac{1}{\|u_\varepsilon\|^2}\right). \tag{3.8}$$

As for N_ε^4, by (3.4) again, we see

$$\begin{aligned}&\int_\Omega u_\varepsilon^{p_\varepsilon}\left(\frac{\partial K}{\partial x_j}\right)\left(\frac{\partial u_\varepsilon}{\partial x_l}\right)\phi^2dx\\&=\int_\Omega u_\varepsilon^{p_\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_l}\right)\phi^2\left\{\sum_{k=1}^N b_{jk}(x_k-x_{0k})+O(|x-x_0|^2)\right\}dx\\&=E_1+E_2\end{aligned}$$

where

$$\begin{aligned}E_1&=\sum_{k=1}^N b_{jk}\int_\Omega u_\varepsilon^{p_\varepsilon}(\frac{\partial u_\varepsilon}{\partial x_l})(x_k-x_{0k})\phi^2dx\\E_2&=\int_\Omega u_\varepsilon^{p_\varepsilon}(\frac{\partial u_\varepsilon}{\partial x_l})O(|x-x_0|^2)\phi^2dx.\end{aligned}$$

By change of variables

$$x=\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon,\quad \frac{\partial u_\varepsilon}{\partial x_l}(x)=\|u_\varepsilon\|^{\frac{p_\varepsilon+1}{2}}\frac{\partial\tilde u_\varepsilon}{\partial y_l}(y),$$

we see

$$
\begin{aligned}
E_1 = &\sum_{k=1}^{N} b_{jk} \|u_\varepsilon\|^{p_\varepsilon+(p_\varepsilon+1)/2-(p_\varepsilon-1)N/2} \times \\
&\times \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_l}\right) \left(\frac{y_k}{\|u_\varepsilon\|^{(p_\varepsilon-1)/2}} + x_{\varepsilon,k} - x_{0k}\right) \phi_\varepsilon^2(y) dy \\
&= F_1 + F_2
\end{aligned}
$$

where

$$
\begin{aligned}
F_1 &= \sum_{k=1}^{N} b_{jk} \|u_\varepsilon\|^{p_\varepsilon+(p_\varepsilon+1)/2-(p_\varepsilon-1)/2-(p_\varepsilon-1)N/2} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_l}\right) y_k \phi_\varepsilon^2(y) dy \\
&= \sum_{k=1}^{N} b_{jk} \|u_\varepsilon\|^{\varepsilon(N/2-1)} \left[\int_{\mathbb{R}^N} U^p(y) \left(\frac{\partial U}{\partial y_l}\right) y_k dy + o(1)\right] \\
&= \sum_{k=1}^{N} b_{jk} \left[\left(\frac{2-N}{N}\right) \int_{\mathbb{R}^N} U^p(y) \frac{|y|^2}{(1+|y|^2)^{N/2}} dy \delta_{kl} + o(1)\right] \\
&= b_{jl} \left(\frac{2-N}{N}\right) \int_{\mathbb{R}^N} \frac{|y|^2}{(1+|y|^2)^{N+1}} dy + o(1) \\
&= -D_N b_{jl} + o(1),
\end{aligned}
$$

where

$$
D_N = \left(\frac{N-2}{N}\right) \int_{\mathbb{R}^N} \frac{|y|^2}{(1+|y|^2)^{N+1}} dy = \left(\frac{N-2}{4N}\right) \sigma_N \frac{\Gamma(N/2)^2}{\Gamma(N)}.
$$

Similarly, we have

$$
\begin{aligned}
F_2 &= \sum_{k=1}^{N} b_{jk} \|u_\varepsilon\|^{2/(N-2)+\varepsilon(N-3)/2} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_l}\right) \phi_\varepsilon^2(y) O(|x_\varepsilon - x_0|) \phi^2 dy \\
&= \sum_{k=1}^{N} b_{jk} \|u_\varepsilon\|^{2/(N-2)} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_l}\right) \phi_\varepsilon^2(y) O(\|u_\varepsilon\|^{-2/(N-2)}) \phi^2 dy \\
&= O(1) \left[\int_{\mathbb{R}^N} U^p(y) \left(\frac{\partial U}{\partial y_l}\right) dy + o(1)\right] \\
&= o(1),
\end{aligned}
$$

when $N \geq 3$. Here, we again used Lemma 2.3 (2.12) and (2.13). Thus, we have

$$E_1 = F_1 + F_2 = -b_{jl} D_N + o(1).$$

On the other hand,

$$\begin{aligned}
E_2 &= \int_\Omega u_\varepsilon^{p_\varepsilon} \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) O(|x - x_0|^2)\phi^2 dx \\
&= \int_\Omega u_\varepsilon^{p_\varepsilon} \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) O(|x - x_\varepsilon|^2)\phi^2 dx + \int_\Omega u_\varepsilon^{p_\varepsilon} \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) O(|x_\varepsilon - x_0|^2)\phi^2 dx \\
&= \|u_\varepsilon\|^{-2/(N-2)+\varepsilon((N-1)/2)} \left[\int_{\mathbb{R}^N} U^p(y) \left(\frac{\partial U}{\partial y_l}\right) |y^2| dy + o(1)\right] \\
&+ \|u_\varepsilon\|^{2/(N-2)+\varepsilon((N-3)/2)} O(|x_\varepsilon - x_0|^2) \left[\int_{\mathbb{R}^N} U^p \left(\frac{\partial U}{\partial y_l}\right) dy + o(1)\right] \\
&= O(\|u_\varepsilon\|^{-2/(N-2)}).
\end{aligned}$$

Thus,

$$\int_\Omega u_\varepsilon^{p_\varepsilon} \left(\frac{\partial K}{\partial x_j}\right) \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) \phi^2 dx = E_1 + E_2 = -b_{jl} D_N + o(1) \tag{3.9}$$

and

$$\begin{aligned}
N_\varepsilon^4 &= c_0 \sum_{j,l=1}^{i-1} a_j a_l \int_\Omega \phi^2 u_\varepsilon^{p_\varepsilon} \left(\frac{\partial K}{\partial x_j}\right) \left(\frac{\partial u_\varepsilon}{\partial x_l}\right) dx \\
&= c_0 D_N \sum_{j,l=1}^{i-1} a_j a_l (-b_{jl}) + o(1).
\end{aligned}$$

Note that the Hessian matrix $\text{Hess}K(x_0) = (b_{jl})_{1 \leq j,l \leq N}$ is negative definite by the assumption (A), so $|\mu_1|$ is the largest among all $|\mu_j|$, $j = 1, \cdots, N$. In conclusion, we have

$$c_0 D_N (\sum_{j=1}^{i-1} a_j^2) |\mu_N| + o(1) \leq N_\varepsilon^4 \leq c_0 D_N (\sum_{j=1}^{i-1} a_j^2) |\mu_1| + o(1). \tag{3.10}$$

Estimate for D_ε^2 is the same as for N_ε^2.

$$D_\varepsilon^2 = a_0 O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right). \tag{3.11}$$

As for D^3_ε, by change of variables we see

$$\begin{aligned}
&\int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1}\phi^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial u_\varepsilon}{\partial x_l}\right) dx \qquad (3.12)\\
&= \|u_\varepsilon\|^{p_\varepsilon - 1 + 2(\frac{p_\varepsilon+1}{2}) - \frac{p_\varepsilon - 1}{2}N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon-1}\phi_\varepsilon^2(y)\left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}\right)\left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_l}\right) dy \\
&= \|u_\varepsilon\|^{4/(N-2)}\left(\int_{\mathbb{R}^N} U^{p-1}\left(\frac{\partial U}{\partial y_j}\right)\left(\frac{\partial U}{\partial y_l}\right) dy + o(1)\right) \\
&= \|u_\varepsilon\|^{4/(N-2)}\left(\frac{\delta_{jl}}{N}\int_{\mathbb{R}^N} U^{p-1}|\nabla U|^2 dy + o(1)\right).
\end{aligned}$$

Here, we have used the fact $\nabla \tilde{u}_\varepsilon \to \nabla U$ in $L^2(\mathbb{R}^N)$ by (2.10). Thus,

$$\begin{aligned}
D^3_\varepsilon &= c_0 p_\varepsilon \sum_{j,l=1}^{i-1} a_j a_l \|u_\varepsilon\|^{4/(N-2)}\delta_{jl}\left(\frac{1}{N}\int_{\mathbb{R}^N} U^{p-1}|\nabla U|^2 dy + o(1)\right) \qquad (3.13)\\
&= c_0 p \sum_{j=1}^{i-1} a_j^2 \|u_\varepsilon\|^{4/(N-2)}\left(\frac{1}{N}\int_{\mathbb{R}^N} U^{p-1}|\nabla U|^2 dy + o(1)\right) \\
&= \|u_\varepsilon\|^{4/(N-2)}\left(\sum_{j=1}^{i-1} a_j^2\right)\left(\frac{c_0 p}{N}\int_{\mathbb{R}^N} U^{p-1}|\nabla U|^2 dy + o(1)\right).
\end{aligned}$$

Note that

$$\int_{\mathbb{R}^N} U^{p-1}|\nabla U|^2 dy = \frac{N(N-2)^2}{8(N+1)}\sigma_N \frac{\Gamma(N/2)^2}{\Gamma(N)}.$$

Now, by testing $(a_0, a_1, \cdots, a_{i-1}) = (0, 1, \cdots, 1)$, we have

$$\begin{aligned}
\max_{v \in W_i} \frac{\int_\Omega |\nabla v|^2 dx}{c_0 p_\varepsilon \int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1} v^2 dx} &= \max_{(a_0, a_1, \cdots, a_{i-1}) \in \mathbb{R}^i}\left\{1 + \frac{N_\varepsilon}{D_\varepsilon}\right\} \\
&\geq 1 + \frac{O(\frac{1}{\|u_\varepsilon\|^2}) + c_0 D_N (\sum_{j=1}^{i-1} a_j^2)|\mu_N| + o(1)}{\|u_\varepsilon\|^{4/(N-2)}}
\end{aligned}$$

by the lower estimate of (3.10).

