Classification of torus manifolds with codimension one extended actions

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Abstract. The purpose of this paper is to classify torus manifolds (M^{2n}, T^n) with codimension one extended G-actions (M^{2n}, G) up to essential isomorphism, where G is a compact, connected Lie group whose maximal torus is T^n. For technical reasons, we do not assume torus manifolds are orientable. We prove that there are seven types of such manifolds. As a corollary, if a non-singular toric variety or a quasitoric manifold has a codimension one extended action then such manifold is a complex projective bundle over a product of complex projective spaces.

1. Introduction

This paper is a continuation of [**Ku3**] devoted to find the natural symmetries of torus manifolds. A *torus manifold*, defined in [**HaMa, Ma**], is an even dimensional, oriented, compact, connected manifold M^{2n} acted on by a half-dimensional torus T^n with non-empty fixed point set. The class of torus manifolds provides a rich and interesting class of T-spaces, because this class contains both of non-singular toric varieties studied by algebraic geometers (see [**Fu, Od**]) and quasitoric manifolds studied by topologists (see [**BuPa, DaJa**]). As is well-known, the n-dimension torus is a maximal compact abelian group which acts on $2n$-dimensional manifolds effectively. On the other hand, there exist torus manifolds whose torus actions are induced from non-abelian group actions, e.g., complex projective spaces or even dimensional spheres (see [**Ku3**]). Namely, the T^n-action on torus manifold M^{2n} do not always become the maximal (compact) symmetry of M^{2n}.

One of fundamental problems in geometry is to find the most natural symmetry on the given space, i.e., the most natural group action on the given space. In order to find natural group actions on torus manifolds, we have studied extended actions of T^n-actions on torus manifolds. In [**Ku3**], we classify torus manifolds with transitive extended G-actions (also see Theorem 2.4 in this paper), where G is a compact, connected Lie group whose maximal torus is T^n. In this case, the principal orbit G/K is M itself. In other words, the codimension of principal orbit of transitive actions is zero, i.e., $\dim M - \dim G/K = 0$. Therefore, we may regard the classification in [**Ku3**] as the classification of torus manifolds induced from codimension zero extended actions. So we are naturally led to study torus manifolds induced from codimension one extended G-actions, i.e., *torus manifolds with codimension one extended actions* (or *torus manifolds induced from cohomogeneity one symmetries*). The purpose of this paper is to classify all such torus manifolds up to essential isomorphism. For technical reasons, we do not assume torus manifolds are orientable as we do in [**Ku3**]. Namely, we classify more general class of T-manifolds with codimension one extended actions.

Let us prepare to state our main theorem. We use the following notations: $\mathcal{S}(a,b) = \prod_{j=1}^{a} S^{2l_i+1} \times \prod_{j=1}^{b} S^{2m_j}$; $G' = \prod_{j=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1)$; the symbol V_α represents the representation space with the scaler representation α of $T^a \times \mathcal{A}$, where $\mathcal{A} \subset (\mathbb{Z}_2)^b \subset \prod_{j=1}^{b} O(2m_j+1)$ generated by diagonal matrices, i.e., if V is a complex (resp. real) space then

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$\alpha : T^a \times \mathcal{A} \to S^1$ (resp. $\alpha : T^a \times \mathcal{A} \to \mathbb{Z}_2$); the symbol $P(V \oplus W)$ (resp. $\mathbb{R}P(V \oplus W)$) represents the complex (resp. real) projective space of the complex (resp. real) vector space $V \oplus W$; and $S(V \oplus W)$ is the unit sphere in $V \oplus W$. The goal of this paper is to prove the following theorem (see Propositions 9.4–9.10, 10.3 and 11.5 for detail):

THEOREM 1.1. *Let (M, T) be a (possibly unoriented) torus manifold and (M, G) be its codimension one extended action, where G is a compact connected Lie group whose maximal torus is T^n. Then, (M, G) is essentially isomorphic to one of the followings:*

	M	G
(1)	$\mathcal{S}(a,b) \times_{T^a \times \mathcal{A}} P(\mathbb{C}^{k_1}_\rho \oplus \mathbb{C}^{k_2})$	$G' \times S(U(k_1) \times U(k_2))$ $(k_1 + k_2 \geq 3)$
(2)	$\mathcal{S}(a,b) \times_{T^a \times \mathcal{A}} S(\mathbb{C}^{k}_\rho \oplus \mathbb{R}_\epsilon)$	$G' \times U(k)$
(3)	$\mathcal{S}(a;b) \times_{T^a \times \mathcal{A}} S(\mathbb{R}^{2k}_\rho \oplus \mathbb{R}_\epsilon)$	$G' \times SO(2k)$ $(k \geq 2)$
(4)	$\mathcal{S}(a,b) \times_{T^a \times \mathcal{A}} S(\mathbb{C}^{k_1}_{\rho_1} \oplus \mathbb{R}^{2k_2-1}_{\rho_2})$	$G' \times U(k_1) \times SO(2k_2-1)$ $(k_2 \geq 2)$
(5)	$\mathcal{S}(a;b) \times_{T^a \times \mathcal{A}} S(\mathbb{R}^{2k_1}_{\rho_1} \oplus \mathbb{R}^{2k_2-1}_{\rho_2}))$	$G' \times SO(2k_1) \times SO(2k_2-1)$ $(k_2 \geq 2)$
(6)	$\mathcal{S}(a,b) \times_{T^a \times \mathcal{A}} \mathbb{R}P(\mathbb{C}^{k_1}_{\rho} \oplus \mathbb{R}^{2k_2-1})$	$G' \times U(k_1) \times SO(2k_2-1)$
(7)	$\mathcal{S}(a;b) \times_{T^a \times \mathcal{A}} \mathbb{R}P(\mathbb{R}^{2k_1}_{\rho} \oplus \mathbb{R}^{2k_2-1}))$	$G' \times SO(2k_1) \times SO(2k_2-1)$

for some subgroup $\mathcal{A} \subset (\mathbb{Z}_2)^b$ and scaler representations $\rho, \epsilon, \rho_1, \rho_2$. Here, $T^a \times \mathcal{A}$ acts on $\mathcal{S}(a;b) \subset \prod_{i=1}^a \mathbb{C}^{l_i+1} \times \prod_{j=1}^b \mathbb{R}^{2m_j+1}$ naturally.

Furthermore, the following statements hold:

- *the manifolds in (1) are orientable if and only if $\mathcal{A} \subset SO(\sum_{j=1}^b 2m_j + b)$;*
- *the manifolds in (2) and (3) are orientable if and only if $\{(a, \epsilon(a)) \in \prod_{j=1}^b O(2m_j+1) \times O(1) \mid a \in \mathcal{A}\} \subset SO(\sum_{j=1}^b 2m_j + b + 1)$;*
- *the manifolds in (4) and (5) are orientable if and only if $\{(a, \rho_2(a)) \in \prod_{j=1}^b O(2m_j + 1) \times O(2k_2-1) \mid a \in \mathcal{A}\} \subset SO(\sum_{j=1}^b 2m_j + b + 2k_2 - 1)$;*
- *the manifolds in (6) and (7) are non-orientable.*

By Theorem 1.1, we have the following corollary:

COROLLARY 1.2. *Let (M, G) be a non-singular toric variety or a quasitoric manifold with codimension one extended G-action. Then, (M, G) is essentially isomorphic to*

$$M = \prod_{i=1}^a S^{2l_i+1} \times_{T^a} P(\mathbb{C}^{k_1}_\rho \oplus \mathbb{C}^{k_2}) \cong \prod_{i=1}^a \mathbb{C}^{l_i+1}_o \times_{(\mathbb{C}^*)^a} P(\mathbb{C}^{k_1}_\rho \oplus \mathbb{C}^{k_2}),$$
$$G = \prod_{i=1}^a SU(l_i+1) \times S(U(k_1) \times U(k_2)),$$

where $\mathbb{C}^{l_i+1}_o = \mathbb{C}^{l_i+1} - \{o\}$ and $\mathbb{C}^* = \mathbb{C} - \{o\}$.

The organization of this paper and the method of classification are as follows. We first, in Section 2 and 3, recall some basic notions needed later and give some examples of torus manifolds with codimension one extended actions. In order to prove Theorem 1.1, we will combine the methods introduced by Alekseevskii-Alekseevskii in [**AlAl**] and Uchida in [**Uc**], and use the main results in [**Ku3**]. Due to [**AlAl**], if (M, G) has a codimension one orbit then M can be constructed from a *primitive* (M_1, G'') with codimension one orbits; roughly speaking, M is equivariantly diffeomorphic to the crossed product $G' \times_{H'} M_1$ for some G' and its subgroup H', where $G = G' \times G''$ and an H'-action on M_1 commutes with the G''-action on M_1. In Section 4, we recall the definition of the primitive G-manifolds introduced in [**AlAl**]. We also show that, for the non-primitive torus manifold $M \cong G' \times_{H'} M_1$, both of G'/H' and M_1 are also torus manifolds (Lemma 4.4). Note that in this case M is an M_1-bundle over G'/H'. The main theorem in [**Ku3**] tells us the possibilities of G'/H' (Theorem 4.5). We next, in Section 5, 6 and 7, classify primitive torus manifolds (M_1, G'') by using the method of [**Uc**] (also see [**Ku2**] for details of this method). As a result, we have that there exist seven types of primitive torus manifolds (Theorem 7.1). In order to construct $G' \times_{H'} M_1$ from homogeneous torus manifold G'/H' and primitive torus manifold (M_1, G''), we next, in Section 8, 9 and 10, analyze H'-actions on M_1 which commute with the given G''-action on M_1. Then, we get the classification table in Theorem 1.1 of torus manifolds with codimension one extended actions up to essential isomorphism. Finally, in Section 11, we give some relations with moment-angle manifolds introduced in [**BuPa, DaJa**] (also see [**BoMe**]) and prove the orientability of torus manifolds appearing in the table of Theorem 1.1.

2. Preliminary

In this section, we recall some basic notations and facts needed later. We refer the reader to the following papers and books for further details: [**HaMa, Ma**] for toric topology; [**MiTo**] for classical Lie theory; [**Br, Ka**] for transformation group theory; and the paper [**Ku3**].

2.1. General terminologies and notations. We first recall general terminologies and notations.

Throughout of this paper, the symbol T^n represents an n-dimensional, compact, abelian group, i.e., T^n is a product of n circles $(S^1)^n$, we call it an n-dimensional torus or a torus. The symbol (M, G) represents the space M with G-action. If it is needed to indicate the action explicitly, we shall write (M, G) as (M, G, φ) with the action φ. In this paper, we assume all G-actions are smooth.

The symbol G_x represents the isotropy subgroup of $x \in M$, $G(x)$ represents the orbit of x, and M/G represents the orbit space. We denote the set of fixed points of (M, G) by M^G. A maximal orbit in (M, G) is called a *principal orbit*. Let $G(x)$ be a non-principal orbit in (M, G). If the dimension of $G(x)$ is strictly less than that of principal orbits, $G(x)$ is called a *singular orbit*. Otherwise, $G(x)$ is called an *exceptional orbit* (see Example 3.10).

