On projective bundles over small covers (a survey)

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Abstract. In this article we survey results of a joint paper of the author with Z. Lü which is devoted to studying constructions of projective bundles over small covers. In order to construct all projective bundles from certain basic projective bundles, we introduce a new operation which is a combinatorial adaptation of the fibre sum of fibre bundles. By using this operation, we give a topological characterization of all projective bundles over 2-dimensional small covers.

1. Introduction

In the toric topology, the following question asked by Masuda and Suh in [**11**] is still open and interesting problem:

Problem 1 (Cohomological rigidity problem). *Let M and M' be two (quasi)toric manifolds. Are M and M' homeomorphic if $H^*(M) \cong H^*(M')$?*

Small covers are a real analogue of quasitoric manifolds introduced by Davis and Januszkiewicz in [**4**]. In the paper [**10**], Masuda gives counterexamples to cohomological rigidity of small covers, that is, he shows that the cohomology ring does not distinguish between small covers. He classified diffeomorphism types of height 2 (generalized) real Bott manifolds by using the KO-ring structure of real projective spaces, where the *height* 2 *(generalized) real Bott manifold* is the total space of projective bundle of the Whitney sum of some line bundles over real projective space, and he found counterexamples to cohomological rigidity in height 2 (generalized) real Bott manifolds. It follows that projective bundles over small covers are not determined by the cohomology ring only. In this article, we try to study topological types of such projective bundles from a different point of view, as compared to Masuda's approach. Our method is closely related to Orlik-Raymond's method of [**16**]. In this paper, Orlik and Raymond show that 4-dimensional simply connected torus manifolds, i.e., objects satisfying weaker conditions than the corresponding quasitoric manifolds, can be constructed from certain basic torus manifolds by using connected sums. Since small covers (resp. 2-torus manifolds) M^n are real analogues of the quasitoric manifolds (resp. torus manifolds), it seems reasonable to try to apply the Orlik-Raymond method of [**16**] to 2-dimensional small covers. The goal of this article is to adapt their method to projective bundles over 2-dimensional small covers, and introduce the following construction theorem of such projective bundles.

Theorem 1. Let $P(\xi)$ be a projective bundle over 2-dimensional small cover M^2. Then $P(\xi)$ can be constructed from projective bundles $P(\kappa)$ over $\mathbb{R}P^2$ and $P(\zeta)$ over T^2 by using the projective connected sum $\sharp^{\Delta^{k-1}}$.

The organization of this paper is as follows. In Section 2, we recall basic properties of small covers. In Section 3, we present Masuda's counterexample to cohomological rigidity. In Section 4, we study the structure of projective bundles over small covers and introduce new characteristic functions. In Section 5, we recall the construction theorem of 2-dimensional small covers and show

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a topological classification of projective bundles over basic 2-dimensional small covers. In Section 6, we present our main theorem. Finally, in Section 7, as an appendix, we show a topological classification of projective bundles over 1-dimensional small covers.

2. Basic properties of small covers

We first recall the definition of small covers and some basic facts.

A convex n-dimensional polytope is called *simple* if the number of its *facets* (that is, codimension-one faces) meeting at every vertex is equal to n. Let $\mathbb{Z}_2 = \{-1, 1\}$ be the 2-element group.

2.1. Definition of small covers. An n-dimensional closed smooth manifold M is called a *small cover* over a simple convex n-polytope P if M has a $(\mathbb{Z}_2)^n$-action such that

- **(a):** the $(\mathbb{Z}_2)^n$-action is *locally standard*, i.e., locally the same as the standard $(\mathbb{Z}_2)^n$-action on $\mathbb{R}^n$, and
- **(b):** its orbit space is homeomorphic to P; the corresponding orbit projection map $\pi : M \to P$ is constant on $(\mathbb{Z}_2)^n$-orbits and maps every rank k orbit (i.e., every orbit isomorphic to $(\mathbb{Z}_2)^k$) to an interior point of a k-dimensional face of the polytope P, $k = 0, \ldots, n$.

We can easily show that π sends $(\mathbb{Z}_2)^n$-fixed points in M to vertices of P by using the above condition (b). We often call P an *orbit polytope* of M. We will use the symbol $(M, (\mathbb{Z}_2)^n)$ to denote an n-dimensional small cover M with $(\mathbb{Z}_2)^n$-action.

Let us look at three examples of small covers.

Example 2.1. Let $\mathbb{R}P^n$ be the n-dimensional real projective space and $[x_0 : x_1 : \cdots : x_n]$ be an element of $\mathbb{R}P^n$ represented by the standard projective coordinate. We define a $(\mathbb{Z}_2)^n$-action as follows:

$$[x_0 : x_1 : \cdots : x_n] \xrightarrow{(t_1,\ldots,t_n)} [x_0 : t_1x_1 : \cdots : t_nx_n],$$

where $(t_1, \ldots, t_n) \in (\mathbb{Z}_2)^n$. Then, one can easily check that this action is locally standard and the orbit polytope is the n-dimensional simplex Δ^n, i.e., the above $(\mathbb{R}P^n, (\mathbb{Z}_2)^n)$ is a small cover.

