Existence and non-existence for nonlinear Schrödinger equations

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0. Introduction

In this report, we will introduce the results of my paper [**S**]. In [**S**], we consider the one dimensional case of the following nonlinear Schrödinger equations:

$$\begin{aligned} &-u'' + (1 + b(x))u = f(u) \quad \text{in } \mathbf{R}, \\ &u \in H^1(\mathbf{R}). \end{aligned} \tag{$*$}$$

Here, we assume that the potential $b(x) \in C(\mathbf{R}, \mathbf{R})$ satisfies the following assumptions:

(b.1) $1 + b(x) \geq 0$ for all $x \in \mathbf{R}$.

(b.2) $\lim_{|x|\to\infty} b(x) = 0$.

(b.3) There exist $\beta_0 > 2$ and $C_0 > 0$ such that $b(x) \leq C_0 e^{-\beta_0 |x|}$ for all $x \in \mathbf{R}$.

We set $F(u) = \int_0^u f(\tau)\, d\tau$ and assume that the nonlinearity $f(u)$ satisfies

(f.1) There exists $\eta_0 > 0$ such that $\lim_{|u|\to\infty} \frac{f(u)}{|u|^{1+\eta_0}} = 0$.

(f.2) There exists $u_0 > 0$ such that

$$\begin{aligned} F(u) &< \frac{1}{2}u^2 \quad \text{for all } u \in (0, u_0), \\ F(u_0) &= \frac{1}{2}u_0^2, \quad f(u_0) > u_0. \end{aligned}$$

(f.3) There exists $\mu_0 > 2$ such that $0 < \mu_0 F(u) \leq u f(u)$ for all $u \neq 0$.

The conditions (f.1) and (f.2) are sufficient conditions for the following equation to have an unique positive solution:

$$-u'' + u = f(u) \quad \text{in } \mathbf{R}, \quad u \in H^1(\mathbf{R}). \tag{0.1}$$

From (b.2), the equation $-u'' + u = f(u)$ appears as a limit when $|x|$ goes to ∞ in $(*)$. The condition (f.3) is so called Ambrosetti-Rabinowitz condition, which guarantees the boundedness of (PS)-sequences for the functional corresponding to the equation $(*)$ and (0.1).

To state an our result about the existence of solutions for $(*)$, we also need the following assumption for $b(x)$.

(b.4) There exists $x_0 \in \mathbf{R}$ such that

$$\overline{\lim_{r\to\infty}} \int_{-r}^{r} b(x-x_0)e^{2|x|}\, dx \in [-\infty, 2).$$

Our first theorem is the following.

Theorem 0.1. *Assume that (b.1)–(b.4) and (f.1)–(f.3) hold. Then $(*)$ has at least a positive solution.*

When we prove Theorem 0.1 in [**S**], it is important to estimate interaction of $\omega(x-R)$ and $\omega(x+R)$ for large $R >> 1$. Here, $\omega(x)$ is an unique solution of (0.1) with $u(0) = \max_{x\in\mathbf{R}} u(x)$. When we estimate interaction of $\omega(x-R)$ and $\omega(x+R)$, we naturally get the conditions (b.4) as a sufficient condition for $(*)$ to have a nontrivial solutions.

In next section, we will mainly give the outline of the proof of Theorem 0.1. In respect to details of the proof of Theorem 0.1, see [**S**].

We must remark that, for the case function $b(x)$ is contained in nonlinearity or higher dimensional cases, there exist non-trivial solutions without conditions like (b.4). In fact, Bahri-Li [**BaL**] showed that there exists a positive solution of

$$-\Delta u + u = (1-b(x))|u|^{p-1}u \quad \text{in } \mathbf{R}^N, \qquad u \in H^1(\mathbf{R}^N), \tag{0.2}$$

where $N \geq 3$, $1 < p < \frac{N+2}{N-2}$ and $b(x) \in C(\mathbf{R},\mathbf{R})$ satisfies the following conditions:

(b.1)' $1 - b(x) \geq 0$ for all $x \in \mathbf{R}^N$.
(b.2)' $\lim_{|x|\to\infty} b(x) = 0$.
(b.3)' There exist $\beta_0 > 2$ and $C_0 > 0$ such that $b(x) \leq C_0 e^{-\beta_0|x|}$ for all $x \in \mathbf{R}^N$.