Thus we have some $C_0 > 0$ such that

$$\max_{v \in W_i} \frac{\int_\Omega |\nabla v|^2 dx}{c_0 p_\varepsilon \int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1} v^2 dx} \geq 1 + \frac{C_0}{\|u_\varepsilon\|^{4/(N-2)}} \qquad (3.14)$$

for any $\varepsilon > 0$ small.

Let $(a_{0,\varepsilon}, a_{1,\varepsilon}, \cdots, a_{i-1,\varepsilon}) \in \mathbb{R}^i$ be a maximizer of $\max_{(a_0,a_1,\cdots,a_{i-1})\in\mathbb{R}^i} \left\{1 + \frac{N_\varepsilon}{D_\varepsilon}\right\}$. Since

$$\left\{1 + \frac{N_\varepsilon}{D_\varepsilon}\right\}_{(a_0,a_1,\cdots,a_{i-1})=(1,0,\cdots,0)} = 1/p_\varepsilon$$

is the first eigenvalue $\lambda_{1,\varepsilon}$, and so $(1, 0, \cdots, 0)$ is not a maximizer, we may assume that $\sum_{j=1}^{i-1} a_{j,\varepsilon}^2 = 1$. In this case, by using (3.14), we claim that $a_{0,\varepsilon}^2$ is uniformly bounded in ε. Indeed, if $a_{0,\varepsilon}^2 \to \infty$ as $\varepsilon \to 0$, then for some $\varepsilon > 0$ small, we have by (3.8), (3.10), (3.11) and (3.13),

$$\begin{aligned}
&\max_{v\in W_i} \frac{\int_\Omega |\nabla v|^2 dx}{c_0 p_\varepsilon \int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1} v^2 dx} \\
&= 1 + \frac{a_{0,\varepsilon}^2 c_0(1-p_\varepsilon)\int_\Omega K(x) u_\varepsilon^{p_\varepsilon+1} dx + a_{0,\varepsilon}(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}) + (\frac{1}{\|u_\varepsilon\|^2}) + O(1)}{c_0 p_\varepsilon a_{0,\varepsilon}^2 \int_\Omega K(x) u_\varepsilon^{p_\varepsilon+1} dx + a_{0,\varepsilon}(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}) + \|u_\varepsilon\|^{4/(N-2)}(\sum_{j=1}^{i-1} a_{\varepsilon,j}^2) O(1)} \\
&= 1 + \frac{a_{0,\varepsilon}^2(-C) + O(1)}{\|u_\varepsilon\|^{4/(N-2)}\{\frac{a_{0,\varepsilon}^2}{\|u_\varepsilon\|^{4/(N-2)}} O(1) + O(1)\}} \\
&\leq 1 + \frac{-C}{\|u_\varepsilon\|^{4/(N-2)}}
\end{aligned}$$

for some $C > 0$, which contradicts (3.14). Thus we have checked $a_{0,\varepsilon}^2 = O(1)$.

From the above claim and estimates above again, we have

$$\begin{aligned}
&\max_{v\in W_i} \frac{\int_\Omega |\nabla v|^2 dx}{c_0 p_\varepsilon \int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1} v^2 dx} \\
&= \left\{1 + \frac{N_\varepsilon}{D_\varepsilon}\right\}\Big|_{(a_0,a_1,\cdots,a_{i-1})=(a_{0,\varepsilon},a_{1,\varepsilon},\cdots,a_{i-1,\varepsilon})} \\
&= 1 + \frac{a_{0,\varepsilon}^2 c_0(1-p_\varepsilon)\int_\Omega K(x) u_\varepsilon^{p_\varepsilon+1} dx + a_{0,\varepsilon} O(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}) + O(\frac{1}{\|u_\varepsilon\|^2}) + N_\varepsilon^4}{c_0 p_\varepsilon a_{0,\varepsilon}^2 \int_\Omega K(x) u_\varepsilon^{p_\varepsilon+1} dx + a_{0,\varepsilon} O(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}) + D_\varepsilon^3} \\
&\leq 1 + \frac{N_\varepsilon^4 + o(1)}{O(1) + D_\varepsilon^3} \\
&\leq 1 + \frac{c_0 D_N |\mu_1| + o(1)}{\|u_\varepsilon\|^{4/(N-2)}\left(\frac{c_0 p}{N}\int_{\mathbb{R}^N} U^{p-1}|\nabla U|^2 dy + o(1)\right)} \\
&\leq 1 + \frac{C_1 + o(1)}{\|u_\varepsilon\|^{4/(N-2)}},
\end{aligned}$$

where

$$C_1 = \frac{c_0 D_N |\mu_1|}{\frac{c_0 p}{N} \int_{\mathbb{R}^N} U^{p-1} |\nabla U|^2 dy} = \frac{2(N+1)}{N(N+2)} |\mu_1|. \tag{3.15}$$

This proves (3.1).

To prove (3.2), first by using (3.1), we notice that $\lambda_{i,\varepsilon} \to \Lambda \in [0,1]$ for some Λ as $\varepsilon \to 0$. As in the derivation of (2.20), we have $\tilde{v}_{i,\varepsilon} \to V$ in $C^1_{loc}(\mathbb{R}^N)$ for some $V \not\equiv 0$ and V is a solution of

$$\begin{cases} -\Delta V = \Lambda c_0 p U^{p-1} V \text{ in } \mathbb{R}^N, \\ V \in D^{1,2}(\mathbb{R}^N). \end{cases}$$

By Theorem 2.4, we conclude that, if $\Lambda < 1$, then we must have that $\Lambda = 1/p$ and $V = U$. However, this leads to a contradiction because $v_{i,\varepsilon}$ is orthogonal to $v_{1,\varepsilon} = u_\varepsilon / \|u_\varepsilon\|$ for $i \geq 2$. Indeed, by (1.4), we have

$$\begin{aligned} &\int_\Omega K(x) u_\varepsilon^{p_\varepsilon} v_{i,\varepsilon} dy = 0 \\ &\Rightarrow 0 = \int_{\Omega_\varepsilon} K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_{i,\varepsilon} dy \to \int_{\mathbb{R}^N} U^p V dy, \end{aligned}$$

thus this leads to $\int_{\mathbb{R}^N} U^{p+1} dy = 0$, which is absurd. Therefore we conclude that $\Lambda = 1$ and the proof of Proposition 3.1 is finished. □

Lemma 3.2 *Let $i \in \mathbb{N}$ be such that $\lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} = 1$. If b_i in (2.19) of Lemma 2.6 is not 0, then we have*

$$\lambda_{i,\varepsilon} - 1 = \frac{C_2 + o(1)}{\|u_\varepsilon\|^{4/(N-2)}}, \quad (N \geq 4), \tag{3.16}$$

$$\lambda_{i,\varepsilon} - 1 = \frac{C_2 + o(1)}{\|u_\varepsilon\|^2}, \quad (N = 3), \tag{3.17}$$

where

$$C_2 = \begin{cases} \frac{4(N+1)}{N(N^2-4)} |\Delta K(x_0)|, & (N \geq 5), \\ 40\pi^2 R(x_0) + \frac{5}{12} |\Delta K(x_0)|, & (N = 4), \\ \frac{2^9}{15} R(x_0), & (N = 3). \end{cases} \tag{3.18}$$

Proof. First we treat the case $N \geq 4$. We use the integral identity (2.1) in Lemma 2.1 with $y = x_0$. The LHS of (2.1) can be written as

$$\frac{1}{\|u_\varepsilon\|^3}\int_{\partial\Omega}(x-x_0)\cdot\nu\left(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial\nu}\right)\left(\frac{\partial\|u_\varepsilon\|^2 v_{i,\varepsilon}}{\partial\nu}\right)ds_x$$
$$=\frac{1}{\|u_\varepsilon\|^3}\left[-(N-2)^2\sigma_N^2 b_i\int_{\partial\Omega}(x-x_0)\cdot\nu\left(\frac{\partial G}{\partial\nu}(x,x_0)\right)^2 ds_x+o(1)\right]$$
$$=\frac{1}{\|u_\varepsilon\|^3}\left[-(N-2)^3\sigma_N^2 R(x_0)b_i+o(1)\right]. \tag{3.19}$$

Here we have used (2.14), (2.21) and Lemma 2.2 (2.5).