Example 2.2. We define the $\mathbb{Z}_2$-action on $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$ by the antipodal action on the first S^1-factor and the complex conjugation on the second S^1-factor. Let $\mathcal{K}^2 = S^1 \times_{\mathbb{Z}_2} S^1$ be the orbit space of this action. Then, $\mathcal{K}^2$ is the 2-dimensional manifold with the following $(\mathbb{Z}_2)^2$-action:

$$[x_1 + \sqrt{-1}y_1, x_2 + \sqrt{-1}y_2] \xrightarrow{(t_1,t_2)} [x_1 + t_1\sqrt{-1}y_1, x_2 + t_2\sqrt{-1}y_2],$$

where $[x_1 + \sqrt{-1}y_1, x_2 + \sqrt{-1}y_2] \in \mathcal{K}^2$ and $(t_1, t_2) \in (\mathbb{Z}_2)^2$. Then, one can easily check that $(\mathcal{K}^2, (\mathbb{Z}_2)^2)$ is a small cover whose orbit polytope is the 2-dimensional square I^2, where $I = [0, 1]$ is the interval. Note that $\mathcal{K}^2$ is an S^1-bundle over S^1; more precisely, we see that $\mathcal{K}^2$ is diffeomorphic to the Klein bottle.

Example 2.3. Let M_i be an n_i-dimensional small cover with $(\mathbb{Z}_2)^{n_i}$-action and P_i be its orbit polytope. Then, the product of manifolds with actions

$$\left(\prod_{i=1}^{a} M_i, \prod_{i=1}^{a} (\mathbb{Z}_2)^{n_i}\right)$$

is a small cover whose orbit polytope is $\prod_{i=1}^{a} P_i$, where $a \in \mathbb{N}$ and $\dim \prod_{i=1}^{a} M_i = \sum_{i=1}^{a} n_i$.

2.2. Construction of small covers. From another point of view, for a given simple polytope P, the small cover M with orbit projection $\pi : M \to P$ can be reconstructed by using the *characteristic function* $\lambda : \mathcal{F} \to (\mathbb{Z}/2\mathbb{Z})^n$, where $\mathcal{F}$ is the set of all facets in P and $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. In this subsection, we review this construction (see [**2, 4**] for detail).

We first recall the characteristic function of the small cover M over P. Due to the definition of a small cover $\pi : M \to P$, we have that $\pi^{-1}(\mathrm{int}(F^{n-1}))$ consists of $(n-1)$-rank orbits, in other words, the isotropy subgroup at $x \in \pi^{-1}(\mathrm{int}(F^{n-1}))$ is $K \subset (\mathbb{Z}_2)^n$ such that $K \cong \mathbb{Z}_2$, where $\mathrm{int}(F^{n-1})$ is the interior of the facet F^{n-1}. Hence, the isotropy subgroup at x is determined by a primitive vector $v \in (\mathbb{Z}/2\mathbb{Z})^n$ such that $(\mathbf{-1})^v$ generates the subgroup K, where $(\mathbf{-1})^v = ((-1)^{v_1}, \ldots, (-1)^{v_n})$

for $v = (v_1, \ldots, v_n) \in (\mathbb{Z}/2\mathbb{Z})^n$. In this way, we obtain a function λ from the set of facets of P, denoted by $\mathcal{F}$, to vectors in $(\mathbb{Z}/2\mathbb{Z})^n$. We call such $\lambda : \mathcal{F} \to (\mathbb{Z}/2\mathbb{Z})^n$ a *characteristic function* or *coloring* on P. We often describe λ as the $m \times n$-matrix $\Lambda = (\lambda(F_1) \cdots \lambda(F_m))$ for $\mathcal{F} = \{F_1, \ldots, F_m\}$, and we call this matrix a *characteristic matrix.* Since the $(\mathbb{Z}_2)^n$-action is locally standard, a characteristic function has the following property (called *the property* $(\star)$):

$(\star)$**:** if $F_1 \cap \cdots \cap F_n \neq \emptyset$ for $F_i \in \mathcal{F}$ $(i = 1, \ldots, n)$, then $\{\lambda(F_1), \ldots, \lambda(F_n)\}$ spans $(\mathbb{Z}/2\mathbb{Z})^n$.

It is interesting that one also can construct small covers by using a given n-dimensional simple convex polytope P and a characteristic function λ with the property $(\star)$. Next, we mention the construction of small covers by using P and λ. Let P be an n-dimensional simple convex polytope. Suppose that a characteristic function $\lambda : \mathcal{F} \to (\mathbb{Z}/2\mathbb{Z})^n$ which satisfies the above property $(\star)$ is defined on P. Small covers can be constructed from P and λ as the quotient space $(\mathbb{Z}_2)^n \times P/\sim$, where the symbol $\sim$ represents an equivalence relation on $(\mathbb{Z}_2)^n \times P$ defined as follows: $(t, x) \sim (t', y)$ if and only if $x = y \in P$ and

$$\begin{array}{ll} t = t' & \text{if } x \in \mathrm{int}(P); \\ t^{-1}t' \in \langle (\mathbf{-1})^{\lambda(F_1)}, \cdots, (\mathbf{-1})^{\lambda(F_k)} \rangle \cong (\mathbb{Z}_2)^k & \text{if } x \in \mathrm{int}(F_1 \cap \cdots \cap F_k), \end{array}$$

where $\langle (\mathbf{-1})^{\lambda(F_1)}, \cdots, (\mathbf{-1})^{\lambda(F_k)} \rangle \subset (\mathbb{Z}_2)^n$ denotes the subgroup generated by $(\mathbf{-1})^{\lambda(F_i)}$ for $i = 1, \ldots, k$. The small cover $(\mathbb{Z}_2)^n \times P/\sim$ is usually denoted by $M(P, \lambda)$.