Two manifolds with group actions (M, G, φ) and (M', G', φ') are said to be *weakly equivariantly diffeomorphic* if there exist an isomorphism $\psi : G \to G'$ and a diffeomorphism $f : M \to M'$ such that $f(\varphi(g, x)) = \varphi'(\psi(g), f(x))$ for all $(g, x) \in G \times M$; if the isomorphism ψ is identity, then (M, G, φ) and (M', G', φ') are said to be *equivariantly diffeomorphic*.

We call $N = \cap_{x \in M} G_x$ the *kernel* of (M, G). A G-action (M, G) is said to be *almost effective* (resp. *effective*) if the kernel of (M, G) is finite (resp. identity). Let N be the kernel of (M, G). Then, the induced action $(M, G/N)$ is always effective, and we call it the *induced effective action of* (M, G). If two induced effective actions of (M, G) and (M', G') are weakly equivariantly diffeomorphic, then (M, G) and (M', G') are said to be *essentially isomorphic*.

Let $(X \times Y, G)$ be the diagonal G-manifold of (X, G) and (Y, G). We denote its orbit space by $X \times_G Y$. If G acts on X freely, i.e., $G_x = \{e\}$ for all $x \in X$ where $e \in G$ is the identity element, then we may regard $X \times_G Y$ as the Y-bundle over X/G, i.e., there exists the following fibration:

$$Y \longrightarrow X \times_G Y \longrightarrow X/G.$$

2.2. Torus manifold. Let us define a torus manifold.

Definition 2.1. Let M^{2n} be a smooth, $2n$-dimensional, compact manifold. We say (M^{2n}, T^n) a *torus manifold* if an n-dimensional (half dimensional) torus action on M^{2n} is almost effective and there exists a fixed point.

In this paper, a torus manifold (M^{2n}, T^n) is often denoted by (M, T) or M simply. By definition, a torus manifold satisfies that M^T is finite and its principal orbit is T^n itself.

A compact, connected, codimension two T-invariant submanifold of M without boundary is called *characteristic* if it is a connected component of the set fixed pointwise by a certain circle subgroup of T and contains at least one T-fixed point. There exist only finitely many characteristic submanifolds and they are orientable if M is orientable.

Remark 2.2. The concept of a torus manifold is an ultimate (topological) generalization of toric theory. However, in this paper we do not use this theory, i.e., we do not use a *multi-fan*. Hence, our definition of torus manifold becomes rather briefer than that in [**HaMa, Ma**]. For example, we do not need to assume an *omniorientation* of the torus manifold and characteristic submanifolds.

Furthermore, because we would like to classify torus manifolds with codimension one extended actions up to essential isomorphism, we assume a T-action on M is almost effective. For technical reasons, we do not assume M is orientable. Namely, torus manifold in this paper contains more general T-manifolds than those in [**HaMa, Ma**].

2.3. Facts from classical Lie theory and the previous paper. In this paper, we will classify (M, G) up to essential isomorphism. In this subsection, we recall the facts from classical Lie theory and the paper [**Ku3**].

For any compact, connected Lie group G, there exists a finite covering, homomorphism (see [**MiTo**, Section 5]):

$$\widetilde{G} = G_1 \times \cdots \times G_k \xrightarrow{c} G, \tag{1}$$

where G_i is a compact, (simply) connected, simple Lie group, or a torus, for $i = 1, \ldots, k$. Let N be the kernel of c. Then, N is a finite normal subgroup in $G_1 \times \cdots \times G_k$. Because c is a surjective homomorphism, we have

$$G \simeq (G_1 \times \cdots \times G_k)/N.$$

Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{G} \times M & & \\ {\scriptstyle c\times \mathrm{Id}}\downarrow & \searrow^{\widetilde{\varphi}} & \\ G \times M & \xrightarrow{\varphi} & M \end{array}$$

where $\mathrm{Id} : M \to M$ is the identity map. Namely, there exists the lift $(M, \widetilde{G}, \widetilde{\varphi})$ of (M, G, φ). Moreover, one can easily see that $(M, \widetilde{G})$ of (M, G) are essentially isomorphic.

A *rank* of G is the dimension of a maximal torus subgroup of G. As is well known, the following lemma holds for a maximal rank subgroup H^o of G (see [**MiTo**, Theorem 7.2]).

LEMMA 2.3. *Let G_i $(i = 1, \ldots, k)$ be compact, connected Lie groups and let G be their product. Assume H^o is a compact, connected, maximal rank subgroup in G. Then $H^o = H_1 \times \cdots \times H_k$, where H_i is a maximal rank subgroup in G_i.*

We next recall the results of the paper [**Ku3**]. Let (M, T, φ) be a torus manifold. Suppose T is a maximal torus subgroup of a compact, connected Lie group G. If there exists an action $\Phi : G \times M \to M$ such that the restricted T-action $\Phi|_{T\times M}$ is the given φ, then we call (M, G, Φ) an *extended G-action* of (M, T, φ).

Let (M, G) be an extended G-action of (M, T). If there is a principal G-orbit $G(x)$ such that $\dim G(x) = \dim M^{2n} - k = 2n - k$, then we call (M^{2n}, G) a *codimension k extended G-action* of (M, T), where an integer k satisfies $0 \le k \le n$. In particular, if a torus manifold (M, T) has a codimension 1 extended G-action, then we call (M, T) a *torus manifold with codimension one extended action* (or *torus manifold induced from cohomogeneity one action*).

Let $\mathbb{Z}_2$ be the subgroup

$$\{I_{2m_j+1},\ -I_{2m_j+1}\} \subset O(2m_j + 1),$$

where $O(m)$ is the orthogonal group and I_m is its identity element. Note that $\mathbb{Z}_2$ acts on the $2m_j$-dimensional sphere $S^{2m_j} \subset \mathbb{R}^{2m_j+1}$ canonically (we call this action the *antipodal action* on sphere). Let $\mathcal{A}$ be a subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$. Then $\mathcal{A}$ acts on $\prod_{j=1}^{b} S^{2m_j}$ through the canonical $\prod_{j=1}^{b} \mathbb{Z}_2$-action on $\prod_{j=1}^{b} S^{2m_j}$, i.e., the product of antipodal actions. For codimension 0 extended G-actions, we have the following classification results (see [**Ku3**, Theorem 1]):

THEOREM 2.4. *Let (M^{2n}, T^n) be a torus manifold, and G a compact, connected Lie group whose maximal torus is T^n. Suppose (M^{2n}, T^n) extends to a codimension 0 extended G-action. Then (M^{2n}, G) is essentially isomorphic to*

$$\left(\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \frac{\prod_{j=1}^{b} S^{2m_j}}{\mathcal{A}},\ \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1) \right),$$

where the above group acts on M^{2n} in the natural way, and $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n$.

REMARK 2.5. In [**Ku3**, Theorem 1], we used $PU(l+1)$ instead of $SU(l+1)$ as the transformation group, where $PU(l+1)$ is defined as the quotient of $SU(l+1)$ by its center $Z(SU(l+1))$. However, $(\prod \mathbb{C}P(l),\ \prod SU(l+1))$ is essentially isomorphic to $(\prod \mathbb{C}P(l),\ \prod PU(l+1))$ (see [**Ku3**, Example 2.7]). So we may change $PU(l+1)$'s into $SU(l+1)$'s.

3. Structure of orbit space M/G and orbits of fixed points M^T

Henceforth, (M, T) represents a $2n$-dimensional torus manifold and (M, G) represents its codimension 1 extended action, where G is a compact connected Lie group with maximal torus T.

In this section, we analyze the orbit space M/G of (M, G).

3.1. Structure of orbit space M/G. By the definition of torus manifold, there exist non-empty isolated fixed points M^T. We first consider a G-orbit on a fixed point $p \in M^T$. Because $p \in M^T$ is fixed by the T-action, we have

$$T \subset G_p \subset G. \tag{2}$$

Therefore, we have rank $G_p^o =$ rank $G = n$, where rank G represents the dimension of maximal torus of G. Hence, as is well known (see e.g. [**GHZ**, Theorem 1.1 (2), (3)]), the dimension of $G/(G_p)^o$ is even and

$$\dim G/G_p = \dim G/(G_p)^o.$$

It follows that there exists at least one singular orbit in (M, G). Hence, together with the fact from transformation group theory (see e.g. [**Br**, 8.2 Theorem in Chapter IV] or [**Uc**, Lemma 1.2.1]), we obtain the following lemma:

LEMMA 3.1. *Suppose that (M^{2n}, T^n) extends to (M, G) with codimension 1 orbits. Then, the orbit space M/G is homeomorphic to the interval $[-1, 1]$ such that orbits over the interior $(-1, 1)$ are principal orbits G/K and two orbits G/K_1 and G/K_2 over the boundary $\{-1, 1\}$ (respectively) are singular or exceptional. (We may assume G/K_1 is a singular orbit.)*

Furthermore, there exists a closed, invariant tubular neighborhood X_s (of G/K_s for $s = 1,\ 2$) such that

$$M = X_1 \cup X_2$$

and

$$X_1 \cap X_2 = \partial X_1 = \partial X_2 \cong G/K.$$

Figure 1 shows the structure of (M, G).

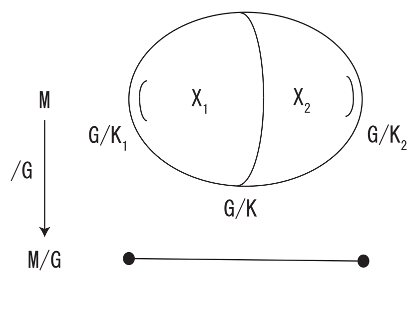

FIGURE 1. The orbit structure of $(M,\ G)$ with codimension 1 orbits.

Once we have the orbits G/K_1 and G/K_2 in Lemma 3.1, their tubular neighborhoods X_1 and X_2 can be computed by using the following *differentiable slice theorem*, or the *slice theorem* for short (see, e.g., [**Br, Ka**]).

Theorem 3.2 (differentiable slice theorem). *Let G be a compact Lie group and M a smooth G-manifold. Then, for all $x \in M$, there is a closed G-invariant neighborhood X of the orbit $G(x) \cong G/G_x$ such that $X \cong G \times_{G_x} D_x$ as a G-diffeomorphism. Here, the G_x-action on $G \times D_x$ is defined as follows: G_x canonically acts on G as a subgroup of G; and on a closed disk D_x through an orthogonal representation $\sigma : G_x \to O(D_x)$, where $O(D_x)$ is an orthogonal group of $D_x \subset \mathbb{R}^N$ ($N = \dim D_x = \dim M - \dim G(x)$).*

In Theorem 3.2, we call σ a *slice representation* of G_x. We identify a tubular neighborhood X of $G(x)$ with $G \times_{G_x} D_x$.

3.2. G-orbits of T-fixed points. Let $p \in M^T$. Using the slice theorem, the tangent space $T_p(M)$ can be regarded as an orthogonal T-representation space. We call it a *tangential representation space*, or simply a *tangential representation* on p. Let α_i be a representation from T to $S^1 \simeq SO(2)$, i.e., $\alpha_i : T \to S^1 \simeq SO(2) (\in Hom(T, S^1) \simeq \mathbb{Z}^n)$, and let $V(\alpha_i) \simeq \mathbb{R}^2$ be the irreducible representation space of α_i. The following lemma tells us the structure of tangential representations on fixed points in torus manifold (M, T).