For one dimensional case, Spradlin [**Sp**] proved that there exists a positive solution of the equation

$$-u'' + u = (1-b(x))f(u) \quad \text{in } \mathbf{R}, \qquad u \in H^1(\mathbf{R}). \tag{0.3}$$

They assumed that $b(x) \in C(\mathbf{R},\mathbf{R})$ satisfies $1 - b(x) \geq 0$ in $\mathbf{R}$ and (b.2)–(b.3) and $f(u)$ satisfies (f.1)–(f.3) and (f.4) $\frac{f(u)}{u}$ is an increasing function for all $u > 0$.

Moreover, we can easily apply the computations in [**BaL**] to the following equation which is a higher dimensional version of $(*)$.

$$-\Delta u + (1 + b(x))u = |u|^{p-1}u \quad \text{in } \mathbf{R}^N, \qquad u \in H^1(\mathbf{R}^N). \tag{0.4}$$

From this application, we see that (0.4) also has at least a positive solution when $N \geq 3$, $1 < p < \frac{N+2}{N-2}$ and $b(x)$ satisfies $1 + b(x) \geq 0$ in $\mathbf{R}^N$ and (b.2)'–(b.3)'.

From the above results, it seems that Theorem 0.1 holds without condition (b.4). However (b.4) is an essential assumption for $(*)$ to have non-trivial solutions. In what follows, we will show a result about the non-existence of nontrivial solutions for $(*)$.

In next our result, we will assume that $b(x)$ satisfies the following condition:

(b.5) There exist $\mu > 0$ and $m_2 \geq m_1 > 0$ such that

$$m_1 \mu e^{-\mu|x|} \leq b(x) \leq m_2 \mu e^{-\mu|x|} \quad \text{for all } x \in \mathbf{R}.$$

Here, we remark that, if (b.5) holds for $\mu > 2$, then $b(x)$ satisfies (b.1)–(b.3) and

$$\frac{2\mu}{\mu - 2} m_1 \leq \int_{-\infty}^{\infty} b(x) e^{2|x|}\, dx \leq \frac{2\mu}{\mu - 2} m_2.$$

Thus, when $m_2 < 1$ and μ is very large, the condition (b.4) also holds.

Our second result is the following:

Theorem 0.2. *Assume that (b.5) holds and $f(u) = |u|^{p-1}u$ $(p > 1)$.*

(i) If $m_1 > 1$, there exists $\mu_1 > 0$ such that $(*)$ does not have non-trivial solution for all $\mu \geq \mu_1$.

(ii) If $m_2 < 1$, there exists $\mu_2 > 0$ such that $(*)$ has at least a non-trivial solution for all $\mu \geq \mu_2$.

From Theorem 0.2, we see that Theorem 0.1 does not hold except for condition (b.4). This is a drastically different situation from the higher dimensional cases. This is one of the interesting points in our results.

We remark that the condition (b.4) implies $\overline{\lim}_{r\to\infty} \int_{-r}^{r} b(x)\, dx < 2$ and the assumption of (ii) of Theorem 0.2 also means $\int_{-\infty}^{\infty} b(x)\, dx < 2$. Thus we expect that the difference from existence and non-existence of non-trivial solutions of $(*)$ depends on the quantity of integrate of $b(x)$.