Let us look at two examples.

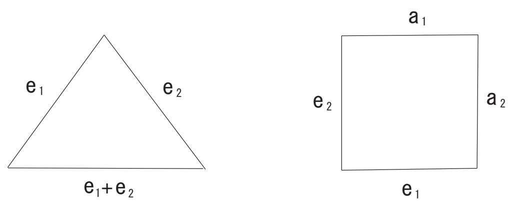

FIGURE 1. Projective characteristic functions. In the left triangle and the right square, functions satisfy $(\star)$ on each vertex, where e_1, e_2 are canonical basis of $(\mathbb{Z}/2\mathbb{Z})^2$ and $a_1, a_2 \in (\mathbb{Z}/2\mathbb{Z})^2$.

As is well known, in Figure 1, the left example corresponds to $\mathbb{R}P^2$ with $(\mathbb{Z}_2)^2$-action as in Example 2.1 and the right example corresponds to the following two small covers: (1) if $a_1 = e_1$ and $a_2 = e_2$ then the small cover is $S^1 \times S^1 (\simeq T^2)$ with the diagonal $\mathbb{Z}_2 \times \mathbb{Z}_2$-action (see Example 2.3); (2) otherwise, the small cover is $\mathcal{K}^2$ in Example 2.2.

Summand up, we have the following relationship:

Small covers with $(\mathbb{Z}_2)^n$-actions	$\longrightarrow$ $\longleftarrow$	Simple, convex polytopes with characteristic functions

2.3. Equivariant cohomology and ordinary cohomology of small covers. In this subsection, we recall the equivariant cohomologies and ordinary cohomologies of small covers (see [**2, 4**] for detail). Let $M = M(P, \lambda)$ be an n-dimensional small cover. We denote the facet of P by $\mathcal{F} = \{F_1, \ldots, F_m\}$ such that $\cap_{i=1}^n F_i \neq \emptyset$. Then, we may take the characteristic functions on $F_1, \ldots, F_n$ as

$$\lambda(F_i) = e_i,$$

where $e_1, \ldots, e_n$ are the standard basis of $(\mathbb{Z}/2\mathbb{Z})^n$. That is, we can write the characteristic matrix as

$$\Lambda = (I_n \mid \Lambda'),$$

where I_n is the $(n \times n)$-identity matrix and Λ' is an $(\ell \times n)$-matrix, where $\ell = m - n$.

An *equivariant cohomology* of G-manifold X is defined by the ordinary cohomology of $EG \times_G X$, where EG is an universal space of G, and denote it by $H^*_G(X)$. In this paper, we assume the coefficient of cohomology is $\mathbb{Z}/2\mathbb{Z}$. Due to [**4**], the ring structure of equivariant cohomology of small cover M is given by the following formula:

$$H^*_{(\mathbb{Z}_2)^n}(M) \cong \mathbb{Z}/2\mathbb{Z}[\tau_1, \ldots, \tau_m]/\mathcal{I},$$

where the symbol $\mathbb{Z}/2\mathbb{Z}[\tau_1, \ldots, \tau_m]$ represents the polynomial ring generated by the degree 1 elements τ_i ($i = 1, \ldots, m$), and the ideal $\mathcal{I}$ is generated by the following monomial elements:

$$\prod_{i \in I} \tau_i,$$

where I takes every subgroup of $\{1, \ldots, m\}$ such $\cap_{i\in I} F_i = \emptyset$. On the other hand, the ordinary cohomology ring of M is given by

$$H^*(M) \cong H^*_{(\mathbb{Z}_2)^n}(M)/\mathcal{J},$$

where the ideal $\mathcal{J}$ is generated by the following degree 1 homogeneous elements:

$$\tau_i + \lambda_{i1}x_1 + \cdots + \lambda_{i\ell}x_\ell,$$

for $i = 1, \ldots, n$. Here, $(\lambda_{i1} \cdots \lambda_{i\ell})$ is the ith row vector of Λ' ($i = 1, \ldots, n$), and $x_j = \tau_{n+j}$ ($j = 1, \ldots, \ell$).

3. Motivation (Masuda's counterexamples)

In this section, we recall a motivation example, more precisely Masuda's counterexample mentioned in the Introduction.

3.1. Projective bundles associated with vector bundles. Given a k-dimensional real vector bundle ξ over M, we denote by $E(\xi)$ its total space, $\widetilde{\rho} : E(\xi) \to M$ the vector bundle projection, and $F_x(\xi)$ the fibre over $x \in M$, thus $\widetilde{\rho}^{-1}(x) = F_x(\xi)$. Then it is known that the space $P(\xi)$ whose points are the 1-dimensional vector subspaces in the fibre $F_x(\xi)$ for all $x \in M$ is the total space of a fibre bundle over M, with fibre homeomorphic to $(k-1)$-dimensional real projective space. We denote the projection of this fibre bundle, known as the *projective bundle of the vector space* ξ, by $\rho : P(\xi) \to M$, and its fibre $\rho^{-1}(x)$ is often denoted by $P_x(\xi)$.

3.2. Masuda's example. Let $M(q) = P(q\gamma \oplus (b-q)\epsilon)$ be the projective bundle of $q\gamma \oplus (b-q)\epsilon$, where γ is the tautological line bundle, ϵ is the trivial bundle over $\mathbb{R}P^a$, and $0 \le q \le b$; here a and b are fixed positive integers.

In [**10**], Masuda proves the following theorem.