Lemma 3.3. *Let (M, T) be a torus manifold and $p \in M^T$. Then, the tangential representation on p decomposes into 1-dimensional representations as follows:*

$$T_p(M) \simeq V(\alpha_1) \oplus \cdots \oplus V(\alpha_n)$$

such that $\{\alpha_1, \ldots, \alpha_n\}$ spans a space $\mathrm{Hom}(T, S^1) \otimes \mathbb{R} \simeq \mathbb{R}^n$.

Proof. According to the definition of torus manifold, the T-action on M is almost effective. It follows that there is a non-degenerate representation ρ from T^n to the orthogonal group $O(T_p(M)) \simeq O(2n)$, i.e., the image of ρ is also an n-dimensional torus. Moreover, the image of ρ is in the special orthogonal group $SO(2n)$ because T^n is connected. Therefore, the image of ρ and the diagonal maximal torus $SO(2) \times \cdots \times SO(2) \subset SO(2n)$ are conjugate in $SO(2n)$. This gives an equivalence between ρ and $\alpha_1 \oplus \cdots \oplus \alpha_n$ for some $\alpha_i : T^n \to SO(2)$. Moreover, $\{\alpha_1, \ldots, \alpha_n\}$ spans a space $\mathrm{Hom}(T, S^1) \otimes \mathbb{R} \simeq \mathbb{R}^n$ because $\rho(T^n) \subset SO(2n)$ is a maximal torus. □ □

The following lemma is one of the key lemmas to classify (M, G).

Lemma 3.4. *Let G/K_1 be a singular orbit of (M, G) which contains a fixed point of (M, T). Then there exists a subtorus $T' \subset T$ such that $(G/K_1, T')$ is a torus manifold.*

Proof. Let $c : \widetilde{G} \to G$ be the finite covering of G appearing in (1) in Section 2.2, and let $\widetilde{K}_1$ (resp. $\widetilde{T}$) be the identity component of $c^{-1}(K_1)$ (resp. $c^{-1}(T)$). Using (2) in Section 3.1, we also have $\widetilde{T}$ is a maximal torus subgroup of $\widetilde{G}$ and $\widetilde{K}_1$. By Lemma 2.3, there exists the following decomposition:

$$\widetilde{G} = G_1' \times G_1'',\ \widetilde{K}_1 = K_1' \times G_1'',\ \widetilde{T} = T_1' \times T_1'',$$

where G_1' and G_1'' are products of compact, simply connected, simple Lie groups and tori, and G_1'' is the same factor in $\widetilde{G}$ and $\widetilde{K}_1$, i.e., the identity component of the kernel of the $\widetilde{G}$-action on G/K_1. Note that rank $G_1' =$ rank $K_1' = \dim T_1'$ and rank $G_1'' = \dim T_1''$. Because $\widetilde{K}_1 \subset c^{-1}(K_1) \subset \widetilde{G}$, we also have the following decomposition:

$$c^{-1}(K_1) = H_1' \times G_1'',$$

where $K_1' \subset H_1' \subset G_1'$ and the identity component of H_1' is K_1'. Then, the projection c induces the diffeomorphism between G_1'/H_1' and G_1/K_1. Note that T_1'' is the identity component of the kernel of $\widetilde{T}$-action on G_1'/H_1'.

Let us prove that $(G_1'/H_1', T_1')$ is a torus manifold. Because T_1' is a maximal torus of G_1' and $K_1' = (H_1')^o$, as is well known, the T_1'-action on G_1'/H_1' is almost effective and there exist fixed points. Moreover, we have the following decomposition on the fixed point $p \in G/K_1 \cap M^T$:

$$T_pM = T_pG/K_1 \oplus N_pG/K_1,$$

where T_pG/K_1 is the tangent space and N_pG/K_1 is its normal space on p. It follows from Lemma 3.3 that there exists a decomposition

$$T_pM = V(\alpha_1) \oplus \cdots \oplus V(\alpha_n).$$

This implies that we may put

$$\begin{aligned} T_pG/K_1 &= V(\alpha_1) \oplus \cdots \oplus V(\alpha_{n-k_1}); \\ N_pG/K_1 &= V(\alpha_{n-k_1+1}) \oplus \cdots \oplus V(\alpha_n), \end{aligned}$$

for some $k_1 \in \mathbb{N}$. Because T_1'' is the connected component of the kernel of $\widetilde{T}$-action on $G_1'/H_1' \cong G/K_1$, the Lie algebra of T_1'' is spanned by $\{\alpha_{n-k_1+1} \oplus \cdots \oplus \alpha_n\}$. Therefore, we have

$$\dim N_pG/K_1 = 2k_1 = 2\dim T_1''.$$

Because $\dim T_1' + \dim T_1'' = n$, we also have

$$\dim G/K_1 = 2(n-k_1) = 2\dim T_1'.$$

Hence, $(G_1'/H_1', T_1') \cong (G/K_1, T_1')$ is a torus manifold. □ □

Remark 3.5. In Lemma 3.4, if (M,T) is an *oriented* torus manifold then G/K_1 is also oriented; moreover, G/K_1 is the connected component of the intersection of some characteristic submanifolds (see [**Ku4**, Lemma 3.2]).

3.3. Examples. In this subsection, we recall quasitoric manifolds briefly, and give some examples of torus manifolds with codimension one extended actions.

We first recall the definition of quasitoric manifold. Let P^n be a *simple* convex polytope, i.e., precisely n *facets* (codimension-1 faces) of P^n meet at each vertex.

Definition 3.6. If the torus manifold (M^{2n}, T^n) satisfies the following two properties:

(1) T^n-action is *locally standard*, i.e., locally looks like the standard torus representation in $\mathbb{C}^n$;
(2) there is a projection map $\pi : M^{2n} \to P^n$ constant on T^n-orbits which maps every k-dimensional orbit to a point in the interior of k-dimensional face of P^n, $k = 0, \ldots, n$,

then (M^{2n}, T^n) is said to be a *quasitoric manifold*.

Remark 3.7. One can easily show that (M,T) satisfies the condition (1) in the definition of the quasitoric manifolds if and only if $\{\alpha_1, \ldots, \alpha_n\}$ in Lemma 3.3 spans $\mathrm{Hom}(T, S^1) \simeq \mathbb{Z}^n$ for each fixed point.

Example 3.8 shows a quasitoric manifold with codimension one extended action.

Example 3.8. Let $(M,T) = (\mathbb{C}P(2), T^2)$ be the torus manifold defined by the standard multiplication of T^2 on the last two coordinates in $[z_0 : z_1 : z_2] \in \mathbb{C}P(2)$ (also see [**Ku3**, Example 2.2]). This torus manifold has an extended $G = PU(2) \times T^1$-action as follows:

- $PU(2) = U(2)/Z(U(2))$ acts on the first two coordinates $(z_0,\ z_1)$ by the standard multiplication, where $Z(U(2))$ is the center of $U(2)$;
- T^1 acts on the third coordinate z_2 by the standard multiplication.

Now we can easily check (M,T) is a quasitoric manifold (also see the left "triangle" in Figure 2), and (M,G) has codimension 1 orbits $G([1:0:1]) \cong \mathbb{C}P(1) \times S^1$ and two singular orbits $G([1:0:0]) \cong \mathbb{C}P(1)$ and $G([0:0:1]) \cong \{*\}$ (one point).

On the other hand, Example 3.9 is not a quasitoric manifold. However, this is a torus manifold with codimension one extended action.

Example 3.9. Let $(M,\ T) = (S^4,\ T^2)$ be the torus manifold defined by the standard multiplication of $T^2 = SO(2) \times SO(2)$ on $S^4 \cap \mathbb{R}^4$, where $S^4 \subset \mathbb{R}^4 \oplus \mathbb{R}$ (also see [**Ku3**, Example 2.3]). Now we can check (M,T) is not a quasitoric manifold because its orbit space is not a convex polytope (see the right "half-moon" in Figure 2, this half-moon is not a convex polytope).

Let $(x,y) \in S^4 \subset \mathbb{R}^2 \oplus \mathbb{R}^3$. This torus manifold has an extended $G = T^1 \times SO(3)$-action as follows:

- $T^1 \simeq SO(2)$ acts on $x \in \mathbb{R}^2$ standardly;
- $SO(3)$ also acts on $y \in \mathbb{R}^3$ standardly.

One can easily see that (M, G) has codimension 1 orbits $G(e_1, f_1) \cong S^1 \times S^2$ and two singular orbits $G(e_1,\ 0) \cong S^1$ and $G(0,\ f_1) \cong S^2$, where $e_1 = (1,\ 0) \in \mathbb{R}^2$ and $f_1 = (1,\ 0,\ 0) \in \mathbb{R}^3$.

Figure 2 shows the image of Examples 3.8 and 3.9.

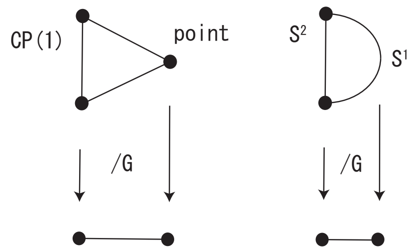

FIGURE 2. The left triangle shows the orbit space $\mathbb{C}P(2)/T^2$, the right half-moon also shows the orbit space S^4/T^2, and interval shows the orbit space of $\mathbb{C}P(2)/(PU(2) \times T^1)$ and $S^4/(T^1 \times SO(3))$.

We also give the following example which has an exceptional orbit:

EXAMPLE 3.10. Let (S^4, T^2) be a torus manifold defined in Example 3.9. Then, we may naturally define the product action of two copies $(S^4 \times S^4, T^2 \times T^2)$, and this is a torus manifold with 4 fixed points. If N and S denote the 2 fixed points in (S^4, T^2), then the 4 fixed points in $(S^4 \times S^4, T^2 \times T^2)$ can be denoted by (N, N), (N, S), (S, N) and (S, S).

Let $\mathbb{Z}_2$ be the group generated by $(-I_5, -I_5)$, where $-I_5$ is the antipodal involution on $S^4 \subset \mathbb{R}^5$ and I_5 is the identity map on $\mathbb{R}^5$. We note that $-I_5$ does not preserve an orientation on S^4; however, $(-I_5, -I_5)$ preserves an orientation on $S^4 \times S^4$. Now we may consider the following manifold

$$(S^4 \times S^4)/\mathbb{Z}_2 = S^4 \times_{\mathbb{Z}_2} S^4.$$

Since $(-I_5, -I_5)$ preserves an orientation of $S^4 \times S^4$ and $(-I_5, -I_5)$ commutes with $T^2 \times T^2$-action on $S^4 \times S^4$, we have that $S^4 \times_{\mathbb{Z}_2} S^4$ is an oriented manifold equipped with $T^2 \times T^2$-action induced from $(S^4 \times S^4, T^2 \times T^2)$. Moreover, there are 2 fixed points denoted by $[N : N] = [S : S]$ and $[N : S] = [S : N]$. Therefore, $(S^4 \times_{\mathbb{Z}_2} S^4, T^2 \times T^2)$ is an oriented torus manifold (also see Theorem 2.4).