We can obtain this expectation from another viewpoint, which is a perturbation problem. Setting $b_\mu(x) = m\mu e^{-\mu|x|}$, $b_\mu(x)$ satisfies (b.5) and, when $\mu \to \infty$, $b_\mu(x)$ converges to the delta function $2m\delta_0$ in distribution sense. Thus $(*)$ approaches to the equation

$$-u'' + (1 + 2m\delta_0)u = |u|^{p-1}u \quad \text{in } \mathbf{R}, \quad u \in H^1(\mathbf{R}), \tag{0.5}$$

in distribution sense. Here, if u is a solution of (0.5) in distribution sense, we can see that u is of C^2-function in $\mathbf{R} \setminus \{0\}$ and continuous in $\mathbf{R}$ and u satisfies

$$u'(+0) - u'(-0) = 2mu(0). \tag{0.6}$$

Moreover, since u is a homoclinic orbit of $-u'' + u = f(u)$ in $(-\infty, 0)$ or $(0, \infty)$, respectively, u satisfies

$$-\frac{1}{2}u'(x)^2 + \frac{1}{2}u(x)^2 - \frac{1}{p+1}|u(x)|^{p+1} = 0 \quad \text{for} \quad x \neq 0. \tag{0.7}$$

When $x \to \pm 0$ in (0.7), from (f.1), we find

$$u'(-0) = -u'(+0), \quad |u'(\pm 0)| < |u(0)|. \tag{0.8}$$

Thus, from (0.6) and (0.8), it easily see that (0.5) has an unique positive solution when $|m| < 1$ and (0.5) has no non-trivial solutions when $|m| \geq 1$. Therefore we can regard Theorem 0.2 as results of a perturbation problem of (0.5).

To prove Theorem 0.2, we develop the shooting arguments which used in [**BE**]. Bianchi and Egnell [**BE**] argued about the existence and non-existence of radial solutions for

$$-\Delta u = K(|x|)|u|^{\frac{N+2}{N-2}}, \quad u > 0 \quad \text{in } \mathbf{R}^N, \quad u(x) = O(|x|^{2-N}) \quad \text{as } |x| \to \infty. \tag{0.9}$$

Here $N \geq 3$ and $K(|x|)$ is a radial continuous function. Roughly speaking, they reduce (0.9) to an ordinary differential equation and considered two solutions for two initial value problems of that ordinary differential equation from $-\infty$ and 0. And, examining whether those solutions has suitable matchings at $r = 1$, they argued about the existence and non-existence of radial solutions.

In [**S**], to prove Theorem 0.2, we also consider two initial value problems from $\pm\infty$, that is, for $\lambda_1, \lambda_2 > 0$, we consider the following two problems:

$$\begin{aligned} &-u'' + (1 + b(x))u = f(u), \\ &\lim_{x \to -\infty} e^{-x} u(x) = \lim_{x \to -\infty} e^{-x} u'(x) = \lambda_1, \end{aligned} \tag{0.10}$$

and

$$
\begin{aligned}
&-u'' + (1+b(x))u = f(u), \\
&\lim_{x\to\infty} e^x u(x) = -\lim_{x\to\infty} e^x u(x) = \lambda_2.
\end{aligned} \tag{0.11}
$$

We can prove (0.10) and (0.11) have an unique solution respectively and write those unique solutions as $u_1(x;\lambda_1)$ and $u_2(x;\lambda_2)$ respectively. We set

$$
\begin{aligned}
\Gamma_1 &= \{(u_1(0;\lambda_1), u_1'(0;\lambda_1)) \in \mathbf{R}^2 \mid \lambda_1 > 0\}, \\
\Gamma_2 &= \{(u_2(0;\lambda_2), u_1'(0;\lambda_2)) \in \mathbf{R}^2 \mid \lambda_2 > 0\}.
\end{aligned}
$$

Then, $\Gamma_1 \cap \Gamma_2 = \emptyset$ is equivalent to the non-existence of solutions for $(*)$. Thus it is important to study shapes of Γ_1 and Γ_2. In respect to the details of proofs of Theorem 0.2, see [**S**].

In next section, we state about the outline of the proof of Theorem 0.1 in [**S**].

1. The outline of the proof of Theorem 0.1

In this section, we state the outline of the proof of Theorem 0.1. We will developed a variational approach which was used in [**BaL**] and [**Sp**].