THEOREM 3.1 (Masuda). *The following two statements hold:*

(1) $H^*(M(q); \mathbb{Z}/2\mathbb{Z}) \cong H^*(M(q'); \mathbb{Z}/2\mathbb{Z})$ *if and only if* $q' \equiv q$ *or* $b - q \mod 2^{h(a)}$*;*
(2) $M(q) \simeq M(q')$ *if and only if* $q' \equiv q$ *or* $b - q \mod 2^{k(a)}$*,*

where $h(a) = \min\{n \in \mathbb{N} \cup \{0\} \mid 2^n \ge a\}$ *and* $k(a) = \#\{n \in \mathbb{N} \mid 0 < n < a \text{ and } n \equiv 0, 1, 2, 4 \mod 8\}$.

Therefore, in view of Proposition 4.1, the following example gives a counterexample to cohomological rigidity of small covers.

COROLLARY 3.2. Let $M(q) = P(q\gamma \oplus (17 - q)\epsilon)$ be the above projective bundle over $\mathbb{R}P^{10}$, i.e., $a = 10$ and $b = 17$. Then $H^*(M(0); \mathbb{Z}/2\mathbb{Z}) \cong H^*(M(1); \mathbb{Z}/2\mathbb{Z})$ but $M(0) \not\simeq M(1)$.

PROOF. By the definition of $h(a)$ and $k(a)$, we have $h(10) = 4$, $k(10) = 5$. Because $b = 17$, we have $0 \equiv 17 - 1 \mod 2^{h(10)} = 16$. Therefore, the cohomology rings of $M(0)$ and $M(1)$ are isomorphic. However, $0 \not\equiv 17 - 1 \mod 2^{k(10)} = 32$. It follows that $M(0)$ and $M(1)$ are not homeomorphic. $\square$

4. Projective bundles over small covers

In this section, we introduce some notations and basic results for projective bundles over small covers. We first recall the definition of a G-equivariant vector bundle over a G-space M (also see the notations in Section 3). A *G-equivariant vector bundle* is a vector bundle ξ over G-space M together with a lift of the G-action to $E(\xi)$ by fibrewise linear transformations, i.e., $E(\xi)$ is also a G-space, the projection $E(\xi) \to M$ is G-equivariant and the induced fibre isomorphism between $F_x(\xi)$ and $F_{gx}(\xi)$ is linear, where $x \in M$ and $g \in G$.

Henceforth, we assume M is an n-dimensional small cover, and ξ is a k-dimensional, $(\mathbb{Z}_2)^n$-equivariant vector bundle over M. One can easily show that the following proposition gives a criterion for the projective bundle $P(\xi)$ to be a small cover (e.g. see [**9**]).

PROPOSITION 4.1. The projective bundle $P(\xi)$ of ξ is a small cover if and only if the vector bundle ξ decomposes into the Whitney sum of line bundles, i.e., $\xi \equiv \gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \gamma_k$.

By using the fact that $P(\xi \otimes \gamma) \simeq P(\xi)$ (homeomorphic) for all line bundles γ (e.g. see [**11**]) and the above Proposition 4.1, we have the following corollary.

COROLLARY 4.2. Let M be a small cover, and ξ be the Whitney sum of some k line bundles over M. Then the small cover $P(\xi)$ is homeomorphic to

$$P(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \epsilon),$$

where the vector bundles γ_i ($i = 1, \ldots, k-1$) are line bundles and ϵ is the trivial line bundle over M.

In this article, a *projective bundle over a small cover* means the projective bundle in Corollary 4.2 (also see Section 1).

4.1. Structures of projective bundles over small covers. In this section, we show the structure of projective bundles over small covers. First, we recall the moment-angle complex of small covers. Let P be a simple, convex polytope and $\mathcal{F}$ a set of its facets $\{F_1, \cdots, F_m\}$. We denote $\mathcal{Z}_P$ as the manifolds

$$\mathcal{Z}_P = (\mathbb{Z}_2)^m \times P/\sim,$$

where $(t, p) \sim (t', p)$ is defined by $t^{-1}t' \in \prod_{p \in F_i} \mathbb{Z}_2(i)$ ($\mathbb{Z}_2(i) \subset (\mathbb{Z}_2)^m$ is the subgroup generated by the ith factor), and we call it a *moment-angle manifold* of P. We note that if $P = M^n/(\mathbb{Z}_2)^n$ then there is a subgroup $K \subset (\mathbb{Z}_2)^m$ such that $K \cong (\mathbb{Z}_2)^{m-n}$ and K acts freely on $\mathcal{Z}_P$. Therefore, we can denote the small cover $M = \mathcal{Z}_P/(\mathbb{Z}_2)^\ell$ for $\ell = m - n$.