This action extends to the canonical $G = SO(5) \times SO(4)$-action on $S^4 \times_{\mathbb{Z}_2} S^4$. Then we have the following three orbit types: $G([e_1 : e_1]) = (SO(5) \times SO(4))/(SO(4) \times SO(4)) = S^4$; $G([e_1 : e_2]) \cong (SO(5) \times SO(4))/(SO(4) \times SO(3) \times \mathbb{Z}_2) \cong S^4 \times_{\mathbb{Z}_2} S^3$; and $G([e_1 : e_1 + e_2]) = (SO(5) \times SO(4))/(SO(4) \times SO(3)) = S^4 \times S^3$. Here, $e_1, \ldots, e_5$ are the canonical basis of $\mathbb{R}^5$. Therefore, in this case there are one singular orbit S^4, principal orbits $S^4 \times S^3$, and the exceptional orbit $S^4 \times_{\mathbb{Z}_2} S^3$.

4. Crossed product of (M_1, G_1) by G/H and Primitive manifolds

In this section, we introduce a *primitive manifold*. This notion, which was first introduced by Alekseevskii-Alekseevskii in [**AlAl**], plays an important role in the classification of torus manifolds with extended actions. In this paper, we slightly modify the original definition in [**AlAl**].

In order to define it, we first define the following notion:

DEFINITION 4.1. Let M_1 be a compact connected manifold, G a Lie group, H a closed subgroup of G. Then, the G-manifold $M = G \times_H M_1$ is said to be a *crossed product* of M_1 by G/H, where H acts on M_1 by representation $\mu : H \to \mathrm{Diff}(M_1)$ such that $\ker \mu$ does not contain any normal subgroup of G.

Here, the symbol $\mathrm{Diff}(M_1)$ represents the set of all diffeomorphisms on M_1.

REMARK 4.2. If M is a crossed product of M_1 by G/H, then M is a fibre bundle whose base space is G/H and fibre is M_1 (see Section 2.1). Therefore, we may regard M_1 as a submanifold of M. Furthermore, if (M_1, H, μ) has a codimension one principal orbit H/J for some subgroup $J \subset H$, then (M, G) also has a codimension one principal orbit $G \times_H (H/J) \cong G/J$.

Now we may define a *primitive manifold.*

DEFINITION 4.3. A G-manifold (M, G) is said to be *primitive* if there is no submanifold $M_1(\neq \{*\})$ whose nontrivial crossed product by G/H for any $H \subset G$, i.e., $G \times_H M_1$ such that $\mu : H \to \mathrm{Diff}(M_1)$ is non-trivial, is G-diffeomorphic to M.

We call a torus manifold (M, T) with primitive extended G-action (M, G) a *primitive torus manifold* in this paper.

Let us prove the following 2^{nd} key lemma

LEMMA 4.4. *Let (M, T) be a non-primitive torus manifold, i.e., $M \cong G \times_H M_1$ for some non-trivial subgroup H of G, where H acts on M_1 via non-trivial $\mu : H \to \mathrm{Diff}(M_1)$. Then, there exists the decomposition $T \simeq T' \times T''$ such that $(G/H, T')$ and (M_1, T'') are torus manifolds, where $T'' \subset H$ acts on M_1 via μ.*

PROOF. We first prove that $T \subset H \subset G$. Let $\pi : M \to G/H$ be the projection. Because the projection π is a T-equivariant map, we have $\pi(M^T) \subset (G/H)^T$, i.e., there exists a fixed point in $(G/H, T)$. Therefore, there exists an element $gH \in G/H$ such that $TgH = gH$. It follows that $g^{-1}Tg \subset H$. Hence, we have that rank $G =$ rank H. In particular, by taking conjugation, we may assume $T \subset H \subset G$.

It follows from the method similar to that demonstrated in Section 2.2 that we may assume

$$\begin{aligned} G &= G' \times G'', \\ H &= H' \times G'' \end{aligned}$$

where G' is a product of connected, simple compact Lie groups, and H' is its maximal rank subgroup. Then, we may devide T into $T' \times T''$, where T' is a maximal torus of G' and T'' is that of G''. Because T' acts on $G/H \cong G'/H'$ almost effectively, we have $2\dim T' \leq \dim G/H$. On the other hand, T'' also acts on M_1 almost effectively, because T'' acts on G/H trivially and T acts on M almost effectively.

Asume $2\dim T' < \dim G/H$. Then, we have $2\dim T'' > \dim M_1$ because (M, T) is a torus manifold. However, this gives a contradiction to that T'' acts on M_1 almost effectively. Therefore, we have that

$$2\dim T' = \dim G/H \quad \text{and} \quad 2\dim T'' = \dim M_1.$$

Hence, $(G/H, T')$ is a torus manifold. If $M_1^{T''} = \emptyset$, then we can easily see that $M^T = \emptyset$. This gives a contradiction to that (M, T) is a torus manifold. Hence, (M_1, T'') is also a torus manifold. This establishes the statement. □ □

Using Remark 4.2 and Lemma 4.4 together with Theorem 2.4, we have the following theorem:

THEOREM 4.5. *Let (M, T) be a torus manifold with codimension one extended G-action. Assume (M, G) is not primitive. Then, there exist the following two manifolds: the torus submanifold (M_1, T'') with codimension one extended G''-action such that (M_1, G'') is primitive; and the homogeneous torus manifold $(G'/H', T')$, and (M, G) is essential isomorphic to*

$$\begin{aligned} &M \cong G \times_{(H' \times G'')} M_1, \\ &G = G' \times G'' \simeq \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1) \times G'', \end{aligned}$$

where G acts on the G-factor in M naturally and

$$H' = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S} \subset \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1)$$

where

$$\prod_{j=1}^{b} SO(2m_j) \subset \mathcal{S} \subset \prod_{j=1}^{b} S(O(2m_j) \times O(1)).$$

Here, in Theorem 4.5, the quotient space $G \times_{(H' \times G'')} M_1$ is defined by the following $(H' \times G'')$-actions: on G naturally; and on M_1 by the product of the G''-action on M_1 and an H'-action on M_1 defined by representation

$$\mu : H' \to \mathrm{Diff}_{G''}(M_1),$$

where $\mathrm{Diff}_{G''}(M_1)$ is the set of all G''-equivariant diffeomorphisms on M_1.

Now we note the following lemma:

LEMMA 4.6. *If (X, G) and (Y, H) are essentially isomorphic, then*

$$\mathrm{Diff}_G(X) \simeq \mathrm{Diff}_H(Y).$$

PROOF. One can easily check that if (X, G) and (Y, H) are weakly equivariantly diffeomorphic then $\mathrm{Diff}_G(X) \simeq \mathrm{Diff}_H(Y)$. Therefore, it is enough to show that $\mathrm{Diff}_G(X) = \mathrm{Diff}_{G/N}(X)$, where N is the kernel of (X, G, φ) (see Section 2.1).

Let $f \in \mathrm{Diff}_G(X)$. By definition, the following diagram is commute:

$$\begin{array}{ccc} G \times X & \xrightarrow{\varphi} & X \\ {\scriptstyle \mathrm{Id}\times f}\downarrow & & \downarrow{\scriptstyle f} \\ G \times X & \xrightarrow{\varphi} & X \end{array}$$

Let $\varphi_N : G/N \times X \to X$ be the induced effective action. By definition and the commutative diagram above, for $[g] \in G/N$, we have

$$f(\varphi_N([g], x)) = f(\varphi(g, x)) = \varphi(g, f(x)) = \varphi_N([g], f(x)).$$

It follows that $\mathrm{Diff}_G(X) \subset \mathrm{Diff}_{G/N}(X)$. On the other hand, $f \in \mathrm{Diff}_{G/N}(X)$ satisfies that

$$f(\varphi_N([g], x)) = \varphi_N([g], f(x)) = \varphi(g, f(x)).$$

It follows from $f(\varphi_N([g], x)) = f(\varphi(g, x))$ that $f \in \mathrm{Diff}_G(X)$, i.e., $\mathrm{Diff}_G(X) \supset \mathrm{Diff}_{G/N}(X)$. This establishes $\mathrm{Diff}_G(X) = \mathrm{Diff}_{G/N}(X)$. □ □

Due to Theorem 4.5 and Lemma 4.6, in order to classify (M, T) with codimension one extended actions, it is enough to classify the followings:

(1) primitive manifolds (M_1, G''), whose restricted maximal torus T''-action (M_1, T'') is a torus manifold, up to essential isomorphism;
(2) representations $\mu : H' \to \mathrm{Diff}_{G''}(M_1)$.

Henceforth, we call a torus manifold (M_1, T'') whose codimension one extended action (M_1, G'') is primitive a *primitive torus manifold*.

We first classify the primitive torus manifolds in Section 5 to 6. To ahieve this, we need to use the following key lemma:

LEMMA 4.7. *Let (M, G) be a codimension one extended action of torus manifold (M, T), and K_1, K_2 be non-principal isotropy subgroups. Suppose that there exists a proper subgroup H in G such that $K_1 \cup K_2 \subset H$. Then, there exists a submanifold M_1 with codimension one H-action and $M \cong G \times_H M_1$, i.e., (M, G) is not primitive.*

In other words, if (M, G) is a primitive manifold and there exists a subgroup $H \subset G$ satisfies $K_1 \cup K_2 \subset H$, then $H = G$.

PROOF. Using Lemma 3.1 and Theorem 3.2, there exists a decomposition $M \cong X_1 \cup X_2$ where $X_i \cong G \times_{K_1} D_i$. Because of the assumption that $H \neq G$ and $K_i \subset H$, we have $G \times_{K_1} D_i \cong G/H \times_H (H \times_{K_i} D_i)$. Put $M_1 = H \times_{K_1} D_1 \cup H \times_{K_2} D_2$. Then, we may regard $M_1 \subset M$. Moreover, M_1 has the cohomogeneity one H-action by restricting (M, G) to (M_1, H). Hence, we have $M \cong G \times_H M_1$. □ □

5. Isotropy subgroups (K_1, K) in (general) torus manifolds

In order to classify primitive torus manifolds (M_1, T''), in this section, we characterize (G, K_1, K) appearing in the general torus manifold (M, G) with codimension one extended G-actions (with possibly representations $K \subset K_1 \subset G$). Here, we assume $G/K_1 \cap M^T \neq \emptyset$.