In what follows, since we seek positive solutions of $(*)$, without loss of generalities, we assume $f(u) = 0$ for $u < 0$. To prove Theorem 0.1, we seek non-trivial critical points of the functional

$$
I(u) = \frac{1}{2}||u||^2_{H^1(\mathbf{R})} + \frac{1}{2}\int_{-\infty}^{\infty} b(x)u^2\,dx - \int_{-\infty}^{\infty} F(u)\,dx \in C^1(H^1(\mathbf{R}), \mathbf{R}),
$$

whose critical points are positive solutions of $(*)$. Here we use the following notations:

$$
\begin{aligned}
||u||^2_{H^1(\mathbf{R})} &= ||u'||^2_{L^2(\mathbf{R})} + ||u||^2_{L^2(\mathbf{R})}, \\
||u||^p_{L^p(\mathbf{R})} &= \int_{\mathbf{R}} |u|^p\,dx \quad \text{for} \quad p > 1.
\end{aligned}
$$

From (f.1)–(f.2), we can see that $I(u)$ satisfies a mountain pass geometry, that is, $I(u)$ satisfies

- (i) $I(0) = 0$.
- (ii) There exist $\delta > 0$ and $\rho > 0$ such that $I(u) \geq \delta$ for all $||u||_{H^1(\mathbf{R})} = \rho$.
- (iii) There exists $u_0 \in H^1(\mathbf{R})$ such that $I(u_0) < 0$ and $||u_0||_{H^1(\mathbf{R})} > \rho$.

From the mountain pass geometry (i)–(iii), we can define a standard minimax value $c > 0$ by

$$
\begin{aligned}
&c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)), \\
&\Gamma = \{\gamma(t) \in C([0,1], H^1(\mathbf{R})) \mid \gamma(0) = 0,\ I(\gamma(1)) < 0\}.
\end{aligned} \tag{1.1}
$$

And, by a standard way, we can construct $(PS)_c$-sequence $(u_n)_{n=1}^\infty$, that is, $(u_n)_{n=1}^\infty$ satisfies

$$\begin{aligned} &I(u_n) \to c \qquad (n \to \infty), \\ &I'(u_n) \to 0 \qquad \text{in} \quad H^{-1}(\mathbf{R}) \qquad (n \to \infty). \end{aligned}$$

Moreover, since $(u_n)_{n=1}^\infty$ is bounded in $H^1(\mathbf{R})$ from (f.3), $(u_n)_{n=1}^\infty$ has a subsequence $(u_{n_j})_{j=1}^\infty$ which weakly converges to some u_0 in $H^1(\mathbf{R})$. If $(u_{n_j})_{j=1}^\infty$ strongly converges to u_0 in $H^1(\mathbf{R})$, c is a non-trivial critical value of $I(u)$ and our proof is completed. However, since the embedding $L^p(\mathbf{R}) \subset H^1(\mathbf{R})$ $(p > 1)$ is not compact, there may not exist a subsequence $(u_{n_j})_{j=1}^\infty$ which strongly converges in $H^1(\mathbf{R})$. Therefore, in our situation, we don't know c is a critical value.

In our situation, from the lack of the compactness mentioned the above, we must use the concentration-compactness approach as [**BaL**] and [**Sp**]. In the concentration-compactness approach, we examine in detail what happens in bounded (PS)-sequence. When we state the concentration-compactness argument for the (PS)-sequences of $I(u)$, the limit problem (0.1) plays an important role. Setting

$$I_0(u) = \frac{1}{2}||u||^2_{H^1(\mathbf{R})} - \int_{-\infty}^{\infty} F(u)\, dx \in C^1(H^1(\mathbf{R}), \mathbf{R}),$$

the critical points of $I_0(u)$ correspond to the solutions of limit problem (0.1). The equation (0.1) has an unique positive solution, identifying ones which obtain by translations. Thus let $\omega(x)$ be an unique positive solution of (0.1) with $\max_{x \in \mathbf{R}} \omega(x) = \omega(0)$ and we set $c_0 = I_0(\omega)$. Since I_0 also satisfies the mountain pass geometry (i)–(iii), we see $c_0 > 0$ and c_0 is an unique non-trivial critical value.