On the other hand, the RHS of (2.1) $= I_1 + I_2$, where

$$I_1=(1-\lambda_{i,\varepsilon})c_0p_\varepsilon\int_\Omega K(x)u_\varepsilon^{p_\varepsilon-1}w_\varepsilon v_{i,\varepsilon}dx,$$
$$I_2=c_0\int_\Omega\left((x-x_0)\cdot\nabla K(x)\right)u_\varepsilon^{p_\varepsilon}v_{i,\varepsilon}dx,$$

and $w_\varepsilon(x)=(x-x_0)\cdot\nabla u_\varepsilon+\frac{2}{p_\varepsilon-1}u_\varepsilon$. Denote

$$\tilde{w}_\varepsilon(y)=\frac{1}{\|u_\varepsilon\|}w_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon\right)$$
$$=y\cdot\nabla_y\tilde{u}_\varepsilon(y)+\frac{2}{p_\varepsilon-1}\tilde{u}_\varepsilon(y)+\|u_\varepsilon\|^{(p_\varepsilon-1)/2}(x_\varepsilon-x_0)\cdot\nabla_y\tilde{u}_\varepsilon(y) \tag{3.20}$$

for $y\in\Omega_\varepsilon$. By (2.10) and (2.12) for $N\geq 4$ in Lemma 2.3, we see

$$\tilde{w}_\varepsilon\to y\cdot\nabla U+\frac{N-2}{2}U=\left(\frac{2}{p-1}\right)\frac{1-|y|^2}{(1+|y|^2)^{N/2}},\quad\text{in } C^1_{loc}(\mathbb{R}^N).$$

Thus,

$$I_1=(1-\lambda_{i,\varepsilon})c_0p_\varepsilon\|u_\varepsilon\|^{p_\varepsilon-(p_\varepsilon-1)N/2}\int_{\Omega_\varepsilon}K_\varepsilon(y)\tilde{u}_\varepsilon^{p_\varepsilon-1}\tilde{w}_\varepsilon\tilde{v}_{i,\varepsilon}(y)dy$$
$$=(1-\lambda_{i,\varepsilon})c_0p\|u_\varepsilon\|^{-1}\times$$
$$\left[\int_{\mathbb{R}^N}U^{p-1}\left(y\cdot\nabla U+\frac{2}{p-1}U\right)\left(\sum_{j=1}^N a_{i,j}\frac{y_j}{(1+|y|^2)^{N/2}}+b_i\frac{1-|y|^2}{(1+|y|^2)^{N/2}}\right)dy+o(1)\right]$$
$$=(1-\lambda_{i,\varepsilon})\|u_\varepsilon\|^{-1}b_ic_0p\left(\frac{N-2}{2}\right)\left[\int_{\mathbb{R}^N}U^{p-1}\frac{(1-|y|^2)^2}{(1+|y|^2)^N}dy+o(1)\right]$$
$$=(1-\lambda_{i,\varepsilon})\|u_\varepsilon\|^{-1}\left[b_iD+o(1)\right].$$

Here,

$$D = c_0 p\left(\frac{N-2}{2}\right)\int_{\mathbb{R}^N} U^{p-1}\frac{(1-|y|^2)^2}{(1+|y|^2)^N}dy = \frac{N(N^2-4)}{4(N+1)}\sigma_N\frac{\Gamma(N/2)^2}{\Gamma(N)}. \tag{3.21}$$

Also, inserting (3.4) into I_2, we have

$$\begin{aligned} I_2 &= c_0\int_\Omega\left(\sum_{j,k=1}^N b_{jk}(x_j - x_{0j})(x_k - x_{0k}) + O(|x - x_0|^3)\right)u_\varepsilon^{p_\varepsilon} v_{i,\varepsilon}dx \\ &= C_1 + C_2 + C_3 + C_4, \end{aligned}$$

where

$$C_1 = c_0\|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - (p_\varepsilon - 1)}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_{i,\varepsilon}(y)\sum_{j,k=1}^N b_{jk}y_j y_k dy,$$

$$C_2 = 2c_0\|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N - (\frac{p_\varepsilon - 1}{2})}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_{i,\varepsilon}(y)\sum_{j,k=1}^N b_{jk}y_j(x_{\varepsilon k} - x_{0k})dy,$$

$$C_3 = c_0\|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_{i,\varepsilon}(y)\sum_{j,k=1}^N b_{jk}(x_{\varepsilon j} - x_{0j})(x_{\varepsilon k} - x_{0k})dy,$$

$$C_4 = c_0\|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{2})N}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_{i,\varepsilon}(y)\left(O\left(\left|\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + x_\varepsilon - x_0\right|^3\right)\right)dy.$$

Again by (2.12), (2.13), (2.16), (2.18) and the dominated convergence theorem, we see

$$\begin{aligned}
C_2 &= O(\|u_\varepsilon\|^{\frac{-N}{N-2}+\frac{N-1}{2}\varepsilon}) \times O\left(\int_{\mathbb{R}^N} U^p V_i(y)|y|dy + o(1)\right) \times o(\|u_\varepsilon\|^{-\frac{2}{N-2}}) \\
&= o(\|u_\varepsilon\|^{-(N+2)/(N-2)}), \\
C_3 &= O(\|u_\varepsilon\|^{-1+\frac{N-2}{2}\varepsilon}) \times O\left(\int_{\mathbb{R}^N} U^p V_i(y)dy + o(1)\right) \times o(\|u_\varepsilon\|^{-\frac{4}{N-2}}) \\
&= o(\|u_\varepsilon\|^{-(N+2)/(N-2)}), \\
C_4 &= O(\|u_\varepsilon\|^{-1+\frac{N-2}{2}\varepsilon}) \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_{i,\varepsilon}(y) \left(O\left(\Big| \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} \Big|^3 \right) + O(|x_\varepsilon - x_0|^3) \right) \\
&= O(\|u_\varepsilon\|^{-1}) \times O(\|u_\varepsilon\|^{-\frac{6}{N-2}}) \times O\left(\int_{\mathbb{R}^N} U^p V_i(y)(|y|^3+1)dy + o(1)\right) \\
&= O(\|u_\varepsilon\|^{-\frac{N+4}{N-2}}) = o(\|u_\varepsilon\|^{-(N+2)/(N-2)})
\end{aligned}$$

as $\varepsilon \to 0$. Note that the integral $\int_{\mathbb{R}^N} U^p V_i(y)(|y|^3+1)dy < \infty$ if $N \geq 4$.

As for C_1, we calculate

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_{i,\varepsilon}(y) \sum_{j,k=1}^N b_{jk} y_j y_k dy \to \int_{\mathbb{R}^N} U^p(y) V_i(y) \sum_{j,k=1}^N b_{jk} y_j y_k dy \\
&= \int_{\mathbb{R}^N} U^p(y) \left(\sum_{l=1}^N a_{i,l} \frac{y_l}{(1+|y|^2)^{N/2}} + b_i \frac{1-|y|^2}{(1+|y|^2)^{N/2}} \right) \sum_{j,k=1}^N b_{jk} y_j y_k dy \\
&= \int_{\mathbb{R}^N} U^p(y) b_i \frac{1-|y|^2}{(1+|y|^2)^{N/2}} \sum_{j=1}^N b_{jj} y_j^2 dy \\
&= \frac{1}{N} \int_{\mathbb{R}^N} U^p(y) b_i \frac{1-|y|^2}{(1+|y|^2)^{N/2}} \Delta K(x_0) |y|^2 dy \\
&= \frac{1}{N} b_i \Delta K(x_0) \left(\int_{\mathbb{R}^N} \frac{|y|^2}{(1+|y|^2)^{N+1}} dy - \int_{\mathbb{R}^N} \frac{|y|^4}{(1+|y|^2)^{N+1}} dy \right) \\
&= \frac{1}{N(N-2)} b_i |\Delta K(x_0)| \sigma_N \frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)}, \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Note that by the oddness of the integrand,

$$\int_{\mathbb{R}^N} U^p(y) \frac{1}{(1+|y|^2)^{N/2}} y_j y_k y_l dy = 0$$

for any $j, k, l \in \{1, \cdots, N\}$, so the integral involving $a_{i,l}$ terms in V_i must vanish. Thus,

$$\begin{aligned}
C_1 &= c_0 \|u_\varepsilon\|^{-(N+2)/(N-2)+(N/2)\varepsilon} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_{i,\varepsilon}(y) \sum_{j,k=1}^{N} b_{jk} y_j y_k dy \\
&= \left[b_i |\Delta K(x_0)| \sigma_N \frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)} + o(1) \right] \|u_\varepsilon\|^{-(N+2)/(N-2)} \\
&= (b_i A + o(1)) \|u_\varepsilon\|^{-(N+2)/(N-2)}
\end{aligned}$$

where

$$A = |\Delta K(x_0)| \sigma_N \frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)}. \tag{3.22}$$

Therefore, for $N \geq 4$, we have

$$\frac{-(N-2)^3 \sigma_N^2}{\|u_\varepsilon\|^3} b_i \left[R(x_0) + o(1) \right] = (1 - \lambda_{i,\varepsilon}) \frac{b_i D + o(1)}{\|u_\varepsilon\|} + \frac{b_i A + o(1)}{\|u_\varepsilon\|^{(N+2)/(N-2)}} \tag{3.23}$$

where A and D are as in (3.22) and (3.21).