5.1. Singular isotropy subgroup K_1. We first classify the pair (G, K_1) in the general case. At first, we prepare the following lemma needed later (also see [**Ku4**, Corollary 5.4]):

LEMMA 5.1. *The following two statements hold:*

- *if the connected subgroup H in $O(2l)$ acts on S^{2l-1} transitively and its rank is l, i.e.,* rank $H = l$*, then $H \simeq U(l)$ or $SO(2l)$ in $O(2l)$;*
- *if the connected subgroup H in $O(2l-1)$ acts on S^{2l-2} transitively, then $H \simeq SO(2l-1)$ or $H \simeq G_2$ and $l = 4$, where G_2 is the exceptional Lie group and $G_2/SU(3) \cong S^6$.*

PROOF. Using the classification results of transitive actions on sphere (e.g. see [**HsHs**, Section 1] or [**AlAl**, Table 1]), we can easily get the statement. □ □

By Lemma 3.4, we may assume that the orbit G/K_1 is a torus manifold. Therefore, we can put $\dim G/K_1 = 2n - 2k_1$, where $2n = \dim M$. Moreover, with the method similar to that demonstrated in Section 2.3, there is the following decomposition:

$$G = G_1' \times G_1'' \supset K_1 \supset K_1^o = (K_1')^o \times G_1'' \supset T = T_1' \times T_1'', \tag{3}$$

where G_1' and G_1'' are products of compact, connected, simple Lie groups and tori, and T_1' and T_1'' are their maximal tori, respectively. Using the decomposition (3), we also have

$$K_1 = K_1' \times G_1''.$$

Together with Theorem 2.4, there are the following identifications up to conjugation:

$$\begin{aligned} G_1' &= \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1); \\ (K_1')^o &= \prod_{i=1}^{a} S(U(1)\times U(l_i)) \times \prod_{j=1}^{b} SO(2m_j); \\ K_1' &= \prod_{i=1}^{a} S(U(1)\times U(l_i)) \times \mathcal{S}, \end{aligned} \tag{4}$$

where $\mathcal{S}$ is a subgroup which satisfies that

$$\prod_{j=1}^{b} SO(2m_j) \subset \mathcal{S} \subset \prod_{j=1}^{b} S(O(1)\times O(2m_j)).$$

REMARK 5.2. Note that $SO(3) \approx SU(2)$, i.e., locally isomorphic, and the covering map $SU(2) \to SO(3)$ preserves $S(U(1) \times U(1))$ to $SO(2)$. Therefore, we may regard $SU(2)$ as $SO(3)$ up to essential isomorphism. Namely, we may assume that $l_i \geq 2$, for all $i = 1, \ldots, a$, up to essential isomorphism in the identification (4) above.

To get G_1'', we analyze the slice representation of the tubular neighborhood $X_1 = G \times_{K_1} D^{2k_1}$ (see Theorem 3.2), where D^{2k_1} is the $2k_1$-dimensional disk. In our case, the slice representation can be denoted by the following homomorphism:

$$\sigma_1 : K_1 = K_1' \times G_1'' \to O(2k_1).$$

Due to the decomposition (3), G_1'' is in the kernel of the G-action on G/K_1. Hence, G_1'' acts on $D^{2k_1} \subset N_pG/K_1$ almost effectively via σ_1, because G acts on M almost effectively. Note that rank $G_1' =$ rank $K_1' = n - k_1$ and rank $G_1'' = k_1$, because G_1'/K_1' is a $(2n-2k_1)$-dimensional torus manifold. Therefore, we have that

$$\sigma_1(T_1'') = T^{k_1} \subset O(2k_1).$$

We also have that $\sigma_1(G''_1)$ acts on $\partial D^{2k_1} = S^{2k_1-1}$ transitively, because (M, G) has codimension 1 extended action. It follows from Lemma 5.1 that

$$G''_1 \approx \sigma_1(G''_1) \simeq U(k_1) \quad \text{or} \quad SO(2k_1).$$

Here, the symbol $X \approx Y$ represents that X and Y are locally isomorphic, i.e., the surjective homomorphism $\sigma_1 : G''_1 \to \sigma_1(G''_1)$ induces the isomorphism of Lie algebras. If $k_1 = 1$, then we may regard $SO(2)$ as $U(1)$. Hence, we have

$$G''_1 = SU(k_1) \times T^1$$

or

$$G''_1 = SO(2k_1) \text{ and } k_1 \geq 2,$$

up to essential isomorphism.

This establishes the following classification of all pairs of G and the singular isotropy subgroup K_1:

LEMMA 5.3. *Let (M, G) be a codimension one extended action of the torus manifold (M, T). Let G/K_1 be a singular orbit such that $G/K_1 \cap M^T \neq \emptyset$. Then, we may regard G and K_1 as $G = G'_1 \times G''_1$ and $K_1 = K'_1 \times G''_1$ such that*

$$\begin{aligned} G'_1 &= \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1), \\ K'_1 &= \prod_{i=1}^{a} S(U(1) \times U(l_i)) \times \mathcal{S}, \end{aligned}$$

and

$$G''_1 = SU(k_1) \times T^1 \quad \textit{or} \quad SO(2k_1) \text{ (and } k_1 \geq 2).$$

5.2. Property of principal isotropy subgroup K. In this subsection, we classify the principal isotropy subgroup K. Note that $\sigma_1^{-1}(O(2k_1-1)) = K$ because K_1 acts on $S^{2k_1-1} \cong K_1/K$ transitively via the slice representation σ_1. Therefore, we need to compute the slice representation σ_1. To do this, we first define the natural projections of $G = \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j + 1) \times G''_1$ as follows:

$$\begin{aligned} p_i &: G \to SU(l_i+1) \quad \text{for } i = 1, \ \ldots, \ a; \\ q_j &: G \to SO(2m_j+1) \quad \text{for } j = 1, \ \ldots, \ b. \end{aligned}$$

We also prepare the following notations for the sake of brevity. Put

$$\tau_I = \left(\begin{pmatrix} t_1 & 0 \\ 0 & A_1 \end{pmatrix}, \ \cdots, \ \begin{pmatrix} t_a & 0 \\ 0 & A_a \end{pmatrix} \right) \in \prod_{i=1}^{a} S(U(1) \times U(l_i)),$$

where $A_i \in U(l_i)$ and $\det A_i^{-1} = t_i$ for $i \in I = \{1, \ldots, a\}$. By changing the order of $\{1, \ \ldots, \ b\}$, we may regard the first part $J_1 = \{1, \ \ldots, \ b_1\}$ in $\{1, \ \ldots, \ b\}$ as the set satsfing that $m_j = 1$ and $q_j(K) = SO(2m_j) = SO(2)$. Now we may define the following two notations:

$$\begin{aligned} \nu_{J_1} &= (u_1, \ \ldots, \ u_{b_1}) \\ &\in \prod_{j=1}^{b_1} SO(2m_j) \subset \prod_{j=1}^{b_1} S(O(1) \times O(2m_j)) \end{aligned}$$

where $u_j \in SO(2m_j) = SO(2)$; and

$$\begin{aligned} \nu_{J_2} &= \left(\begin{pmatrix} x_{b_1+1} & 0 \\ 0 & X_{b_1+1} \end{pmatrix}, \ \cdots, \ \begin{pmatrix} x_b & 0 \\ 0 & X_b \end{pmatrix} \right) \\ &\in \mathcal{S} \subset \prod_{j=b_1+1}^{b} S(O(1) \times O(2m_j)), \end{aligned}$$

where $X_j \in O(2m_j)$ and $\det X_j = x_j$ for $j \in J_2 = \{b_1+1, \ldots, b\}$.

The following lemma tells us the σ_1-images of τ_I, ν_{J_1} and ν_{J_2}.

LEMMA 5.4. *The following two statements hold:*

- *if $G_1'' = SU(k_1) \times T^1$, then the following equations hold:*
$$\begin{aligned} \sigma_1(\tau_I) &= t_1^{r_1} \cdots t_a^{r_a} \in S^1; \\ \sigma_1(\nu_{J_1}) &= u_1^{s_1} \cdots u_{b_1}^{s_{b_1}} \in S^1; \\ \sigma_1(\nu_{J_2}) &\in \{\pm I_{2k_1}\} \subset O(2k_1), \end{aligned}$$
for some $r_i, s_j \in \mathbb{Z}$.
- *if $G_1'' = SO(2k_1)$, then the following equations hold:*
$$\begin{aligned} \sigma_1(\tau_I) &= I_{2k_1}; \\ \sigma_1(\nu_{J_1}) &= I_{2k_1}; \\ \sigma_1(\nu_{J_2}) &\in \{\pm I_{2k_1}\} \subset O(2k_1). \end{aligned}$$

PROOF. Because $\sigma_1(G_1'') = U(k_1)$ or $SO(2k_1)$ by Section 5.1 and K_1' is in the centralizer of G_1'' in K_1 by Lemma 5.3, there are the following relations:
$$\sigma_1(K_1') \subset Z_{O(2k_1)}(U(k_1)) = Z(U(k_1)) = S^1 \quad (\text{if } G_1'' = SU(k_1) \times T^1)$$
where S^1 is the center of $U(k_1)$, i.e., the diagonal subgroup whose all entries are the same; and
$$\sigma_1(K_1') \subset Z_{O(2k_1)}(SO(2k_1)) = Z(SO(2k_1)) = \{\pm I_{2k_1}\} \quad (\text{if } G_1'' = SO(2k_1))$$
where I_{2k_1} is the identity element of $O(2k_1)$, $Z_G(K)$ is the centralizer of K in G, and $Z(K)$ is the center of K. It follows from the above relations that one can easily check the statements for τ_I and ν_{J_1}.

We will check the statements for ν_{J_2}. If $G_1'' = SO(2k_1)$, then the statement for $\sigma_1(\nu_{J_2}) \in \{\pm I_{2k_1}\}$ is straightforward, because $\sigma_1(K_1') \subset Z(SO(2k_1)) = \{\pm I_{2k_1}\}$.

Assume $G_1'' = SU(k_1) \times T^1$. Because $\nu_{J_2} \in \mathcal{S}$ and $\mathcal{S}$ satisfies that
$$\prod_{j=b_1+1}^{b} SO(2m_j) = \mathcal{S}^o \subset \mathcal{S} \subset \prod_{j=b_1+1}^{b} S(O(1) \times O(2m_j)),$$
it is enough to prove $\sigma_1(\nu_{J_2}) = I_{2k_1}$ for $\nu_{J_2} \in \mathcal{S}^o$. Let $X_j \in S(O(1) \times O(2m_j))$ be the j^{th}-factor of $\nu_{J_2} \in \mathcal{S}^o$. If $m_j \geq 2$, then one can easily check $\sigma_1(X_j) \in \{\pm I_{2k_1}\} \subset U(k_1) \subset O(2k_1)$ because $\sigma_1(K_1') \subset S^1$.