For the bounded (PS)-sequences of $I(u)$, we have the following:

Proposition 1.1. *Suppose (b.1)–(b.2) and (f.1)–(f.2) holds. If $(u_n)_{n=1}^\infty$ is a bounded (PS)-sequence of $I(u)$, then there exist a subsequence $n_j \to \infty$, $k \in \mathbf{N} \cup \{0\}$, k-sequences $(x_j^1)_{j=1}^\infty, \cdots, (x_j^k)_{j=1}^\infty \subset \mathbf{R}$, and a critical point u_0 of $I(u)$ such that*

$$\begin{aligned} &I(u_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty), \\ &\left\| u_{n_j}(x) - u_0(x) - \sum_{\ell=1}^{k} \omega(x - x_j^\ell) \right\|_{H^1(\mathbf{R})} \to 0 \qquad (j \to \infty), \\ &|x_j^\ell - x_j^{\ell'}| \to \infty \quad (j \to \infty) \quad (\ell \neq \ell'), \\ &|x_j^\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \cdots, k). \end{aligned}$$

Proof. See [**JT1**].

If the minimax value c satisfies $c \in (0, c_0)$, from Proposition 1.1, we see that $I(u)$ has at least a non-trivial critical point. In fact, let $(u_n)_{n=1}^{\infty}$ be a bounded $(\mathrm{PS})_c$-sequence of $I(u)$, from Proposition 1.1, there exists a subsequence $n_j \to \infty$, $k \in \mathbf{N} \cup \{0\}$ and a critical point u_0 of $I(u)$ such that

$$I(u_{n_j}) \to I(u_0) + kc_0 \quad (j \to \infty).$$

Here, if $u_0 = 0$, we get $I(u_{n_j}) \to kc_0$ as $j \to \infty$. However this contradicts to the fact that $I(u_n) \to c \in (0, c_0)$ as $n \to \infty$. Thus $u_0 \neq 0$ and u_0 is a non-trivial critical point of $I(u)$. From the above argument, we have the following corollary.

Corollary 1.2. *Suppose $I(u)$ has no non-trivial critical points and let $(u_n)_{n=1}^{\infty}$ be a (PS)-sequence of $I(u)$. Then, only kc_0's ($k \in \mathbf{N} \cup \{0\}$) can be limit points of $\{I(u_n) \,|\, n \in \mathbf{N}\}$.*

Remark 1.3. Corollary 1.2 essentially depends on the uniqueness of the positive solution of (0.1).

As mentioned the above, when $c \in (0, c_0)$, $I(u)$ has at least a non-trivial critical point. However, unfortunately, under the condition (b.1)–(b.4), it may be $c = c_0$. Thus we need consider another minimax value. To define another minimax value, we use a path $\gamma_0(t) \in C(\mathbf{R}, H^1(\mathbf{R}))$ which is defined as follows: for small $\epsilon_0 > 0$, we set

$$h(x) = \begin{cases} \omega(x) & x \in [0, \infty], \\ x^4 + u_0 & x \in [-\epsilon_0, 0), \\ \epsilon_0^4 + u_0 & x \in (-\infty, -\epsilon_0), \end{cases}$$

$$\gamma_0(t)(x) = \begin{cases} h(x-t) & x \geq 0, \\ h(-x-t) & x < 0. \end{cases}$$

Here, we remark that u_0 was given in (f.2). This path $\gamma_0(t)$ was introduced in [**JT2**]. Choosing a proper $\epsilon_0 > 0$ sufficiently small, $\gamma_0(t)$ achieves the mountain pass value of $I_0(u)$ and satisfies the followings:

Lemma 1.4. *Suppose (f.1)–(f.2) hold. Then $\gamma_0(t)$ satisfies*

(i) $\gamma_0(0)(x) = \omega(x)$.

(ii) $I_0(\gamma_0(t)) < I_0(\omega) = c_0$ *for all* $t \neq 0$.

(iii) $\lim_{t \to -\infty} ||\gamma_0(t)||_{H^1(\mathbf{R})} = 0$, $\lim_{t \to \infty} ||\gamma_0(t)||_{H^1(\mathbf{R})} = \infty$.