Note that $(N+2)/(N-2) < 3$ when $N \geq 5$. Thus dividing (3.23) by $\frac{b_i D}{\|u_\varepsilon\|} \neq 0$, we obtain

$$(\lambda_{i,\varepsilon} - 1) = \frac{1}{\|u_\varepsilon\|^{\frac{4}{N-2}}} \left(C_2 + o(1) \right), \tag{3.24}$$

where

$$C_2 = A/D = \frac{4(N+1)}{N(N^2-4)} |\Delta K(x_0)|.$$

Also when $N = 4$, we have by (3.23),

$$\frac{-2^3 \sigma_4^2}{\|u_\varepsilon\|^3} b_i \left[R(x_0) + o(1) \right] = (1 - \lambda_{i,\varepsilon}) \frac{b_i D + o(1)}{\|u_\varepsilon\|} + \frac{b_i A + o(1)}{\|u_\varepsilon\|^3},$$

which implies, with noting $\sigma_4 = 2\pi^2$,

$$\begin{aligned}
(\lambda_{i,\varepsilon} - 1) &= \frac{1}{\|u_\varepsilon\|^2} \left(\frac{A + 8\sigma_4^2 R(x_0)}{D} + o(1) \right) \\
&= \frac{1}{\|u_\varepsilon\|^2} \left(\frac{5}{12} |\Delta K(x_0)| + 40\pi^2 R(x_0) + o(1) \right).
\end{aligned} \tag{3.25}$$

Next, we treat the case $N = 3$. We use the integral identity (2.1) in Lemma 2.1 with $y = x_\varepsilon$. The estimate for the LHS of (2.1) is the same as in the case $N \geq 4$. On the other hand, the RHS of (2.1) $= I_1 + I_3$, where

$$I_1 = (1 - \lambda_{i,\varepsilon}) c_0 p_\varepsilon \int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1} w_\varepsilon v_{i,\varepsilon} dx,$$
$$I_3 = c_0 \int_\Omega \left((x - x_\varepsilon) \cdot \nabla K(x)\right) u_\varepsilon^{p_\varepsilon} v_{i,\varepsilon} dx,$$

and $w_\varepsilon(x) = (x - x_\varepsilon) \cdot \nabla u_\varepsilon + \frac{2}{p_\varepsilon - 1} u_\varepsilon$. Estimate for I_1 is the same as $N \geq 4$. For the estimate of I_3, we use a Taylor expansion of K around x_ε:

$$K(x) = K(x_\varepsilon) + \nabla K(x_\varepsilon) \cdot (x - x_\varepsilon) + \frac{1}{2} \sum_{j,k=1}^N b_{jk}^\varepsilon (x_j - (x_\varepsilon)_j)(x_k - (x_\varepsilon)_k) + O(|x - x_\varepsilon|^3)$$

where $b_{jk}^\varepsilon = \frac{\partial^2 K}{\partial x_j \partial x_k}(x_\varepsilon)$. We put

$$(x - x_\varepsilon) \cdot \nabla K(x) = (x - x_\varepsilon) \cdot \nabla K(x_\varepsilon) + \sum_{j,k=1}^N b_{jk}^\varepsilon (x_j - (x_\varepsilon)_j)(x_k - (x_\varepsilon)_k) + O(|x - x_\varepsilon|^3)$$

into I_3 to obtain that $I_3 = J_1 + J_2 + J_3$, where

$$J_1 = c_0 \int_\Omega \left((x - x_\varepsilon) \cdot \nabla K(x_\varepsilon)\right) u_\varepsilon^{p_\varepsilon} v_{i,\varepsilon} dx,$$
$$J_2 = c_0 \int_\Omega \left(\sum_{j,k=1}^N b_{jk}^\varepsilon (x_j - (x_\varepsilon)_j)(x_k - (x_\varepsilon)_k) \right) u_\varepsilon^{p_\varepsilon} v_{i,\varepsilon} dx,$$
$$J_3 = c_0 \int_\Omega O(|x - x_\varepsilon|^3) u_\varepsilon^{p_\varepsilon} v_{i,\varepsilon} dx.$$

By change of variables as before, we see,

$$\begin{aligned} J_1 &= c_0 \|u_\varepsilon\|^{p_\varepsilon - (p_\varepsilon - 1)/2 - (p_\varepsilon - 1)N/2} \int_{\Omega_\varepsilon} \nabla K(x_\varepsilon) \cdot y \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_{i,\varepsilon} dy \\ &= c_0 \|u_\varepsilon\|^{-N/(N-2)} \left[\int_{\mathbb{R}^N} \nabla K(x_0) \cdot y U^p(y) V_i(y) dy + o(1) \right] \\ &= \|u_\varepsilon\|^{-N/(N-2)} \times o(1) = o(\|u_\varepsilon\|^{-N/(N-2)}), \end{aligned}$$

$$
\begin{aligned}
J_2 &= c_0\|u_\varepsilon\|^{p_\varepsilon-(p_\varepsilon-1)-(p_\varepsilon-1)N/2}\sum_{j,k=1}^{N} b_{jk}^{\varepsilon}\int_{\Omega_\varepsilon} y_j y_k \tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_{i,\varepsilon}dy \\
&= c_0\|u_\varepsilon\|^{-(N+2)/(N-2)}\sum_{j,k=1}^{N} b_{jk}\left[\int_{\mathbb{R}^N} y_j y_k U^p(y)V_i(y)dy + o(1)\right] \\
&= \|u_\varepsilon\|^{-(N+2)/(N-2)}\sum_{j=1}^{N} b_{jj}\left[\frac{\delta_{jk}}{N}\int_{\mathbb{R}^N} |y|^2 U^p(y)V_i(y)dy + o(1)\right] \\
&= \|u_\varepsilon\|^{-(N+2)/(N-2)}\left[b_i|\Delta K(x_0)|\sigma_N\frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)} + o(1)\right] \\
&= o(\|u_\varepsilon\|^{-N/(N-2)}),
\end{aligned}
$$

and

$$
\begin{aligned}
|J_3| &\le c_0\mathrm{diam}(\Omega)\int_\Omega O(|x-x_\varepsilon|^2)u_\varepsilon^{p_\varepsilon}|v_{i,\varepsilon}|dx \\
&= O(\|u_\varepsilon\|^{p_\varepsilon-(p_\varepsilon-1)N/2-(p_\varepsilon-1)})\left[\int_{\mathbb{R}^N}|y|^2U^p|V_i(y)|dy + o(1)\right] \\
&= O(\|u_\varepsilon\|^{-(N+2)/(N-2)}) = o(\|u_\varepsilon\|^{-3})
\end{aligned}
$$

for $N=3$. Note that $|y|^2U^p|V_i(y)| \in L^1(\mathbb{R}^3)$. In conclusion, we see $I_3 = o(\|u_\varepsilon\|^{-3})$ when $N=3$. Thus, we obtain

$$
\frac{-\sigma_3^2}{\|u_\varepsilon\|^3}b_i\left[R(x_0)+o(1)\right] = (1-\lambda_{i,\varepsilon})\frac{b_iD+o(1)}{\|u_\varepsilon\|} + o(\frac{1}{\|u_\varepsilon\|^3}) \tag{3.26}
$$

where

$$
D = \frac{3(3^2-4)}{4(3+1)}\sigma_3\frac{\Gamma(3/2)^2}{\Gamma(3)} = \frac{15}{32}\pi^2. \tag{3.27}
$$

(3.26) and (3.27) implies that

$$
\lambda_{i,\varepsilon}-1 = \frac{C_2+o(1)}{\|u_\varepsilon\|^2},
$$

where

$$
C_2 = \frac{\sigma_3^2R(x_0)}{D} = \frac{2^9}{15}R(x_0).
$$

This proves Lemma 3.2. □

Now, by Proposition 3.1 and Lemma 3.2, we have

$$\frac{C_2 + o(1)}{\|u_\varepsilon\|^{4/(N-2)}} = \lambda_{i,\varepsilon} - 1 \le \frac{C_1 + o(1)}{\|u_\varepsilon\|^{4/(N-2)}}$$

if $b_i \neq 0$ when $N \ge 4$. From this, we must have $C_2 \le C_1$. By (3.15) and (3.18), this implies

$$\frac{4(N+1)}{N(N^2-4)}|\Delta K(x_0)| \le \frac{2(N+1)}{N(N+2)}|\mu_1|,$$

when $N \ge 5$, which is equivalently $|\Delta K(x_0)| \le \frac{(N-2)}{2}|\mu_1|$. On the other hand, the assumption (A) states that $|\Delta K(x_0)| > \frac{(N-2)}{2}|\mu_1|$, a contradiction.

Also when $N = 4$, we must have

$$20\sigma_4 R(x_0) + \frac{5}{12}|\Delta K(x_0)| \le \frac{5}{12}|\mu_1|,$$

which is impossible because $|\Delta K(x_0)| = |\mu_1| + |\mu_2| + \cdots + |\mu_N| > |\mu_1|$ and $R(x_0) > 0$.

When $N = 3$, we must have

$$\frac{C_2 + o(1)}{\|u_\varepsilon\|^2} = \lambda_{i,\varepsilon} - 1 \le \frac{C_1 + o(1)}{\|u_\varepsilon\|^4},$$

but this is impossible again since $C_2 > 0$.

Thus we have $b_i = 0$ in (2.19) and (1.6) in Theorem 1.1 is proved.