Since $[M; B\mathbb{Z}_2] = H^1(M; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^\ell$ (see [**4, 17**]), we see that all line bundles γ can be written as follows:

$$\gamma \equiv \mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} \mathbb{R}_\rho,$$

where $(\mathbb{Z}_2)^\ell$ acts on $\mathbb{R}_\rho = \mathbb{R}$ by some representation $\rho : (\mathbb{Z}_2)^\ell \to \mathbb{Z}_2$. Moreover, its Stiefel-Whitney class is $w(\mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} \mathbb{R}) = 1 + \delta_1 x_1 + \cdots + \delta_\ell x_\ell$ where $(\delta_1, \cdots, \delta_\ell) \in (\mathbb{Z}/2\mathbb{Z})^\ell$ is induced by a representation $(\mathbb{Z}_2)^\ell \to \mathbb{Z}_2$, i.e., $(-1, \cdots, -1) \mapsto (-1)^{\delta_1} \cdots (-1)^{\delta_\ell}$, and $x_1, \ldots, x_\ell$ are the degree 1 generators of $H^*(M)$ induced in Section 2.3. Therefore, by using Corollary 4.2, all projective bundles of small covers are as follows:

$$P(\xi) = \mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} (\mathbb{R}^k - \{0\})/\mathbb{R}^* = \mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} \mathbb{R}P^{k-1}, \tag{4.1}$$

where

$$\xi = \mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} \mathbb{R}^k$$

with the $(\mathbb{Z}_2)^\ell$ representation space $\mathbb{R}^k = \mathbb{R}_{\alpha_1} \oplus \cdots \oplus \mathbb{R}_{\alpha_k}$ such that

$$\alpha_i : (\mathbb{Z}_2)^\ell \to \mathbb{Z}_2$$

where $i = 1, \cdots, k$ and α_k is the trivial representation. Then we may denote the projective bundle of a small cover by

$$\mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} \mathbb{R}P^{k-1} = P(\gamma_1 \oplus \cdots \gamma_{k-1} \oplus \epsilon),$$

where $\gamma_i = \mathcal{Z}_P \times_{(\mathbb{Z}_2)^\ell} \mathbb{R}_{\alpha_i}$ $(i = 1, \cdots, k-1)$ satisfies that $w(\gamma_i) = 1 + \delta_{1i}x_1 + \cdots + \delta_{\ell i}x_\ell$ for $(\delta_{1i}, \cdots, \delta_{\ell i}) \in (\mathbb{Z}/2\mathbb{Z})^\ell$ which is induced by the representation $\alpha_i : (\mathbb{Z}_2)^\ell \to \mathbb{Z}_2$.

Let $(I_n \mid \Lambda) \in M(m, n; \mathbb{Z}/2\mathbb{Z})$ be the characteristic matrix of M. Using the above construction of projective bundles and computing their characteristic functions, we have the following proposition.

PROPOSITION 4.3. Let $P(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \epsilon)$ be the projective bundle over M. Then its orbit polytope is $P^n \times \Delta^{k-1}$, and its characteristic matrix is as follows:

$$\begin{pmatrix} I_n & O & \Lambda & \mathbf{0} \\ O & I_{k-1} & \Lambda_\xi & \mathbf{1} \end{pmatrix}, \tag{4.2}$$

where $P^n = M/(\mathbb{Z}_2)^n$ and

$$\Lambda_\xi = \begin{pmatrix} \delta_{11} & \cdots & \delta_{\ell 1} \\ \vdots & \ddots & \vdots \\ \delta_{1,k-1} & \cdots & \delta_{\ell,k-1} \end{pmatrix}.$$

4.2. New characteristic function of projective bundles over 2-dimensional small covers. In order to show the main result, we introduce a new characteristic function (matrix). As an easy case, we only consider the case when $n = 2$. Let $(I_2 \mid \Lambda)$ be the characteristic matrix of M^2, where $\Lambda \in M(2, \ell; \mathbb{Z}/2\mathbb{Z})$ for $\ell = m - 2$ (m is the number of facets of $P^2 = M/(\mathbb{Z}_2)^2$). By using Proposition 4.3, the characteristic matrix of $P(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \epsilon)$ is

$$\begin{pmatrix} I_2 & O & \Lambda & \mathbf{0} \\ O & I_{k-1} & \Lambda_\xi & \mathbf{1} \end{pmatrix}, \tag{4.3}$$

where

$$\begin{pmatrix} \Lambda \\ \Lambda_\xi \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{\ell 1} \\ \lambda_{12} & \cdots & \lambda_{\ell 2} \\ \delta_{11} & \cdots & \delta_{\ell 1} \\ \vdots & \ddots & \vdots \\ \delta_{1,k-1} & \cdots & \delta_{\ell,k-1} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_\ell \\ \mathbf{b}_1 & \cdots & \mathbf{b}_\ell \end{pmatrix},$$

where $\mathbf{a}_i \in (\mathbb{Z}/2\mathbb{Z})^2$ and $\mathbf{b}_i \in (\mathbb{Z}/2\mathbb{Z})^{k-1}$ for $i = 1, \cdots, \ell$. Therefore, in order to determine the projective bundles over M^2, it is sufficient to consider the following characteristic function: for an m-gon P^2 and its facets $\mathcal{F}$, the function

$$\lambda_P : \mathcal{F} = \{F_1, \cdots, F_m\} \to (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^{k-1}$$

satisfies that $\lambda_P(F_1) = e_1 \times \mathbf{0}$, $\lambda_P(F_2) = e_2 \times \mathbf{0}$ (denote them simply $e_1 = (1, 0)$, $e_2 = (0, 1)$, respectively) and

$$\det(\lambda_P(F_i)\ \lambda_P(F_j)\ X_1\ \cdots\ X_{k-1}) = 1 \tag{4.4}$$

for $F_i \cap F_j \neq \emptyset$ $(i \neq j)$ and all $\{X_1, \cdots, X_{k-1}\} \subset \{\mathbf{0} \times e'_1, \cdots, \mathbf{0} \times e'_{k-1}, \mathbf{0} \times \mathbf{1}\}$, where e'_i $(i = 1, \cdots, k-1)$ is the canonical basis in $(\mathbb{Z}/2\mathbb{Z})^{k-1}$. We call this function a *projective characteristic function* (we can easily generalize this notion to the general dimension n, not only 2), and (P^2, λ_P) corresponds with the projective bundle over the 2-dimensional small cover (over P^2). The Figure 2 is an illustration of projective characteristic functions.