Proof. See [**JT2**].

Now, for $R > 0$, we consider a path $\gamma_R \in C(\mathbf{R}^2, H^1(\mathbf{R}))$ which is defined by

$$\gamma_R(s,t)(x) = \max\{\gamma_0(s)(x+R),\ \gamma_0(t)(x-R)\}.$$

In our proof of Theorem 0.1 in [**S**], the following proposition is a key proposition.

Proposition 1.5. *Suppose (b.1)–(b.3) and (f.1)–(f.2) hold. Then, for any $L > 0$, we have*

$$\lim_{R\to\infty} e^{2R}\left\{\max_{(s,t)\in[-L,L]^2} I(\gamma_R(s,t)) - 2c_0\right\} = \frac{\lambda_0^2}{2}\left(\overline{\lim_{r\to\infty}}\int_{-r}^{r} b(x)e^{2|x|}\,dx - 2\right).$$

Here $\lambda_0 = \lim_{x\to\pm\infty}\omega(x)e^{|x|}$.

Proof. See [**S**].

By using a translation, without loss of generalities, we assume $x_0 = 0$ in (b.4). If (b.4) with $x_0 = 0$ holds, from Proposition 1.5, for any $L > 0$, there exists $R_0 > 0$ such that

$$\max_{(s,t)\in[-L,L]^2} I(\gamma_{R_0}(s,t)) < 2c_0.$$

To prove the Theorem 0.1, we also need a map $m : H^1(\mathbf{R})\setminus\{0\} \to \mathbf{R}$ which is defined by the following: for any $u \in H^1(\mathbf{R})\setminus\{0\}$, a function

$$T_u(s) = \int_{-\infty}^{\infty} \tan^{-1}(x-s)|u(x)|^2\,dx : \mathbf{R}\to\mathbf{R}$$

is strictly decreasing and $\lim_{s\to\infty} T_u(s) = -||u||^2_{L^2(\mathbf{R})} < 0$ and $\lim_{s\to-\infty} T_u(s) = ||u||^2_{L^2(\mathbf{R})} > 0$. Thus, from the theorem of the intermediate value, $T_u(s)$ has an unique $s = m(u)$ such that $T_u(m(u)) = 0$. We also find that $m(u)$ is of continuous by the implicit function theorem to $(u,s)\mapsto T_u(s)$. The map $m(u)$ was introduced in [**S**]. We remark that $m(u)$ is regarded as a kind of center of mass of $|u(x)|^2$ and we can check the followings.

Lemma 1.6. *We have*

(i) $m(\gamma_0(t)) = 0$ *for all* $t\in\mathbf{R}$.
(ii) $m(\gamma_R(s,t)) > 0$ *for all* $-R < s < t < R$.
(iii) $m(\gamma_R(s,t)) < 0$ *for all* $-R < t < s < R$.

Proof. Since $\gamma_0(t)(x)$ is a even function, we have (i). We Note that

$$\gamma_R(s,t)(x) = \begin{cases} \gamma_0(s)(x+R) & \text{for } x\in(-\infty,\frac{s-t}{2}], \\ \gamma_0(t)(x-R) & \text{for } x\in(\frac{s-t}{2},\infty). \end{cases}$$

Since $\gamma_R(s,s)(x)$ is also a even function, we have

$$m(\gamma_R(s,s)) = 0 \quad \text{for all} \quad s\in\mathbf{R},$$

and we get (ii)–(iii). ∎

In what follows, we will complete the proof of Theorem 0.1.

Proof of Theorem 0.1. First of all, we defined a minimax value $c_1 > 0$ by

$$c_1 = \inf_{\gamma\in\Gamma_1} \max_{t\in[0,1]} I(\gamma(t)),$$
$$\Gamma_1 = \{\gamma(t) \in C([0,1], H^1(\mathbf{R})) \,|\, \gamma(0) = 0,\ I(\gamma(1)) < 0,\ |m(\gamma(t))| < 1\}.$$

Noting $\Gamma_1 \subset \Gamma$, we have

$$0 < c \le c_1.$$

Since Γ_1 is not invariant by standard deformation flows of $I(u)$, c_1 may not be a critical point of $I(u)$. We will use c_1 to divide the case. We divide the case into the following three cases:

(i) $c_1 < c_0$.
(ii) $c_1 = c_0$.
(iii) $c_1 > c_0$.