To prove other claims in Theorem 1.1, we need

Lemma 3.3 *For $i = 2, \cdots, N+1$, let $b_i = 0$ and $\vec{a}_i = (a_{i,1}, \cdots, a_{i,N}) \neq 0$ in (2.19). Then we have*

$$\|u_\varepsilon\|^{2+2/(N-2)} v_{i,\varepsilon} \to \sigma_N \sum_{j=1}^{N} a_{i,j} \left(\frac{\partial G}{\partial z_j}(x,z) \right) \Big|_{z=x_0} \tag{3.28}$$

in $C^1_{loc}(\overline{\Omega} \setminus \{x_0\})$.

Proof. Argue as Lemma 3.3 in [5]. □

Now, we prove (1.8). We return to (2.2):

$$\int_{\partial\Omega}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial v_{i,\varepsilon}}{\partial\nu}\right)ds_x=(1-\lambda_{i,\varepsilon})c_0p_\varepsilon\int_\Omega K(x)u_\varepsilon^{p_\varepsilon-1}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)v_{i,\varepsilon}dx$$
$$+c_0\int_\Omega\left(\frac{\partial K}{\partial x_j}\right)u_\varepsilon v_{i,\varepsilon}dx.$$

By (2.14) and Lemma 3.3, we see

$$\begin{aligned}LHS=&\frac{1}{\|u_\varepsilon\|^{3+2/(N-2)}}\int_{\partial\Omega}\left(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial\|u_\varepsilon\|^{2+2/(N-2)}v_{i,\varepsilon}}{\partial\nu}\right)ds_x\\ =&\frac{1}{\|u_\varepsilon\|^{3+2/(N-2)}}\left[(N-2)\sigma_N^2\sum_{k=1}^N a_{i,k}\int_{\partial\Omega}\left(\frac{\partial G}{\partial x_j}\right)\frac{\partial}{\partial\nu_x}\left(\frac{\partial G}{\partial z_k}\right)(x,x_0)ds_x+o(1)\right]\\ =&\frac{1}{\|u_\varepsilon\|^{3+2/(N-2)}}\left[\frac{N-2}{2}\sigma_N^2\sum_{k=1}^N a_{i,k}\frac{\partial^2 R}{\partial z_j\partial z_k}(z)\Big|_{z=x_0}+o(1)\right],\end{aligned}$$

where we have used (2.7).

On the other hand, write $RHS=I+II$ where

$$\begin{aligned}I&=(1-\lambda_{i,\varepsilon})c_0p_\varepsilon\int_\Omega K(x)u_\varepsilon^{p_\varepsilon-1}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)v_{i,\varepsilon}dx,\\ II&=c_0\int_\Omega\left(\frac{\partial K}{\partial x_j}\right)u_\varepsilon^{p_\varepsilon}v_{i,\varepsilon}dx.\end{aligned}$$

As before, we have

$$\begin{aligned}I&=(1-\lambda_{i,\varepsilon})c_0p_\varepsilon\|u_\varepsilon\|^{p_\varepsilon-1+(p_\varepsilon+1)/2-(p_\varepsilon-1)N/2}\int_{\Omega_\varepsilon}K_\varepsilon(y)\tilde u_\varepsilon^{p_\varepsilon-1}\left(\frac{\partial\tilde u_\varepsilon}{\partial y_j}\right)\tilde v_{i,\varepsilon}dy\\ &=\frac{(1-\lambda_{i,\varepsilon})}{\|u_\varepsilon\|^{(N-4)/(N-2)}}c_0p\left[\int_{\mathbb{R}^N}U^{p-1}\left(\frac{\partial U}{\partial y_j}\right)\sum_{k=1}^N a_{i,k}\frac{y_k}{(1+|y|^2)^{N/2}}dy+o(1)\right]\\ &=\frac{(1-\lambda_{i,\varepsilon})}{\|u_\varepsilon\|^{(N-4)/(N-2)}}\frac{c_0p}{2-N}\left[\sum_{k=1}^N a_{i,k}\int_{\mathbb{R}^N}U^{p-1}\left(\frac{\partial U}{\partial y_j}\right)\left(\frac{\partial U}{\partial y_k}\right)dy+o(1)\right]\\ &=\frac{(\lambda_{i,\varepsilon}-1)}{\|u_\varepsilon\|^{(N-4)/(N-2)}}\frac{c_0p}{N(N-2)}a_{i,j}\left[\int_{\mathbb{R}^N}U^{p-1}|\nabla U|^2dy+o(1)\right].\end{aligned}$$

As for II, we see by (2.10) and (2.19) (with $b_i = 0$) that

$$\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_{i,\varepsilon}(y) \to U^p(y)V(y) = \sum_{k=1}^N a_{i,k}\frac{y_k}{(1+|y|^2)^{N+1}} = \sum_{k=1}^N a_{i,k}\frac{\partial}{\partial y_k}\left\{\frac{-1}{2N}U^{p+1}(y)\right\}$$

in $C^1_{loc}(\mathbb{R}^N)$. Now, we exploit the solution $\psi_{i,\varepsilon}$ of the linear first order PDE

$$\sum_{k=1}^N a_{i,k}\frac{\partial\psi}{\partial y_k} = \tilde{u}_\varepsilon^{p_\varepsilon}(y)\tilde{v}_{i,\varepsilon}(y) \quad (y \in \mathbb{R}^N), \quad \psi|_{\Gamma_{\vec{a}_i}} = \frac{-1}{2N}U^{p+1}(y),$$

where $\Gamma_{\vec{a}_i} = \{x \in \mathbb{R}^N | x \cdot \vec{a}_i = 0\}$. We see that the solution $\psi_{i,\varepsilon}$ satisfies the estimate $\psi_{i,\varepsilon}(y) = O(|y|^{-2N+1})$ as $|y| \to \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_\varepsilon(y) = O(U^{p+1}(y)) = O(|y|^{-2N})$ by (2.16) and (2.18),

$$\psi_{i,\varepsilon} \to \frac{-1}{2N}U^{p+1}$$

uniformly on compact subsets of $\mathbb{R}^N$, and

$$\int_{\Omega_\varepsilon}\psi_{i,\varepsilon}(y)dy \to \frac{-1}{2N}\int_{\mathbb{R}^N}U^{p+1}dy = \frac{-1}{2N}\sigma_N C_N$$

by the dominated convergence theorem, where $C_N = \int_0^\infty \frac{r^{N-1}}{(1+r^2)^N}dr = \frac{\Gamma(N/2)^2}{2\Gamma(N)}$. Thus,

$$\begin{aligned}
II &= c_0\|u_\varepsilon\|^{p_\varepsilon-(\frac{p_\varepsilon-1}{2})N}\int_{\Omega_\varepsilon}\left(\frac{\partial K}{\partial x_j}\right)\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon\right)\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_{i,\varepsilon}dy \\
&= c_0\|u_\varepsilon\|^{p_\varepsilon-(\frac{p_\varepsilon-1}{2})N}\int_{\Omega_\varepsilon}\left(\frac{\partial K}{\partial x_j}\right)\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon\right)\sum_{k=1}^N a_{i,k}\frac{\partial\psi_{i,\varepsilon}}{\partial y_k}dy \\
&= -c_0\|u_\varepsilon\|^{p_\varepsilon-(\frac{p_\varepsilon-1}{2})N}\sum_{k=1}^N a_{i,k}\int_{\Omega_\varepsilon}\frac{\partial}{\partial y_k}\left\{\left(\frac{\partial K}{\partial x_j}\right)\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon\right)\right\}\psi_{i,\varepsilon}(y)dy \\
&= -c_0\|u_\varepsilon\|^{p_\varepsilon-(\frac{p_\varepsilon-1}{2})N-(\frac{p_\varepsilon-1}{2})}\sum_{k=1}^N a_{i,k}\int_{\Omega_\varepsilon}\left(\frac{\partial^2 K}{\partial x_j\partial x_k}\right)(x)\Big|_{x=\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon}\psi_{i,\varepsilon}(y)dy \\
&= -c_0\|u_\varepsilon\|^{-N/(N-2)}\left[\sum_{k=1}^N a_{i,k}\frac{\partial^2 K}{\partial x_j\partial x_k}(x_0)\left(\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\psi_{i,\varepsilon}(y)dy\right)+o(1)\right] \\
&= \|u_\varepsilon\|^{-N/(N-2)}\left[\frac{N-2}{2}\sigma_N C_N\sum_{k=1}^N a_{i,k}\frac{\partial^2 K}{\partial x_j\partial x_k}(x_0)+o(1)\right].
\end{aligned}$$

Returning to $LHS = I + II$, multiplying $\|u_\varepsilon\|^{N/(N-2)}$ to both sides, we have

$$\|u_\varepsilon\|^{-2}\left[\frac{N-2}{2}\sigma_N^2\sum_{k=1}^N a_{i,k}\frac{\partial^2 R}{\partial z_j \partial z_k}(x_0)+o(1)\right]$$
$$=(\lambda_{i,\varepsilon}-1)\|u_\varepsilon\|^{4/(N-2)}pa_{i,j}\left[\int_{\mathbb{R}^N}U^{p-1}|\nabla U|^2dy+o(1)\right]$$
$$+\left(\frac{N-2}{2}\right)\sigma_N C_N\sum_{k=1}^N a_{i,k}\frac{\partial^2 K}{\partial x_j \partial x_k}(x_0)+o(1).$$

Thus we obtain

$$\|u_\varepsilon\|^{4/(N-2)}(1-\lambda_{i,\varepsilon}) \to M\eta_i,$$

as $\varepsilon \to 0$, where

$$M=\frac{(\frac{N-2}{2})\sigma_N C_N}{p\int_{\mathbb{R}^N}U^{p-1}|\nabla U|^2dy}=\frac{2(N+1)}{N(N+2)}$$

and

$$\eta_i=\frac{\sum_{k=1}^N a_{i,k}\left(\frac{\partial^2 K}{\partial x_j\partial x_k}(x_0)\right)}{a_{i,j}}, \quad (i=2,\cdots,N+1).$$

This means η_i is an eigenvalue of the matrix Hess$K(x_0)$ and $\vec{a}_i$ is a corresponding eigenvector.