Note that in Figure 2, if we put $b = 0$ and $b_1 = b_2 = 0 \in (\mathbb{Z}/2\mathbb{Z})^{k-1}$, then this gives ordinary characteristic functions on the triangle and the square. Therefore, we can regard such a forgetful map of $(\mathbb{Z}/2\mathbb{Z})^{k-1}$-part, $f : (P^2, \lambda_P) \to (P^2, \lambda)$, as the equivariant projection $P(\xi) \to M^2$.

5. Basic 2-dimensional small covers and classification of their projective bundles

In order to state the main result, in this section, we introduce a construction theorem for 2-dimensional small covers and two classification results for projective bundles.

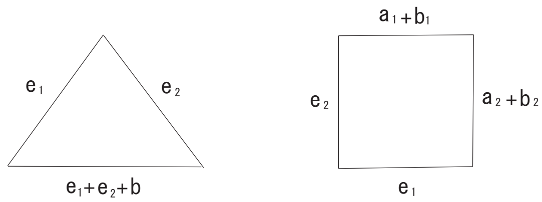

FIGURE 2. Projective characteristic functions. In the left triangle and the right square, functions satisfy (4.4) on each vertex, where $a_1,\ a_2 \in (\mathbb{Z}/2\mathbb{Z})^2$.

5.1. Construction theorem for 2-dimensional small covers. First, we introduce Proposition 5.1. Note that a 2-dimensional small cover can be induced by the 2-dimensional polytope and characteristic functions as those in Figure 3, where $e_1 = (1,\ 0)$, $e_2 = (0,\ 1)$, $e_1 + e_2 = (1,\ 1)$ and the *n-gon* is the 2-dimensional polytope with n facets (or n vertices).

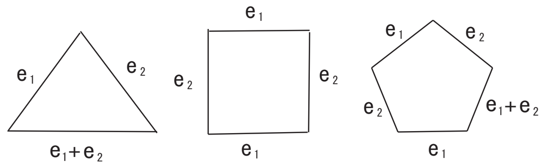

FIGURE 3. The left figure is $(\Delta^2,\ \lambda_0)$ and the middle figure is $(I^2,\ \lambda_0^I)$. We see that $M(\Delta^2,\ \lambda_0) = \mathbb{R}P^2$ and $M(I^2,\ \lambda_0^I) = T^2$, where $\mathbb{R}P^2$ and T^2 have the standard $(\mathbb{Z}_2)^2$-actions. The third figure is the 5-gon with characteristic functions.

We have the following construction theorem for 2-dimensional small covers based on an argument from [**16**] (see [**9**] for detail).

PROPOSITION 5.1 (Construction theorem). Let M^2 be a 2-dimensional small cover. Then M^2 is equivariantly homeomorphic to equivariant connected sum of $\mathbb{R}P^2$ and T^2 with standard $(\mathbb{Z}_2)^2$-actions.

Therefore, the real projective space $\mathbb{R}P^2$ and torus T^2 with standard $(\mathbb{Z}_2)^2$-actions are the basic small covers in the realm of 2-dimensional small covers.

5.2. Topological classification of projective bundles over $\mathbb{R}P^2$ and T^2. Next, we classify the topological types of projective bundles over basic small covers, i.e., $\mathbb{R}P^2$ and T^2.

The classification of projective bundles over $\mathbb{R}P^2$ is known by Masuda's paper [**10**]. Thanks to [**10**], we have that $q \equiv q'$ or $k - q'$ (mod 4) if and only if $S^2 \times_{\mathbb{Z}_2} P(q\gamma \oplus (k-q)\epsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(q'\gamma \oplus (k-q')\epsilon)$. Note that the line bundle over $\mathbb{R}P^2$ is the tautological line bundle γ or the trivial line bundle ϵ. Therefore, by Proposition 4.1, we see that the projective bundles over $\mathbb{R}P^2$ are only these types. Hence, we can easily obtain the following proposition.

PROPOSITION 5.2. Let $P(q\gamma \oplus (k-q)\epsilon)$ be a projective bundle over $\mathbb{R}P^2$. Then its topological type is one of the followings.

(1) The case $k \equiv 0 \mod 4$:
  (a) if $q \equiv 0 \mod 4$, then $P(q\gamma \oplus (k-q)\epsilon) \simeq \mathbb{R}P^2 \times \mathbb{R}P^{k-1}$;
  (b) if $q \equiv 1,\ 3 \mod 4$, then $P(q\gamma \oplus (k-q)\epsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(\gamma \oplus (k-1)\epsilon)$;
  (c) if $q \equiv 2 \mod 4$, then $P(q\gamma \oplus (k-q)\epsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(2\gamma \oplus (k-2)\epsilon)$.
(2) The case $k \equiv 1 \mod 4$:
  (a) if $q \equiv 0,\ 1 \mod 4$, then $P(q\gamma \oplus (k-q)\epsilon) \simeq \mathbb{R}P^2 \times \mathbb{R}P^{k-1}$;
  (b) if $q \equiv 2,\ 3 \mod 4$, then $P(q\gamma \oplus (k-q)\epsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(2\gamma \oplus (k-2)\epsilon)$.
(3) The case $k \equiv 2 \mod 4$:
  (a) if $q \equiv 0,\ 2 \mod 4$, then $P(q\gamma \oplus (k-q)\epsilon) \simeq \mathbb{R}P^2 \times \mathbb{R}P^{k-1}$;
  (b) if $q \equiv 1 \mod 4$, then $P(q\gamma \oplus (k-q)\epsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(\gamma \oplus (k-1)\epsilon)$;

(c) if $q \equiv 3 \mod 4$, then $P(q\gamma \oplus (k-q)\epsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(3\gamma \oplus (k-3)\epsilon)$.