Proof of Theorem 0.1 for the case (i). Since the inequality $c_1 < c_0$ implies $0 < c < c_0$, from Corollary 1.2, we can see $I(u)$ has at least a non-trivial critical point. ∎

Proof of Theorem 0.1 for the case (ii). In this case, if $c < c_1 = c_0$, then $I(u)$ has at least a non-trivial critical point from Corollary 1.2. Thus we may consider the case $c = c_1 = c_0$. In this case, for any $\epsilon > 0$, there exists $\gamma_\epsilon(t) \in \Gamma_1$ such that

$$c \le \max_{t\in[0,1]} I(\gamma_\epsilon(t)) < c + \epsilon.$$

Since $\gamma_\epsilon \in \Gamma_1 \subset \Gamma$ and Γ is an invariant set by standard deformation flows of $I(u)$, by a standard Ekland principle, there exists $u_\epsilon \in H^1(\mathbf{R})$ such that

$$c \le I(u_\epsilon) \le \max_{t\in[0,1]} I(\gamma_\epsilon(t)) < c + \epsilon,$$
$$||I'(u_\epsilon)|| < 2\sqrt{\epsilon},$$
$$\inf_{t\in[0,1]} ||u_\epsilon - \gamma_\epsilon(t)||_{H^1(\mathbf{R})} < \epsilon. \tag{1.2}$$

Then, from Proposition 1.1, there exist a subsequence $\epsilon_j \to 0$, $k \in \mathbf{N}\cup\{0\}$, k-sequences $(x_j^1)_{j=1}^\infty, \cdots, (x_j^k)_{j=1}^\infty \subset \mathbf{R}$, and a critical point u_0 of $I(u)$ such that

$$I(u_{\epsilon_j}) \to I(u_0) + kc_0 \quad (j \to \infty), \tag{1.3}$$
$$\left|\left| u_{\epsilon_j}(x) - u_0(x) - \sum_{\ell=1}^k \omega(x - x_j^\ell) \right|\right|_{H^1(\mathbf{R})} \to 0 \qquad (j \to \infty),$$
$$|x_j^\ell - x_j^{\ell'}| \to \infty \quad (j \to \infty) \quad (\ell \ne \ell'),$$
$$|x_j^\ell| \to \infty \quad (j \to \infty) \quad (\ell = 1, 2, \cdots, k).$$

Now, if $u_0 \neq 0$, our proof is completed. So we suppose $u_0 = 0$. Then, from (1.3), it must be $k = 1$. Thus, we have

$$\begin{aligned} &\left|\left|u_{\epsilon_j}(x) - \omega(x - x_j^1)\right|\right|_{H^1(\mathbf{R})} \to 0 \qquad (j \to \infty). \\ &|x_j^1| \to \infty \quad (j \to \infty). \end{aligned} \tag{1.4}$$

On the other hand, we remark that, since $m(\omega) = 0$ and m is of continuous, there exists $\delta > 0$ such that

$$|m(u)| < 1 \quad \text{for all} \quad u \in B_\delta(\omega) = \{v \in H^1(\mathbf{R}) \,|\, ||v - \omega||_{H^1(\mathbf{R})} < \delta\}.$$

Thus, from (1.2) and (1.4), for some $\epsilon_0 \in (0, \frac{\delta}{2})$ and $t_0 \in [0, 1]$, we have

$$|m(\gamma_{\epsilon_0}(t_0)) - x_j^1| < 1.$$

This contradicts to $\gamma_{\epsilon_0} \in \Gamma_1$. Therefore $u_0 \neq 0$ and $I(u)$ has at least a non-trivial critical point. ∎