If $i \neq j$, we see that $\vec{a}_i$ and $\vec{a}_j$ is perpendicular to each other in $\mathbb{R}^N$. Indeed, for fixed ε, $v_{i,\varepsilon}$ and $v_{j,\varepsilon}$ is orthogonal in the sense of (1.4):

$$\int_\Omega K(x)u_\varepsilon^{p_\varepsilon-1}v_{i,\varepsilon}v_{j,\varepsilon}dx=0.$$

From this, we have

$$\|u_\varepsilon\|^{p_\varepsilon-1-(p_\varepsilon-1)N/2}\int_{\Omega_\varepsilon}K_\varepsilon(y)\tilde{u}_\varepsilon^{p_\varepsilon-1}\tilde{v}_{i,\varepsilon}\tilde{v}_{j,\varepsilon}dy=0.$$

Multiplying $\|u_\varepsilon\|^2$ to the both sides and letting $\varepsilon \to 0$, we obtain

$$\int_{\mathbb{R}^N}U^{p-1}\left(\sum_{k=1}^N a_{i,k}\frac{y_k}{(1+|y|^2)^{N/2}}\right)\left(\sum_{l=1}^N a_{j,l}\frac{y_l}{(1+|y|^2)^{N/2}}\right)dy$$
$$=0,$$

where we have used (2.10) and (1.6). From this, we have

$$0 = \sum_{k,l=1}^{N} \int_{\mathbb{R}^N} U^{p-1} a_{i,k} a_{j,l} \frac{y_k y_l}{(1+|y|^2)^N} dy$$
$$= \sum_{k=1}^{N} a_{i,k} a_{j,k} \left(\frac{1}{N} \int_{\mathbb{R}^N} U^{p-1} \frac{|y|^2}{(1+|y|^2)^N} dy \right),$$

which implies $\vec{a}_i \cdot \vec{a}_j = 0$. Thus, all η_i is one of N eigenvalues of Hess$K(x_0)$.

Since $1-\lambda_{i,\varepsilon}$ is decreasing in $i \in \mathbb{N}$, we have $\eta_i = \mu_{N+2-i}$ for $i = 2, \cdots, N+1$. This ends the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. First, we prove

Lemma 4.1

$$\lambda_{N+2,\varepsilon} \to 1 \quad as\ \varepsilon \to 0. \tag{5.1}$$

Proof. Since we know $\liminf_{\varepsilon\to 0} \lambda_{N+2,\varepsilon} \geq 1$ by Proposition 3.1 (3.2), we have to check that $\limsup_{\varepsilon\to 0} \lambda_{N+2,\varepsilon} \leq 1$. For this purpose, we use a variational characterization of $\lambda_{N+2,\varepsilon}$ to obtain

$$\lambda_{N+2,\varepsilon} \leq \max_{v\in W} \frac{\int_\Omega |\nabla v|^2 dx}{c_0 p_\varepsilon \int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1} v^2 dx}, \tag{5.2}$$

where $W = \text{span}\{u_\varepsilon, \phi(\frac{\partial u_\varepsilon}{\partial x_1}), \cdots, \phi(\frac{\partial u_\varepsilon}{\partial x_N}), \phi w_\varepsilon\}$, ϕ is a cut-off function as in Lemma 2.8, and, as before, $w_\varepsilon(x) = (x - x_\varepsilon)\cdot\nabla u_\varepsilon + \frac{2}{p_\varepsilon - 1} u_\varepsilon$. For $a_0, a_1, \cdots, a_N, d \in \mathbb{R}$, we set

$$\hat{z}_\varepsilon(x) = \sum_{j=1}^{N} a_j \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) + d w_\varepsilon(x).$$

Direct calculation shows that $\hat{z}_\varepsilon$ satisfies the equation

$$-\Delta \hat{z}_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} \hat{z}_\varepsilon + c_0 u_\varepsilon^{p_\varepsilon} \left(\sum_{j=1}^{N} a_j \left(\frac{\partial K}{\partial x_j}\right) + d(x - x_\varepsilon)\cdot\nabla K(x)\right).$$

We test (5.2) by $v = a_0 u_\varepsilon + \phi \hat{z}_\varepsilon \in W$. As in the proof of Proposition 3.1, we have

$$\max_{v \in W} \frac{\int_\Omega |\nabla v|^2 dx}{c_0 p_\varepsilon \int_\Omega K(x) u_\varepsilon^{p_\varepsilon - 1} v^2 dx} = \max_{a_0, a_1, \cdots, a_N, d} \left\{ 1 + \frac{\hat{N}_\varepsilon}{\hat{D}_\varepsilon} \right\},$$

where $\hat{N}_\varepsilon = \hat{N}_\varepsilon^1 + \cdots + \hat{N}_\varepsilon^{10}$,

$$\hat{N}_\varepsilon^1 = a_0^2 c_0 (1 - p_\varepsilon) \int_\Omega K(x) u_\varepsilon^{p_\varepsilon + 1} dx,$$

$$\hat{N}_\varepsilon^2 = 2 a_0 c_0 (1 - p_\varepsilon) \sum_{j=1}^N a_j \int_\Omega K(x) u_\varepsilon^{p_\varepsilon} \phi \left(\frac{\partial u_\varepsilon}{\partial x_j} \right) dx,$$

$$\hat{N}_\varepsilon^3 = 2 a_0 c_0 (1 - p_\varepsilon) d \int_\Omega K(x) u_\varepsilon^{p_\varepsilon} \phi w_\varepsilon dx,$$

$$\hat{N}_\varepsilon^4 = \sum_{j,k=1}^N a_j a_l \int_\Omega |\nabla \phi|^2 \left(\frac{\partial u_\varepsilon}{\partial x_j} \right) \left(\frac{\partial u_\varepsilon}{\partial x_k} \right) dx,$$

$$\hat{N}_\varepsilon^5 = 2d \sum_{j=1}^N \int_\Omega |\nabla \phi|^2 \left(\frac{\partial u_\varepsilon}{\partial x_j} \right) w_\varepsilon dx,$$

$$\hat{N}_\varepsilon^6 = d^2 \int_\Omega |\nabla \phi|^2 w_\varepsilon^2 dx,$$

$$\hat{N}_\varepsilon^7 = c_0 \sum_{j,k=1}^N a_j a_k \int_\Omega u_\varepsilon^{p_\varepsilon} \left(\frac{\partial K}{\partial x_j} \right) \left(\frac{\partial u_\varepsilon}{\partial x_k} \right) \phi^2 dx,$$

$$\hat{N}_\varepsilon^8 = c_0 d \sum_{j=1}^N a_j \int_\Omega u_\varepsilon^{p_\varepsilon} \left(\frac{\partial u_\varepsilon}{\partial x_j} \right) (x - x_\varepsilon) \cdot \nabla K(x) \phi^2 dx,$$

$$\hat{N}_\varepsilon^9 = c_0 d \sum_{j=1}^N a_j \int_\Omega u_\varepsilon^{p_\varepsilon} \left(\frac{\partial K}{\partial x_j} \right) w_\varepsilon \phi^2 dx,$$

$$\hat{N}_\varepsilon^{10} = c_0 d^2 \int_\Omega u_\varepsilon^{p_\varepsilon} (x - x_\varepsilon) \cdot \nabla K(x) w_\varepsilon \phi^2 dx,$$

and $\hat{D}_\varepsilon = \hat{D}_\varepsilon^1 + \cdots + \hat{D}_\varepsilon^6$,

$$
\begin{aligned}
\hat{D}_\varepsilon^1 &= a_0^2 c_0 p_\varepsilon \int_\Omega K(x) u_\varepsilon^{p_\varepsilon+1} dx, \\
\hat{D}_\varepsilon^2 &= 2a_0 c_0 p_\varepsilon \sum_{j=1}^N a_j \int_\Omega K(x) u_\varepsilon^{p_\varepsilon} \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \phi dx, \\
\hat{D}_\varepsilon^3 &= 2a_0 c_0 p_\varepsilon d \int_\Omega K(x) u_\varepsilon^{p_\varepsilon} \phi w_\varepsilon dx, \\
\hat{D}_\varepsilon^4 &= c_0 p_\varepsilon \sum_{j,k=1}^N a_j a_k \int_\Omega K(x) u_\varepsilon^{p_\varepsilon-1} \phi^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial u_\varepsilon}{\partial x_k}\right) dx, \\
\hat{D}_\varepsilon^5 &= 2c_0 p_\varepsilon d \sum_{j=1}^N a_j \int_\Omega K(x) u_\varepsilon^{p_\varepsilon-1} \phi^2 \left(\frac{\partial u_\varepsilon}{\partial x_j}\right) w_\varepsilon dx, \\
\hat{D}_\varepsilon^6 &= c_0 p_\varepsilon d^2 \int_\Omega K(x) u_\varepsilon^{p_\varepsilon-1} \phi^2 w_\varepsilon^2 dx.
\end{aligned}
$$

Let $(a_0, a_1, \cdots, a_N, d)$ denote a maximizer of $\max_{a_0,a_1,\cdots,a_N,d}\left\{1 + \frac{\hat{N}_\varepsilon}{\hat{D}_\varepsilon}\right\}$ which is normalized as $a_0^2 + \sum_{j=1}^N a_j^2 + d^2 = 1$. Since the case $a_0 = 1$ is obvious, we consider only the case $|a|^2 + d^2 \neq 0$, where $|a|^2 = \sum_{j=1}^N a_j^2$.