(4) The case $k \equiv 3 \mod 4$:

(a) if $q \equiv 0,\ 3 \mod 4$, then $P(q\gamma \oplus (k-q)\epsilon) \simeq \mathbb{R}P^2 \times \mathbb{R}P^{k-1}$;

(b) if $q \equiv 1,\ 2 \mod 4$, then $P(q\gamma \oplus (k-q)\epsilon) \simeq S^2 \times_{\mathbb{Z}_2} P(\gamma \oplus (k-1)\epsilon)$.

Note that the moment-angle manifold over $\mathbb{R}P^2$ is S^2.

Finally, we classify projective bundles over T^2. Let γ_i be the pull back of the canonical line bundle over S^1 by the ith factor projection $\pi_i : T^2 \to S^1$ $(i = 1,\ 2)$. We can easily show that line bundles over T^2 are completely determined by their first Stiefel-Whitney classes by $[T^2,\ B\mathbb{Z}_2] \cong H^1(T^2;\ \mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$. Therefore, all of the line bundles over T^2 are ϵ, γ_1, γ_2 and $\gamma_1 \otimes \gamma_2$. By the definition of γ_i, we can easily show that $\gamma_i \oplus \gamma_i = \pi_i^*(\gamma \oplus \gamma) = \pi_i^*(2\epsilon) = 2\epsilon$. Therefore, we also have

$$(\gamma_1 \otimes \gamma_2) \oplus (\gamma_1 \otimes \gamma_2) = \gamma_1 \otimes (\gamma_2 \oplus \gamma_2) = \gamma_1 \otimes 2\epsilon = \gamma_1 \oplus \gamma_1 = 2\epsilon.$$

Hence, we have the following proposition (see [**9**] for detail).

Proposition 5.3. Let $P(\xi)$ be a projective bundle over T^2. Then its topological type is one of the following.

(1) The case $k \equiv 0 \mod 2$:

(a) $P(k\epsilon) \simeq T^2 \times \mathbb{R}P^{k-1}$;

(b) $P((\gamma_1 \otimes \gamma_2) \oplus (k-1)\epsilon) \simeq P(\gamma_1 \oplus \gamma_2 \oplus (k-2)\epsilon) \simeq T^2 \times_{(\mathbb{Z}_2)^2} P(\mathbb{R}_{\rho_1} \oplus \mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-2})$;

(c) $P(\gamma_1 \oplus (k-1)\epsilon) \simeq P((\gamma_1 \otimes \gamma_2) \oplus \gamma_2 \oplus (k-2)\epsilon) \simeq T^2 \times_{(\mathbb{Z}_2)^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}}^{k-1})$;

(d) $P(\gamma_2 \oplus (k-1)\epsilon) \simeq P((\gamma_1 \otimes \gamma_2) \oplus \gamma_1 \oplus (k-2)\epsilon) \simeq T^2 \times_{(\mathbb{Z}_2)^2} P(\mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-1})$;

(2) The case $k \equiv 1 \mod 2$:

(a) $P(k\epsilon) \simeq T^2 \times \mathbb{R}P^{k-1}$;

(b) $P((\gamma_1 \otimes \gamma_2) \oplus (k-1)\epsilon) \simeq P(\gamma_1 \oplus (k-1)\epsilon) \simeq P(\gamma_2 \oplus (k-1)\epsilon) \simeq T^2 \times_{(\mathbb{Z}_2)^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}}^{k-1})$;

(c) $P((\gamma_1 \oplus \gamma_2) \oplus (k-2)\epsilon) \simeq P((\gamma_1 \otimes \gamma_2) \oplus \gamma_1 \oplus (k-2)\epsilon) \simeq P((\gamma_1 \otimes \gamma_2) \oplus \gamma_2 \oplus (k-2)\epsilon) \simeq T^2 \times_{(\mathbb{Z}_2)^2} P(\mathbb{R}_{\rho_1} \oplus \mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-2})$,

where $\rho_i : (\mathbb{Z}_2)^2 \to \mathbb{Z}_2$ is the ith projection and $\underline{\mathbb{R}}$ is the trivial representation space.

Note that the moment-angle manifold over T^2 is T^2 itself.

6. Main Theorem

In this section, we state our main theorem. Before doing so, we introduce a new operation.

For two polytopes with projective characteristic functions, we can do the connected sum operation which is compatible with projective characteristic functions as indicated in Figure 4. Then we get a new polytope with the projective characteristic function. We call this operation a

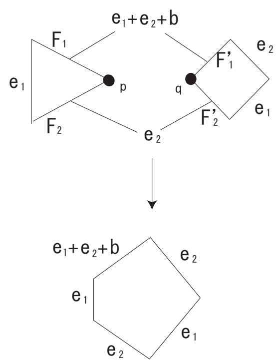

Figure 4. We can construct a connected sum at two vertices with the same projective characteristic function.

projective connected sum and denote it by $\sharp^{\Delta^{k-1}}$.