Therefore, we may assume there exists $j \in \{b_1+1, \ldots, b\}$ such that $m_j = 1$. If $\mathcal{S} = \mathcal{S}^o$, then $m_j \geq 2$ for all $j = b_1+1, \ldots, b$ because of the definitions of J_1 and J_2. Hence, we may also assume $\mathcal{S}/\mathcal{S}^o \neq \{e\}$ when $m_j = 1$. Then, by the the definition of J_1 and J_2, the projection $q_j : S(O(1) \times O(2m_j)) \to S(O(1) \times O(2))$ is surjective. Let $\iota_j : S(O(1) \times O(2)) \to \mathcal{S}$ be an inclusion such that
$$\operatorname{Im} \iota_j \cap SO(2m_j) = SO(2m_j) = SO(2).$$
We will prove that this inclusion ι_j satisfies $\sigma_1 \circ \iota_j(S(O(1) \times O(2))) \subset \{\pm I_{2k_1}\} \subset U(k_1) \subset O(2k_1)$. Let
$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in O(2).$$
Using $J^2 = I_2$ and $\sigma_1(K_1') \subset S^1$, we have the following relation:
$$\sigma_1 \circ \iota_j \left(\begin{pmatrix} -1 & 0 \\ 0 & J \end{pmatrix} \right) \in \{\pm I_{2k_1}\} \subset U(k_1). \tag{5}$$
On the other hand, the following equation holds:
$$\sigma_1 \circ \iota_j \left(\begin{pmatrix} 1 & 0 \\ 0 & X_j \end{pmatrix} \right) = (X_j)^r \in S^1 \subset U(k_1) \subset O(2k_1)$$

for some $r \in \mathbb{Z}$ where $X_j \in SO(2)$, because $SO(2)$ is the abelian group. Hence, by the relation (5), we have that

$$\begin{aligned}
(X_j)^r &= \sigma_1 \circ \iota_j \left(\begin{pmatrix} 1 & 0 \\ 0 & X_j \end{pmatrix} \right) \\
&= \sigma_1 \circ \iota_j \left(\begin{pmatrix} -1 & 0 \\ 0 & J \end{pmatrix} \right) \sigma_1 \circ \iota_j \left(\begin{pmatrix} 1 & 0 \\ 0 & X_j \end{pmatrix} \right) \sigma_1 \circ \iota_j \left(\begin{pmatrix} -1 & 0 \\ 0 & J \end{pmatrix} \right) \\
&= \sigma_1 \circ \iota_j \left(\begin{pmatrix} -1 & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & X_j \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & J \end{pmatrix} \right) \\
&= \sigma_1 \circ \iota_j \left(\begin{pmatrix} 1 & 0 \\ 0 & (X_j)^{-1} \end{pmatrix} \right) \\
&= (X_j)^{-r}.
\end{aligned}$$

It follows that $r = 0$. This establishes $\sigma_1(\mathcal{S}^o) = \{I_{2k_1}\}$. Therefore, it follows from $\mathcal{S}/\mathcal{S}^o \subset (\mathbb{Z}_2)^{b-b_1}$ that we have $\sigma_1(\nu_{J_2}) \in \{\pm I_{2k_1}\}$. □ □

It follows from Lemma 5.4 and $\sigma_1^{-1}(O(2k_1 - 1)) = K$ that we have the following lemma:

LEMMA 5.5. *Fix the slice representation $\sigma_1 : K_1 \to O(2k_1)$. Then, the following two statements hold:*

- *if $G_1'' = SU(k_1) \times T^1$, then K is the following subgroup:*
$$\left\{ \left(\tau_I, \nu_{J_1}, \nu_{J_2} \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \;\middle|\; t_1^{r_1} \cdots t_a^{r_a} u_1^{s_1} \cdots u_{b_1}^{s_{b_1}} \sigma_1(\nu_{J_2}) z^{\gamma} = a^{-1} \right\},$$
where $A \in U(k_1 - 1)$ such that $\det A = a^{-1}$, $z \in T^1$ and $\gamma \in \mathbb{Z}$;
- *if $G_1'' = SO(2k_1)$, then K is the following subgroup:*
$$\left\{ \left(\tau_I, \nu_{J_1}, \nu_{J_2}, \begin{pmatrix} x & 0 \\ 0 & X \end{pmatrix} \right) \;\middle|\; \sigma_1(\nu_{J_2}) \in \{\pm I_{2k_1}\} \text{ such that } \sigma_1(\nu_{J_2}) x = +1 \right\},$$
where $X \in O(2k_1 - 1)$ such that $\det X = x$.

Here, we regard $\sigma_1(\nu_{J_2}) \in \{\pm I_{2k_1}\}$ as $\sigma_1(\nu_{J_2}) \in \{\pm 1\} \subset \mathbb{Z}$ in the relations above.

This establishes the classification of the principal isotropy subgroups.

6. Isotropy subgroups (G, K_1, K_2, K) in primitive torus manifolds

In this section, we characterize (G, K_1, K_2, K) (with possibly inclusions $K \subset K_s \subset G$ for $s = 1, 2$) appearing in primitive manifolds.

6.1. Preliminary. As a preliminary to characterizing such (G, K_1, K_2, K), we show the decomposition of K_2 in this subsection (see Lemma 6.1).

Let H be one of the following proper subgroups in G:

$$\begin{aligned}
H_k^U &= \prod_{i \in I(k)} SU(l_i + 1) \times S(U(1) \times U(l_k)) \times \prod_{j=1}^{b} SO(2m_j + 1) \times G_1''; \\
H_k^O &= \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j \in J(k)} SO(2m_j + 1) \times S(O(1) \times O(2m_k)) \times G_1'',
\end{aligned}$$

where $I(k) = \{1, \ldots, a\} \setminus \{k\}$ and $J(k) = \{1, \ldots, b\} \setminus \{k\}$. Due to the classification of K_1 in Section 5.1 (see Lemma 5.3), we have $K_1 \subset H$ for all k.

Henceforth, we take an isotropy type K_2 as a subgroup of G such that $K \subset K_1 \cap K_2$, where K is the subgroup appearing in Lemma 5.5. Assume the projections of K_2 satisfy one of the following relations for some i or j:

$$\begin{aligned}
p_i(K_2) &\subset S(U(1) \times U(l_i)); \\
q_j(K_2) &\subset S(O(1) \times O(2m_j)).
\end{aligned}$$

Then, we can easily check that $K_2 \subset H_i^U$ or $K_2 \subset H_j^O$. Because $K_1 \subset H_i^U \cap H_j^O$ for all i, j, we have that $K_1 \cup K_2 \subset H_i^U$ or H_j^O for some i or j. Therefore, by virtue of Lemma 4.7, we have that (M, G) is not primitive. Hence, it follows from Lemma 5.5 that

$$SU(l_i) \subset p_i(K) \subset S(U(1) \times U(l_i)) \subsetneq p_i(K_2) \subset SU(l_i + 1);$$
$$SO(2m_j) \subset q_j(K) \subset S(O(1) \times O(2m_j)) \subsetneq q_j(K_2) \subset SO(2m_j + 1),$$

for all $i = 1, \ldots, a$ and $j = 1, \ldots, b$. As is well known, subgroups above are $p_i(K_2) = SU(l_i + 1)$ for $l_i \geq 2$ (see Remark 5.2) and $q_j(K_2) = SO(2m_j + 1)$. Hence, the natural projection $p' : G \to G_1'$ satisfies that

$$p'(K_2) = G_1' = \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1).$$

Now we may prove the following lemma:

Lemma 6.1. *If (M_1, G) is an extended action of a primitive torus manifold and G decomposes into $G_1' \times G_1''$ appearing in Lemma 5.3, then there exists the following decomposition:*

$$K_2 = G_1' \times K_2'',$$

where K_2'' is the image of K_2 by the natural projection $p'' : G \to G_1''$. Furthermore, we have that p'' induces the isomorphism $G/K_2 \cong G_1''/K_2''$.

Proof. By using $p'(K_2) = G_1'$, we have that the surjective map

$$G/K_2 \longrightarrow G_1''/K_2''$$

induced from p'' is isomorphism. Moreover, $K_2(\subset G_1' \times K_2'')$ satisfies that

$$\dim K_2 = \dim G_1' \times K_2''.$$

This implies that $(G_1' \times \{e\}) \cap K_2 = H \times \{e\}$ is a maximal rank subgroup of $G_1' \times \{e\}$ such that G_1'/H is finite. Because G_1' is a product of $SU(l_i + 1)$ and $SO(2m_j + 1)$, i.e., product of simple Lie groups, by using the arguments demonstrated in Section 2, we have $H = G_1'$. Therefore, $G_1' \times \{e\} \subset K_2$. This implies $K_2 = G_1' \times K_2''$. □ □

Now there are the following two cases:

- G/K_2 contains a T-fixed point, i.e., $G/K_2 \cap M^T \neq \emptyset$;
- otherwise, i.e., $G/K_2 \cap M^T = \emptyset$.

From the next subsection, we will analyze each case.

6.2. The case when $G/K_2 \cap M_1^T \neq \emptyset$. Assume G/K_2 contains a T-fixed point. Similarly to the case of G/K_1, we have that G/K_2 is a torus manifold. Note that $G/K_2 \cong G_1''/K_2''$ by Lemma 6.1. Therefore, we can put $\dim G/K_2 = 2n - 2k_2$ for some $k_2 \in \mathbb{N}$. Now there are two cases: $G_1'' = SU(k_1) \times T^1$ and $SO(2k_1)$ $(k_1 \geq 2)$ by Section 5.1.

6.2.1. *The case when $G_1'' = SU(k_1) \times T^1$.* Suppose $G_1'' = SU(k_1) \times T^1$. Because G_1''/K_2'' is a torus manifold, it follows from Lemma 5.5 and $K \subset K_2$ that we may put

$$K_2'' = S(U(1) \times U(k_1 - 1)) \times T^1 \ (k_2 \geq 1)$$

or

$$K_2'' = G_1'' = SU(k_1) \times T^1 \ (k_2 = n).$$

Assume $K_2'' = S(U(1) \times U(k_1 - 1)) \times T^1$. In this case, the kernel of the G-action on $G/K_2 \cong G_1''/K_2''$ contains $G_1' \times T^1$. With the method similar to that demonstrated in Section 5.1, the kernel of the G-action on G/K_2 acts on $K_2/K \cong S^{2k_2-1}$ transitively and almost effectively via the slice representation $\sigma_2 : K_2 = G_1' \times K_2'' \to O(2k_2)$. Together with Lemma 5.1, we have that

$$\sigma_2(G_1' \times T^1) = U(k_2).$$

Moreover, we have that $\ker \sigma_2 \cap (G_1' \times T^1)$ is finite, because the G-action on M_1 is almost effective. Therefore, we may assume $\sigma_2(\{e\} \times T^1) = S^1$, where $\{e\}$ is the identity element in G_1'' and S^1 is the center of $U(k_2)$. Namely, σ_2 induces the representation $\sigma_2' : G_1' \to PU(k_2)$ such that $\ker \sigma_2'$

is finite, where $PU(k_2) = U(k_2)/S^1$. Note that G'_1 is a product of Lie groups. Let us recall the following well-known lemma ([**MoSa**, Theorem I']):

Lemma 6.2. *Let X and Y be two compact connected Lie groups and let $G = X \times_N Y$ where N is a finite normal subgroup of $X \times Y$. If G acts transitively on the n-dimensional sphere S^n then one of the two subgroups of G corresponding to X and Y acts transitively on S^n.*

Due to Lemma 6.2, there is the factor in G'_1 such that the restriction of σ'_2 to this factor induces the surjective homomorphism onto $PU(k_2)$. Together with the fact that $\ker \sigma'_2$ is finite, we have $G'_1 = SU(l_1+1)$, $SO(2m_1+1)$ or $\{e\}$ with $k_2 = 1$. Recall that if $Spin(2m+1) \simeq SU(m+1)$ then $m = 1$ and $Spin(3) \simeq SU(2)$ (see e.g. [**MiTo**]). Therefore, we may put

$$G'_1 = SU(l_1+1) \quad \text{and} \quad k_2 = l_1 + 1 (\geq 1).$$

Here, we note the following two facts: if $k_2 = 1$ then $l_1 = 0$; if $k_2 = 2$ then $G'_1 = SO(3)$, i.e., $m_1 = 1$ but this case can be regarded as $G'_1 = SU(2)$ up to essential isomorphism by Remark 5.2. This establishes that

$$\begin{aligned} G &= SU(k_2) \times SU(k_1) \times T^1, \\ K_1 &= S(U(1) \times U(k_2-1)) \times SU(k_1) \times T^1, \\ K_2 &= SU(k_2) \times S(U(1) \times U(k_1-1)) \times T^1, \end{aligned}$$

and $k_1 + k_2 - 1 = n$.