We calculate

$$
\begin{aligned}
&\int_\Omega K(x) u_\varepsilon^{p_\varepsilon} \phi w_\varepsilon dx = \int_\Omega K(x) u_\varepsilon^{p_\varepsilon} \phi \left((x - x_\varepsilon)\cdot \nabla u_\varepsilon + \frac{2}{p_\varepsilon - 1} u_\varepsilon\right) dx \\
&= \int_\Omega \frac{K\phi}{p_\varepsilon+1} \sum_{j=1}^N \frac{\partial}{\partial x_j}\left\{(x_j - (x_\varepsilon)_j) u_\varepsilon^{p_\varepsilon+1}\right\} - \left(\frac{N}{p_\varepsilon+1} - \frac{2}{p_\varepsilon-1}\right) K(x) u_\varepsilon^{p_\varepsilon+1} \phi dx \\
&= -\frac{1}{p_\varepsilon+1} \int_\Omega \nabla(K\phi)\cdot(x - x_\varepsilon) u_\varepsilon^{p_\varepsilon+1} dx + O(\varepsilon) \int_\Omega K(x) u_\varepsilon^{p_\varepsilon+1} \phi dx.
\end{aligned}
$$

Now, by (2.15) in Theorem 2.3, we see

$$
O(\varepsilon) = \begin{cases} O(\frac{1}{\|u_\varepsilon\|^2}), & N = 3, 4, \\ O(\frac{1}{\|u_\varepsilon\|^{4/(N-2)}}), & N \geq 5. \end{cases}
$$

Denote

$$\begin{aligned}
&\int_\Omega \nabla(K\phi)\cdot(x-x_\varepsilon)u_\varepsilon^{p_\varepsilon+1}dx \\
&= \int_\Omega \nabla K\cdot(x-x_\varepsilon)u_\varepsilon^{p_\varepsilon+1}\phi dx + \int_\Omega \nabla\phi\cdot(x-x_\varepsilon)K(x)u_\varepsilon^{p_\varepsilon+1}dx \\
&=: J_1 + J_2.
\end{aligned}$$

We see $J_2 = O(\frac{1}{\|u_\varepsilon\|^{p_\varepsilon+1}})$ as before. Also, J_1 can be estimated as in the derivation of (3.6):

$$J_1 = O\left(\mathrm{diam}(\Omega)\int_\Omega \phi(x)u_\varepsilon^{p_\varepsilon+1}\left(\frac{\partial K}{\partial x_j}\right)dx\right) = O(\|u_\varepsilon\|^{-2/(N-2)}), \quad (N\geq 3)$$

by (3.5). In conclusion, we have

$$\begin{aligned}
\int_\Omega K(x)u_\varepsilon^{p_\varepsilon}\phi w_\varepsilon dx &= O\left(\frac{1}{\|u_\varepsilon\|^{p_\varepsilon+1}}\right) + O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right) + O(\varepsilon) \\
&= O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right). \qquad (5.3)
\end{aligned}$$

Also by (2.14) as before, we have

$$\int_\Omega |\nabla\phi|^2\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)w_\varepsilon dx = O\left(\frac{1}{\|u_\varepsilon\|^2}\right), \qquad (5.4)$$

$$\int_\Omega |\nabla\phi|^2 w_\varepsilon^2 dx = O\left(\frac{1}{\|u_\varepsilon\|^2}\right) \qquad (5.5)$$

just as (7.9), (7.10) in [2].

Furthermore, as (7.11), (7.12) in [2], we have

$$
\begin{aligned}
&\int_\Omega K(x)u_\varepsilon^{p_\varepsilon-1}\phi^2\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)w_\varepsilon dx\\
&=\int_\Omega K(x)u_\varepsilon^{p_\varepsilon-1}\phi^2\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\left((x-x_\varepsilon)\cdot\nabla u_\varepsilon+\frac{2}{p_\varepsilon-1}u_\varepsilon\right)dx\\
&=\sum_{l=1}^N\int_\Omega K(x)u_\varepsilon^{p_\varepsilon-1}\phi^2(x_l-(x_\varepsilon)_l)\left(\frac{\partial u_\varepsilon}{\partial x_l}\right)\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)dx\\
&+\frac{2}{p_\varepsilon-1}\int_\Omega K(x)u_\varepsilon^{p_\varepsilon}\phi^2\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)dx\\
&=\|u_\varepsilon\|^{p_\varepsilon-1-(p_\varepsilon-1)/2+(p_\varepsilon+1)-(p_\varepsilon-1)N/2}\sum_{l=1}^N\int_{\Omega_\varepsilon}K_\varepsilon(y)\tilde{u}_\varepsilon^{p_\varepsilon-1}\phi_\varepsilon^2(y)y_l\left(\frac{\partial\tilde{u}_\varepsilon}{\partial y_l}\right)\left(\frac{\partial\tilde{u}_\varepsilon}{\partial y_j}\right)dy\\
&+\frac{2}{p_\varepsilon-1}\|u_\varepsilon\|^{p_\varepsilon+(p_\varepsilon+1)/2-(p_\varepsilon-1)N/2}\int_{\Omega_\varepsilon}K_\varepsilon(y)\tilde{u}_\varepsilon^{p_\varepsilon}\phi_\varepsilon^2(y)\left(\frac{\partial\tilde{u}_\varepsilon}{\partial y_j}\right)dy\\
&=\|u_\varepsilon\|^{2/(N-2)}\left(\sum_{l=1}^N\int_{\mathbb{R}^N}U^{p-1}y_l\left(\frac{\partial U}{\partial y_l}\right)\left(\frac{\partial U}{\partial y_j}\right)dy+o(1)\right)\\
&+\frac{2}{p-1}\|u_\varepsilon\|^{2/(N-2)}\left[\int_{\mathbb{R}^N}U^p\left(\frac{\partial U}{\partial y_j}\right)dy+o(1)\right]\\
&=\|u_\varepsilon\|^{2/(N-2)}o(1),
\end{aligned}
\tag{5.6}
$$

$$
\begin{aligned}
&\int_\Omega K(x)u_\varepsilon^{p_\varepsilon-1}\phi^2w_\varepsilon^2dx\\
&=\|u_\varepsilon\|^{p_\varepsilon+1-(p_\varepsilon-1)N/2}\int_{\Omega_\varepsilon}K_\varepsilon(y)\tilde{u}_\varepsilon^{p_\varepsilon-1}\phi_\varepsilon^2(y)\left(y\cdot\nabla\tilde{u}_\varepsilon+\frac{2}{p_\varepsilon-1}\tilde{u}_\varepsilon\right)^2dy\\
&=\left(\frac{N-2}{2}\right)^2\int_{\mathbb{R}^N}U^{p-1}(y)\left(y\cdot\nabla U+\frac{2}{p-1}U\right)^2dy+o(1)\\
&=\left(\frac{N-2}{2}\right)^2\int_{\mathbb{R}^N}U^{p-1}(y)\left(\frac{1-|y|^2}{(1+|y|^2)^{N/2}}\right)^2dy+o(1).
\end{aligned}
\tag{5.7}
$$

Thus by (3.6), (3.7), (5.3), (5.4), (5.5), we have

$$\hat{N}_\varepsilon^2 = \hat{N}_\varepsilon^3 = O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right),$$
$$\hat{N}_\varepsilon^4 = \hat{N}_\varepsilon^5 = \hat{N}_\varepsilon^6 = O\left(\frac{1}{\|u_\varepsilon\|^2}\right).$$

Next, by integration by parts, we see

$$\begin{aligned}
&\int_\Omega u_\varepsilon^{p_\varepsilon}\left(\frac{\partial K}{\partial x_j}\right)\left(\frac{\partial u_\varepsilon}{\partial x_k}\right)\phi^2 dx = \frac{1}{p_\varepsilon+1}\int_\Omega \frac{\partial u_\varepsilon^{p_\varepsilon+1}}{\partial x_k}\left(\frac{\partial K}{\partial x_j}\right)\phi^2 dx \\
&= -\frac{1}{p_\varepsilon+1}\int_\Omega u_\varepsilon^{p_\varepsilon+1}\frac{\partial}{\partial x_k}\left(\left(\frac{\partial K}{\partial x_j}\right)\phi^2\right)dx \\
&= O(1). \qquad (5.8)
\end{aligned}$$

Similarly,

$$\int_\Omega u_\varepsilon^{p_\varepsilon}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)(x-x_\varepsilon)\cdot\nabla K(x)\phi^2 dx = O(1). \qquad (5.9)$$