More precisely, this operation is defined as follows. Let p and q be vertices in 2-dimensional polytopes with projective characteristic functions (P, λ_P) and $(P', \lambda_{P'})$, respectively. Here, we assume that the target space of λ_P and $\lambda_{P'}$ are the same $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^{k-1}$, i.e., the corresponding projective bundles have the same fibre $\mathbb{R}P^{k-1}$. Moreover, we assume that $\lambda_P(F_1) = \lambda_{P'}(F_1')$ and $\lambda_P(F_2) = \lambda_{P'}(F_2')$ for facets $\{F_1, F_2\}$ around p and $\{F_1', F_2'\}$ around q, i.e., $F_1 \cap F_2 = \{p\}$ and $F_1' \cap F_2' = \{q\}$. Then we can do the connected sum of two polytopes P and P' at these vertices by gluing each pair of facets F_i and F_i'. Thus, we get a polytope with projective characteristic functions $(P\sharp^{\Delta^{k-1}}P',\ \lambda_{P\sharp^{\Delta^{k-1}}P'})$ from (P, λ_P) and $(P', \lambda_{P'})$ (also see Figure 4).

Note that, from the geometric point of view, the inverse image of vertices of polytopes with projective characteristic functions corresponds to the projective space $\mathbb{R}P^{k-1}$. Therefore, a geometric interpretation of this operation is an equivariant gluing along the fibre $\mathbb{R}P^{k-1}$, i.e., fibre sum of two fibre bundles.

Now we may state the main theorem of this article (see [**9**] for the proof).

THEOREM 6.1. *Let $P(\xi)$ be a projective bundle over a 2-dimensional small cover M^2. Then $P(\xi)$ can be constructed from one of the projective bundles $P(\kappa)$ over $\mathbb{R}P^2$ in Proposition 5.2 and $P(\zeta)$ over T^2 in Proposition 5.3 by using the projective connected sum $\sharp^{\Delta^{k-1}}$.*

If $k = 1$, then the projective characteristic function is the ordinary characteristic function, i.e., the dimension of fibre is 0. Therefore, we may regard this theorem as a generalization of the construction theorem of 2-dimensional small covers (Proposition 5.1).

7. Appendix: Topological classification of projective bundles over the 1-dimensional small cover

In closing this paper, we explain a topological classification of projective bundles over the 1-dimensional small cover.

It is easy to show that the 1-dimensional small cover is the 1-dimensional circle S^1 and its moment-angle manifold is S^1 itself, i.e., the moment-angle manifold $S^1 \subset \mathbb{R}^2$ has the diagonal free $\mathbb{Z}_2$-action and its quotient is the 1-dimensional small cover S^1. Therefore, the classification problem corresponds with the classification of the following projective bundles of $S^1 \times_{\mathbb{Z}_2} (q\gamma \oplus (k-q)\epsilon)$:

$$S^1 \times_{\mathbb{Z}_2} P(q\gamma \oplus (k-q)\epsilon),$$

where γ is the canonical line bundle and ϵ is the trivial line bundle over S^1 and $0 \le q \le k-1$ (also see [**10**]).

Because $[S^1,\ BO(k)] = \mathbb{Z}_2$, the vector bundles over S^1 can be classified by their first Stiefel-Whitney classes. Therefore, we see that

(1) if $q \equiv 0 \mod 2$, then $S^1 \times_{\mathbb{Z}_2} (q\gamma \oplus (k-q)\epsilon) \equiv S^1 \times_{\mathbb{Z}_2} k\epsilon \simeq S^1 \times \mathbb{R}^k$;
(2) if $q \equiv 1 \mod 2$, then $S^1 \times_{\mathbb{Z}_2} (q\gamma \oplus (k-q)\epsilon) \equiv S^1 \times_{\mathbb{Z}_2} (\gamma \oplus (k-1)\epsilon) \simeq (S^1 \times_{\mathbb{Z}_2} \mathbb{R}) \times \mathbb{R}^{k-1}$.

Because of the following three well-known facts: $P(\xi \otimes \gamma) \simeq P(\xi)$; $H^*(S^1 \times_{\mathbb{Z}_2} P(\xi);\ \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x,\ y]/ < x^2,\ y^k + y^{k-1}w_1(\pi^*\xi) >$ for $\deg x = \deg y = 1$ (by the Borel-Hirzebruch formula, see [**3**]); and the injectivity of induced homomorphism $\pi^* : H^*(S^1;\ \mathbb{Z}/2\mathbb{Z}) \to H^*(P(\xi);\ \mathbb{Z}/2\mathbb{Z})$ of $\pi : P(\xi) \to S^1$ (see [**14**]), we have the following proposition.

PROPOSITION 7.1. Let $P(q\gamma \oplus (k-q)\epsilon)$ be a projective bundle over S^1. Then its topological type is one of the following:

(1) if $k \equiv 1 \mod 2$ or $q \equiv 0 \mod 2$, then $P(q\gamma \oplus (k-q)\epsilon) = S^1 \times \mathbb{R}P^{k-1}$;
(2) otherwise, i.e., if $k \equiv 0 \mod 2$ and $q \equiv 1 \mod 2$, then $P(q\gamma \oplus (k-q)\epsilon) = S^1 \times_{\mathbb{Z}_2} \mathbb{R}P^{k-1}$, where $\mathbb{Z}_2$ acts freely on S^1 and on the first coordinate of $\mathbb{R}P^{k-1}$.

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