Assume $K''_2 = G''_1 = SU(k_1) \times T^1$. With the method similar to that demonstareted as above, we have

$$\sigma_2(G'_1 \times G''_1) = U(k_2) = U(n)$$

such that $\ker \sigma_2$ is finite. Similarly to the case above, $\sigma_2(\{e\} \times T^1) = S^1$ and there are the following two cases:

$$\sigma'_2(G'_1) = PU(n);$$

and

$$\sigma'_2(SU(k_1)) = PU(n),$$

where $\sigma'_2 : G'_1 \times SU(k_1) \to PU(n)$ is the induced representation. Note that $\ker \sigma'_2$ is finite in the both cases. Therefore, if $\sigma'_2(G'_1) = PU(n)$ then we have $k_1 = 1$ and $l_1 + 1 = n$; and if $\sigma'_2(SU(k_1)) = PU(n)$ then we have $k_1 = n$ and $G'_1 = \{e\}$. This establishes that

$$\begin{aligned} G = K_2 &= SU(n) \times T^1, \\ K_1 &= S(U(1) \times U(n-1)) \times T^1 \text{ and } k_1 = 1,\ k_2 = n \end{aligned}$$

or

$$G = K_2 = K_1 = SU(n) \times T^1 \text{ and } k_1 = k_2 = n.$$

Note that when $G = K_2 = SU(n) \times T^1$ and $K_1 = S(U(1) \times U(n-1)) \times T^1$, we may regard this case as the case when $G = SU(k_2) \times SU(k_1) \times T^1$, $K_1 = S(U(1) \times U(k_2-1)) \times SU(k_1) \times T^1$ and $K_2 = SU(k_2) \times S(U(1) \times U(k_1-1)) \times T^1$ with $k_1 = 1$, $k_2 = n$.

6.2.2. *The case when $G''_1 = SO(2k_1)$.* Suppose that $G''_1 = SO(2k_1)$ ($k_1 \geq 2$). Similarly to the case when $G''_1 = SU(k_1) \times T^1$, we have that

$$K''_2 = G''_1 = SO(2k_1)$$

and

$$\sigma_2(G'_1 \times SO(2k_1)) = SO(2k_2)$$

and $\ker \sigma_2$ is finite. Therefore, by using $k_1 \geq 2$ and Lemma 6.2, we have that $k_1 = k_2$ and $G'_1 = \{e\}$. Note that $n = k_2$ because $G/K_2 = \{*\}$. This establishes that

$$G = K_1 = K_2 = SO(2n) \text{ and } k_1 = k_2 = n.$$

Consequently, we have the following proposition:

Proposition 6.3. *Suppose that* (M_1^{2n}, G) *is a primitive torus manifold and* G/K_2 *contains a* T*-fixed point. Then, there are the following three cases:*

(1) $G = SU(k_1) \times SU(k_2) \times T^1$, $K_1 = SU(k_1) \times S(U(1) \times U(k_2 - 1)) \times T^1$, $K_2 = S(U(1) \times U(k_1 - 1)) \times SU(k_2) \times T^1$ *and*

$$K = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & B \end{pmatrix}, a^{-1}b^{-1} \right) \Big| \ \det A = a^{-1}, \det B = b^{-1} \right\}$$

where $A \in U(k_1 - 1)$, $B \in U(k_2 - 1)$ *and* $k_1 + k_2 - 1 = n$*;*

(2) $G = K_1 = K_2 = SU(n) \times T^1$ *and*

$$K = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, a^{-1} \right) \Big| \ \det A = a^{-1} \right\}$$

where $A \in U(n-1)$*;*

(3) $G = K_1 = K_2 = SO(2n)$ *and* $K = SO(2n-1)$.

Proof. By the the argument before this proposition, we have the three possibilities of (G, K_1, K_2) appearing in the statement. So, it is enough to show the principal isotropy subgroups K in each case.

For the 3rd case of (G, K_1, K_2), by using Lemma 5.5, it is straightforward to get K.

For the 1st case of (G, K_1, K_2), by using Lemma 5.5, we have

$$K = \left\{ \left(\begin{pmatrix} t_1 & 0 \\ 0 & A_1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \Big| \ t_1^{r_1} z^{\gamma} = a^{-1} \right\},$$

where $A_1 \in U(k_2 - 1)$ such that $\det A_1 = t_1^{-1}$, $A \in U(k_1 - 1)$ such that $\det A = a^{-1}$, $z \in T^1$ and $r_1, \gamma \in \mathbb{Z}$.

Then, the kernel of the G-action on M_1 contains the subgroup $\{e\} \times \mathbb{Z}_{|\gamma|}$, where $\{e\} \subset SU(k_2) \times SU(k_1)$ and $\mathbb{Z}_{|\gamma|}$ is the cyclic group of order $|\gamma| \geq 1$ or $\mathbb{Z}_0 = T^1$ for $\gamma = 0$. Because G acts on M_1 almost effectively, the case where $\gamma = 0$ does not occur. Moreover, it is easy to check that all the cases where $\gamma \neq 0$ are essentially isomorphic. Hence, we may regard $\gamma = 1$ up to essential isomorphism.

Moreover, in this case, we can also use Lemma 5.5 by interchanging the role of K_1 and K_2. Therefore, by using the arguments above again, we also have

$$K = \left\{ \left(\begin{pmatrix} t_1 & 0 \\ 0 & A_1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \Big| \ a^{r_2} z = t_1^{-1} \right\},$$

for some $r_2 \in \mathbb{Z}$. Hence, we can easily get $r_1 = r_2 = 1$ by using the two K's above. This establishes the 1st case of the statement. Similarly, we can show the 2nd case. □ □

6.3. The case when $G/K_2 \cap M_1^T = \emptyset$, I: preparations. We next assume $G/K_2 \cap M_1^T = \emptyset$. Then, $T \not\subset K_2$, i.e., rank $K_2 <$ rank $G = n$. Because we have rank $K = n - 1$ by virtue of Lemma 5.5, we also have

$$\text{rank } K_2 = n - 1 = \text{rank } K.$$

Therefore, we can put $K_2/K \cong S^{2k_2-2}$, i.e., the $(2k_2 - 2)$-dimensional sphere, and $\dim G/K_2 = 2n - 2k_2 + 1$ for $k_2 \geq 1$.

Recall $K_2 = G_1' \times K_2''$ such that $G/K_2 \cong G_1''/K_2''$ by Lemma 6.1. Moreover, $G_1' = \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1)$ and $K_2'' \subset SU(k_1) \times T^1$ or $SO(2k_1)(= G_1'')$. Therefore,

$$\text{rank } K_2'' = k_1 - 1.$$

We also have that $G_1' \times K_2''$ acts on $K_2/K \cong S^{2k_2-2}$ transitively via $\sigma_2 : K_2 = G_1' \times K_2'' \to O(2k_2 - 1)$.

Using Lemma 6.2, there are the following two cases:

- one of the factors in G_1' acts transitively on S^{2k_2-2};
- K_2'' acts transitively on S^{2k_2-2}.

Moreover, using Lemma 5.1, we have that $\sigma_2(K_2^o) = SO(2k_2 - 1)$ or $\sigma_2(K_2^o) = G_2$ and $k_2 = 4$, where K_2^o is the identity component and G_2 is the exceptional Lie group. The purpose of this subsection is to prove the following lemma:

LEMMA 6.4. *If $G/K_2 \cap M_1^T = \emptyset$, then*

$$\sigma_2(K_2^o) = SO(2k_2 - 1),$$

for some $k_2 \geq 1$.

PROOF. Assume $\sigma_2(K_2^o) = G_2$. Note that G_2 is the simply connected, simple Lie group and G_1' is a product of simple Lie groups or $\{e\}$. Because there are no factors in G_1' which is locally isomorphic to G_2, we have that $\sigma_2((K_2'')^o) = G_2$. Then, we also see that the covering group $\mathcal{K}_2 = \widetilde{(K_2'')^o}$ in Section 2.3 of $(K_2'')^o$ has a G_2-factor. Let $\mathcal{K}_2 = G_2 \times X$, where X is a product of simply connected simple Lie groups and tori. Now there are the following two cases: $K_2'' \subset SU(k_1) \times T^1$ or $SO(2k_1)$.

If $K_2'' \subset SU(k_1) \times T^1$, then $(K_2'')^o$ contains the following group as a maximal rank subgroup by Lemma 5.5:

$$K'' = \left\{ \left(\left(\begin{array}{cc} a & 0 \\ 0 & A \end{array} \right), z \right) \in S(U(1) \times U(k_1 - 1)) \times T^1 \,\middle|\, az^\gamma = 1 \right\},$$

where we can take γ as a non-zero integer because rank $K_2'' = k_1 - 1$. Note that the covering group $\widetilde{K''}$ in Section 2.3 of K'' is $SU(k_1 - 1) \times T^1$. Therefore, by Lemma 2.3, we have that $SU(k_1 - 1) \times T^1 \subset \mathcal{K}_2 = G_2 \times X$. Hence, in this case, we have $k_1 = 4$ and $X = T^1$.

Let $p : G \to SU(k_1) = SU(4) \subset G_1'' = SU(4) \times T^1$ be the natural projection. Recall the covering projection $c : \mathcal{K}_2 \to (K_2'')^o$ appearing in Section 2.3. Then, we have that

$$p(c(SU(3))) \subset p(c(G_2)) \subset p((K_2'')^o) \subset p(G) = SU(4).$$

Using Lemma 5.5 (also see K'' above), we also have

$$p(c(SU(3))) = SU(3) \subset p(c(G_2)) \subset SU(4).$$

Therefore, $p(c(G_2))$ is a non-trivial subgroup in $SU(4)$. Since G_2 is the simple Lie group, we also have that

$$\dim p(c(G_2)) = \dim G_2 = 14.$$

It follows that there exists a subgroup $H \subset SU(4)$ such that $\dim H = 14$. However, this also implies that there exists $H \subset SU(4)$ such that $SU(4)/H \cong S^1$, because $SU(4)$ is compact and $\dim SU(4) = 15$. As is well known, $SU(4)$ can not act on S^1 non-trivially (see e.g. [**Ku4**, Theorem 5.2, 5.3]). Therefore, this gives a contradiction.