Also,

$$\begin{aligned}
&\int_\Omega u_\varepsilon^{p_\varepsilon}\left(\frac{\partial K}{\partial x_j}\right)w_\varepsilon\phi^2 dx \\
&= \|u_\varepsilon\|^{p_\varepsilon+1-(p_\varepsilon-1)N/2}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}\left(\frac{\partial K}{\partial x_j}\right)\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon\right)\tilde{w}_\varepsilon\phi_\varepsilon^2 dy \\
&= \left(\frac{\partial K}{\partial x_j}\right)(x_0)\left[\int_{\mathbb{R}^N}\left(y\cdot\nabla U+\frac{2}{p-1}U\right)U^p dy + o(1)\right] \\
&= o(1), \qquad (5.10)
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega u_\varepsilon^{p_\varepsilon}(x-x_\varepsilon)\cdot\nabla K(x)w_\varepsilon\phi^2 dx \\
&= \|u_\varepsilon\|^{p_\varepsilon+1-(p_\varepsilon-1)/2-(p_\varepsilon-1)N/2}\int_{\Omega_\varepsilon} y\cdot\nabla_x K\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}}+x_\varepsilon\right)\tilde{w}_\varepsilon\tilde{u}_\varepsilon^{p_\varepsilon}\phi_\varepsilon^2 dy \\
&= \|u_\varepsilon\|^{-2/(N-2)}\left[\int_{\mathbb{R}^N}(y\cdot\nabla K(x_0))\left(y\cdot\nabla U+\frac{2}{p-1}U\right)U^p dy + o(1)\right] \\
&= o\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right). \qquad (5.11)
\end{aligned}$$

Thus, we have

$$\hat{N}_\varepsilon^7 = |a|^2 O(1), \quad \hat{N}_\varepsilon^8 = |a||d|O(1),$$
$$\hat{N}_\varepsilon^9 = o(1), \quad \hat{N}_\varepsilon^{10} = O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right).$$

Therefore we have

$$\begin{aligned}
&\hat{N}_\varepsilon = \hat{N}_\varepsilon^1 + \cdots + \hat{N}_\varepsilon^{10} \\
&= a_0^2 c_0 (1-p_\varepsilon) \int_\Omega K(x) u_\varepsilon^{p_\varepsilon+1} dx + O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right) + O\left(\frac{1}{\|u_\varepsilon\|^2}\right) \\
&+ |a|^2 O(1) + |a||d|O(1) + o(1) \\
&\leq |a|^2 O(1) + |a||d|O(1) + o(1).
\end{aligned} \tag{5.12}$$

Next, we estimate $\hat{D}_\varepsilon$ from the below. First, we see

$$\hat{D}_\varepsilon^1 \geq 0.$$

Recall (3.6) and (5.3). Thus we have

$$\hat{D}_\varepsilon^2 = O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right), \quad \hat{D}_\varepsilon^3 = O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right). \tag{5.13}$$

Calculation shows

$$
\begin{aligned}
&\int_\Omega K(x)u_\varepsilon^{p_\varepsilon-1}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\left(\frac{\partial u_\varepsilon}{\partial x_k}\right)\phi^2dx\\
&=\|u_\varepsilon\|^{p_\varepsilon-1+(p_\varepsilon+1)-(p_\varepsilon-1)N/2}\int_{\Omega_\varepsilon}K_\varepsilon(y)\tilde{u}_\varepsilon^{p_\varepsilon-1}\left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}\right)\left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_k}\right)\phi_\varepsilon^2dy\\
&=\|u_\varepsilon\|^{4/(N-2)}\left[\frac{\delta_{jk}}{N}\int_{\mathbb{R}^N}U^{p-1}|\nabla U|^2dy+o(1)\right]\\
&=O(\|u_\varepsilon\|^{4/(N-2)}), \qquad (5.14)\\
&\int_\Omega K(x)u_\varepsilon^{p_\varepsilon-1}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)w_\varepsilon\phi^2dx\\
&=\|u_\varepsilon\|^{p_\varepsilon+(p_\varepsilon+1)/2-(p_\varepsilon-1)N/2}\int_{\Omega_\varepsilon}K_\varepsilon(y)\tilde{u}_\varepsilon^{p_\varepsilon-1}\left(\frac{\partial \tilde{u}_\varepsilon}{\partial y_j}\right)\tilde{w}_\varepsilon\phi_\varepsilon^2dy\\
&=\|u_\varepsilon\|^{2/(N-2)}\left[\int_{\mathbb{R}^N}U^{p-1}\left(\frac{\partial U}{\partial y_j}\right)\left(y\cdot\nabla U+\frac{2}{p-1}U\right)dy+o(1)\right],\\
&=o(\|u_\varepsilon\|^{2/(N-2)}), \qquad (5.15)\\
&\int_\Omega K(x)u_\varepsilon^{p_\varepsilon-1}w_\varepsilon^2\phi^2dx\\
&=\|u_\varepsilon\|^{p_\varepsilon-1+2-(p_\varepsilon-1)N/2}\int_{\Omega_\varepsilon}K_\varepsilon(y)\tilde{u}_\varepsilon^{p_\varepsilon-1}\tilde{w}_\varepsilon^2\phi_\varepsilon^2dy\\
&=\left[\int_{\mathbb{R}^N}\left(y\cdot\nabla U+\frac{2}{p-1}U\right)^2dy+o(1)\right]\\
&=O(1). \qquad (5.16)
\end{aligned}
$$

From (5.13), (5.14), (5.15) and (5.16), we can estimate $\hat{D}_\varepsilon$ from below, just as in Grossi and Pacella [2]:

$$
\begin{aligned}
&\hat{D}_\varepsilon\geq\hat{D}_\varepsilon^2+\cdots+\hat{D}_\varepsilon^6\\
&\geq O\left(\frac{1}{\|u_\varepsilon\|^{2/(N-2)}}\right)+\gamma_1|a|^2\|u_\varepsilon\|^{4/(N-2)}+d\left(\sum_{j=1}^N a_j\right)o(\|u_\varepsilon\|^{2/(N-2)})+\gamma_2d^2\\
&\geq(\gamma_1/2)|a|^2\|u_\varepsilon\|^{4/(N-2)}+(\gamma_2/2)d^2 \qquad (5.17)
\end{aligned}
$$

for some $\gamma_1,\gamma_2>0$. Therefore by (5.12) and (5.17), we have

$$
\limsup_{\varepsilon\to0}\lambda_{N+2,\varepsilon}\leq\limsup_{\varepsilon\to0}\left\{1+\frac{\hat{N}_\varepsilon}{\hat{D}_\varepsilon}\right\}\leq 1+\lim_{\varepsilon\to0}\frac{|a|^2O(1)+|a||d|O(1)+o(1)}{(\gamma_1/2)|a|^2\|u_\varepsilon\|^{4/(N-2)}+(\gamma_2/2)d^2}=1,
$$

since $|a|^2 = \sum_{j=1}^N a_j^2$ and d^2 can not vanish simultaneously. Thus we have proved Lemma 4.1 $\square$

Since we have checked (5.1), we know by Lemma 2.6 that

$$\tilde{v}_{N+2,\varepsilon} \to \sum_{k=1}^N a_{N+2,k} \frac{y_k}{(1+|y|^2)^{N/2}} + b_{N+2} \frac{1-|y|^2}{(1+|y|^2)^{N/2}}$$

in $C^1_{loc}(\mathbb{R}^N)$. Now, for fixed ε, $v_{N+2,\varepsilon}$ and $v_{i,\varepsilon}$ is orthogonal in the sense of (1.4) for $i = 2, \cdots, N+1$. By the same argument as in the last part of the proof of Theorem 1.1, we have that $\vec{a}_{N+2} \cdot \vec{a}_i = 0$ for any $i = 2, \cdots N+1$. Since $\vec{a}_i$ are linearly independent in $\mathbb{R}^N$, we have that $\vec{a}_{N+2} = \vec{0}$. Thus we obtain (1.9).

Also by $b_{N+2} \neq 0$, Lemma 2.7 assures that

$$\|u_\varepsilon\|^2 v_{N+2,\varepsilon} \to -(N-2)\sigma_N b_{N+2} G(\cdot, x_0), \text{ in } C^1_{loc}(\overline{\Omega} \setminus \{x_0\}) \quad \text{as } \varepsilon \to 0.$$

Then, we can repeat the same proof of Lemma 3.2 (with $i = N+2$) to obtain

$$\lambda_{N+2,\varepsilon} - 1 = \frac{1}{\|u_\varepsilon\|^{4/(N-2)}}(C_2 + o(1)), \quad (N \geq 4),$$
$$\lambda_{N+2,\varepsilon} - 1 = \frac{1}{\|u_\varepsilon\|^2}(C_2 + o(1)), \quad (N = 3),$$

where C_2 is defined in (3.18).

Recall (2.15) in Lemma 2.3: Thus, we have Γ in (1.10) is

$$\Gamma = \frac{C_2}{\lim_{\varepsilon \to 0} \varepsilon \|u_\varepsilon\|^{4/(N-2)}}$$
$$= \frac{4(N+1)}{N(N^2-4)} |\Delta K(x_0)| \times \frac{(N-2)^2}{2|\Delta K(x_0)|} = \frac{2(N-2)(N+1)}{N(N+2)}$$

when $N \geq 5$. We argue similarly when $N = 4$ and $N = 3$. This proves Theorem 1.2. $\square$

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