Proof of the Theorem 0.1 for the case (iii). First of all, we set $\delta = \frac{c_1 - c_0}{2} > 0$ and choose $L_0 > 0$ such that

$$\max_{\overline{(s,t) \in D_{2L_0} \setminus D_{L_0}}} I(\gamma_R(s,t)) < c_0 + \delta < c_1 \quad \text{for all} \quad R > 3L_0. \tag{1.5}$$

Here we set $D_L = [L, L] \times [L, L] \subset \mathbf{R}^2$. Next, from Proposition 1.5, we can choose $R_0 > 3L_0$ such that

$$\max_{(s,t) \in D_{L_0}} I(\gamma_{R_0}(s,t)) < 2c_0. \tag{1.6}$$

Here we fix $\gamma_{R_0}(s,t)$ and define the following minimax value:

$$\begin{aligned} &c_2 = \inf_{\gamma \in \Gamma_2} \max_{(s,t) \in D_{2L_0}} I(\gamma(s,t)), \\ &\Gamma_2 = \{\gamma(s,t) \in C(D_{2L_0}, H^1(\mathbf{R})) \,|\, \gamma(s,t) = \gamma_{R_0}(s,t) \text{ for all } (s,t) \in D_{2L_0} \setminus D_{L_0}\}. \end{aligned}$$

Then we have the following lemma.

Lemma 1.7. *We have*

$$0 < c_0 < c_1 \leq c_2 < 2c_0.$$

We postpone the proof of Lemma 1.7 to end of this section. If Lemma 1.7 is true, then Γ_2 is an invariant set by the deformation flows of $I(u)$. Thus $I(u)$ has a (PS)-sequence $(u_n)_{n=1}^\infty$ such that

$$I(u_n) \to c_2 \in (c_0, 2c_0) \quad (n \to \infty).$$

From Corollary 1.2, we can see that $I(u)$ must have at least a non-trivial critical point. Combining the proofs of the cases (i)–(iii), we complete a proof of Theorem 0.1. ∎

Finally we show Lemma 1.7.

Proof of Lemma 1.7. The inequality $c_0 < c_1$ is an assumption of the case (iii). From $\gamma_{R_0} \in \Gamma_2$ and (1.5)–(1.6), $c_2 < 2c_0$ is obvious. Thus we show $c_1 \le c_2$. For any $\gamma(s,t) \in \Gamma_2$, we have

$$m(\gamma(s,t)) > 0 \quad \text{for all} \quad (s,t) \in D_1, \tag{1.7}$$

$$m(\gamma(s,t)) < 0 \quad \text{for all} \quad (s,t) \in D_2. \tag{1.8}$$

Here we set $D_1 = \{(s,t) \in D_{2L_0} \setminus D_{L_0} \,|\, s < t\}$ and $D_2 = \{(s,t) \in D_{2L_0} \setminus D_{L_0} \,|\, s > t\}$. From (1.7)–(1.8), a set $\{(s,t) \in D_{2L_0} \,|\, |m(\gamma(s,t))| < 1\}$ have a connected component which contains a path joining two points $\gamma_{R_0}(-2L_0, -2L_0)$ and $\gamma_{R_0}(2L_0, 2L_0)$. Thus we construct a path $\gamma_1(t) \in \Gamma_1$ such that

$$\{\gamma_1(t) \,|\, t \in [1/3, 2/3]\} \subset \{\gamma(s,t) \,|\, (s,t) \in D_{2L_0}\},$$

$$\max_{t \in [0,1/3] \cup [2/3,1]} I(\gamma_1(t)) \le c_0.$$

Thus we see

$$\begin{aligned} c_1 &\le \max_{t \in [0,1]} I(\gamma_1(t)) \\ &\le \max_{(s,t) \in D_{2L_0}} I(\gamma(s,t)). \end{aligned} \tag{1.9}$$

Since $\gamma(s,t) \in \Gamma_2$ is arbitrary, from (1.9), we have

$$c_1 \le c_2.$$

Thus we get Lemma 1.7. ∎

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