If $K_2'' \subset SO(2k_1)$, then $SO(2k_1 - 1) \subset K_2''$ by virtue of Lemma 5.5. Therefore, by Lemma 2.3, we have that $Spin(2k_1 - 1) \subset \mathcal{K}_2 = G_2 \times X$. Note that $SO(2k_1 - 1) \subset (K_2'')^o$ is a maximal rank subgroup because rank $(K_2'')^o = k_1 - 1$, i.e., $Spin(2k_1 - 1) \subset \mathcal{K}_2 = G_2 \times X$ is a maximal rank subgroup. Using Lemma 2.3 again, it is easy to check that

$$Spin(2k_1 - 1) \subset G_2$$

and

$$X = \{e\}.$$

Hence, with the method similar to that demonstrated in the case $K_2'' \subset SU(k_1) \times T^1$, we have that $k_1 = 2$ and $K_2^o = G_1' \times c(G_2)$. This implies that there exists a subgroup $H \subset SO(4) = G_1''$ such that $c(G_2) = H$; however, this gives a contradiction because $\dim H = \dim c(G_2) = 14$ and $\dim SO(4) = 6$.

This establishes the statement of this lemma. □ □

In order to classify the case when $G/K_2 \cap M_1^T = \emptyset$, we will decompose into the followng two cases:

- G/K_2 is an exceptional orbit;
- G/K_2 is a singular orbit.

Before we will analyze for each case above, we remark the following:

Remark 6.5. Because $G/K_2 \cong G_1''/K_2''$, we see that G_1' is in the kernel of G-action on G/K_2. Together with the assumption that $G = G_1' \times G_1''$ acts on M_1 almost effectively, this implies that G_1' acts on S^{2k_2-2} almost effectively via $\sigma_2 : K_2 = G_1' \times K_2'' \to O(2k_2-1)$. Namely, $\ker\sigma_2 \cap G_1'$ is a finite normal subgroup of G_1'.

6.4. The case when $G/K_2 \cap M_1^T = \emptyset$, II: G/K_2 is an exceptional orbit. Assume $k_2 = 1$, i.e., G/K_2 is an exceptional orbit. Then, we have the following proposition:

Proposition 6.6. *Suppose that (M_1, G) is a primitive torus manifold and G/K_2 does not contain a T-fixed point. If $k_2 = 1$, then there are the following two cases:*

(1) $G = K_1 = SU(n) \times T^1$,

$$K_2 = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \in S(U(1) \times U(n-1)) \times T^1 \,\middle|\, az = \pm 1 \right\}$$

and

$$K = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \in S(U(1) \times U(n-1)) \times T^1 \,\middle|\, az = 1 \right\}.$$

(2) $G = K_1 = SO(2n)$, $K_2 = S(O(1) \times O(2n-1))$ *and* $K = SO(2n-1)$.

Proof. Because G_1' is a product of connected, simple Lie groups and $\ker\sigma_2 \cap G_1'$ is finite (see Remark 6.5) for $\sigma_2 : K_2 = G_1' \times K_2'' \to O(2k_2-1) = O(1) \simeq \mathbb{Z}_2$, we have $G_1' = \{e\}$, i.e.,

$$G = G_1'' = K_1.$$

Therefore, we have $k_1 = n$. Moreover, $K_2 = K_2''$ and $\sigma_2(K_2'') = O(1) \simeq \mathbb{Z}_2$. This implies that $\ker\sigma_2 = K$ and $K_2''/K \simeq \mathbb{Z}_2$. By Lemma 5.3, there are the following two cases:

$$G = G_1'' = SU(n) \times T^1$$

or

$$G = G_1'' = SO(2n).$$

Assume $G = K_1 = SU(n) \times T^1$. Using Lemma 5.5, we have that

$$K = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \in S(U(1) \times U(n-1)) \times T^1 \,\middle|\, az^{\gamma} = 1 \right\},$$

where we can take γ as a non-zero integer because rank $K = n-1$. Moreover, we have that

$$K \subset K_2 \subset N_G(K) = S(U(1) \times U(n-1)) \times T^1,$$

where $N_G(K)$ is the normalizer of K in G. We denote an element in $N_G(K) = S(U(1) \times U(n-1)) \times T^1$ by (a, z) for the sake of brevity, i.e., $K = \{(a, z) \mid az^{\gamma} = 1\}$. Define the representation $\alpha : S(U(1) \times U(n-1)) \times T^1 \to S^1$ by

$$\alpha(a, z) = az^{\gamma}.$$

Then, by definition, $\ker\alpha = K$. Therefore, together with $K_2/K \simeq \mathbb{Z}_2$, we have that $K \subsetneq K_2 \subset \alpha^{-1}(\{\pm 1\})$. It follows that

$$K_2 = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \in S(U(1) \times U(n-1)) \times T^1 \,\middle|\, az^{\gamma} = \pm 1 \right\}.$$

It is easy to see that we may regard $\gamma = 1$ up to essential isomorphism. This establishes the 1st case in the statement.

Assume $G = K_1 = SO(2n)$. Then, by Lemma 5.5, we have that $K = SO(2n-1)$. Because $K \subset N_G(K) = S(O(1) \times O(2n-1))$ and $K_2/K \simeq \mathbb{Z}_2$, we have that $K_2 = S(O(1) \times O(2n-1))$. This establishes the 2nd case in the statement. □ □

6.5. The case when $G/K_2 \cap M_1^T = \emptyset$, III: G/K_2 is a singular orbit. Assume $k_2 > 1$, i.e., G/K_2 is a singular orbit. Because $K_2 = G_1' \times K_2''$, there are the following two cases by using Lemma 6.2 and 6.4:

- $\sigma_2(G_1') = SO(2k_2 - 1)$; or
- $\sigma_2((K_2'')^o) = SO(2k_2 - 1)$.

Suppose that $\sigma_2((K_2'')^o) = SO(2k_2 - 1)$. We first prove this case does not occur (see Lemma 6.7).

Because $\sigma_2(G_1') \subset Z(\sigma_2((K_2'')^o)) = Z(SO(2k_2 - 1)) = \{\pm 1\}$ and G_1' is connected, we have $G_1' \subset \ker \sigma_2$. Because $\ker \sigma_2 \cap G_1'$ is finite (see Remark 6.5), we see that $G_1' = \{e\}$. Therefore, using Lemma 5.3, we have

$$G = K_1 = G_1'' = SU(k_1) \times T^1$$

or

$$G = K_1 = G_1'' = SO(2k_1).$$

Moreover, we have that $k_1 = n$, $K_2 = K_2''$ (by Lemma 6.1) and $\operatorname{rank} K_2^o = \operatorname{rank} K^o = n - 1$.

Because $\sigma_2((K_2'')^o) = \sigma_2(K_2^o) = SO(2k_2 - 1)$, it is easy to check that the covering group $\mathcal{K}_2 = \widetilde{K_2^o}$ in Section 2.3 of K_2^o can be decomposed into as follows:

$$\mathcal{K}_2 = Spin(2k_2 - 1) \times L,$$

where L is a product of simply connected, simple Lie groups and tori. In other words, the covering map

$$c : Spin(2k_2 - 1) \times L \to K_2^o$$

satisfies that $\sigma_2 \circ c(Spin(2k_2 - 1)) = SO(2k_2 - 1)$ and $c(L) \subset \ker \sigma_2$. Because $\operatorname{rank} K^o = \operatorname{rank} K_2^o$, it follows from Lemma 2.3 that $\widetilde{K^o} = Spin(2k_2 - 2) \times L$ and

$$K^o = c(Spin(2k_2 - 2) \times L).$$

We claim the following:

Claim 1. In the conditions above, we have $G = K_1 = G_1'' = SO(2k_1)$.

Proof. Assume $G_1'' = SU(n) \times T^1$. By Lemma 5.5, we have that

$$K^o = K = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \in S(U(1) \times U(n-1)) \times T^1 \,\middle|\, z^\gamma = a^{-1} \right\}$$

where γ is a non-zero integer. Therefore, the covering group $\widetilde{K}$ in Section 2.3 of K can decompose into $SU(n-1) \times T^1$. Hence, in this case, there is an isomorphism between $SU(n-1) \times T^1$ and $Spin(2k_2 - 2) \times L$. As is well known, $SU(l_1) \simeq Spin(l_2)$ if and only if $(l_1, l_2) = (2, 3)$ and $(4, 6)$ (see [**MiTo**]). Together with the assumption $k_2 > 1$, there are just the following two cases:

- $k_2 = 2$ and $L \simeq SU(n-1)$;
- $(n, k_2) = (5, 4)$ and $L \simeq T^1$.

If $k_2 = 2$ and $L \simeq SU(n-1)$, then $\mathcal{K}_2 = Spin(3) \times SU(n-1)$. Let $\iota : K_2 \to G = SU(n) \times T^1$ be the natural inclusion. Then, there exists the representation

$$\iota \circ c : Spin(3) \times SU(n-1) \to SU(n) \times T^1.$$

Because $\iota \circ c(Spin(2) \times SU(n-1)) = \iota \circ c(SU(n-1) \times T^1) = K \subset SU(n) \times T^1$, we have that

$$\iota \circ c(SU(n-1)) = SU(n-1) \subset SU(n)$$

and

$$\iota \circ c(Spin(2) \times \{e\}) \simeq S^1.$$

This also implies that $\iota \circ c(Spin(3)) \subset Z_{SU(n)\times T^1}(SU(n-1)) \subset SU(n) \times T^1$. By an easy computation, we have that $Z_{SU(n)\times T^1}(SU(n-1)) \simeq T^2$. This implies that $\iota \circ c$ provides a representation from $Spin(3)$ to T^2. Because $Spin(3)$ is the simple Lie group and T^2 is the commutative group, such representation is just the trivial representation. This gives a contradiction to $\iota \circ c(Spin(2)) \simeq S^1$.

Therefore, we have $(n, k_2) = (5, 4)$ and $L \simeq T^1$. Then, $\mathcal{K}_2 = Spin(7) \times T^1$ and there is the the sequence

$$K \subset c(Spin(7) \times T^1) = K_2^o \subset G = SU(5) \times T^1.$$

Let $p : SU(5) \times T^1 \to SU(5)$ be the natural projection. Then, we have $p(K) = S(U(1) \times U(4))$ because $\gamma \neq 0$. Because we may regard p is the quotient representation by $\{e\} \times T^1$, the dimension of $p(H)$ is $\dim H - 1$ or $\dim H$ for all subgroup H in G. Therefore, there is the following possibilities of dimension of $p(K_2^o)$:

$$\dim p(K_2^o) = \dim p \circ c(Spin(7) \times T^1) = \dim Spin(7) = 21$$

or

$$\dim p(K_2^o) = \dim p \circ c(Spin(7) \times T^1) = \dim(Spin(7) \times T^1) = 22.$$

On the other hand, we have $S(U(1)\times U(4)) \subset p(K_2^o) \subset SU(5)$. As is well known, $S(U(1)\times U(4))$ is a maximal rank maximal subgroup of $SU(5)$ (see e.g. [**MiTo**]). This implies that $S(U(1)\times U(4)) = p(K_2^o)$ or $p(K_2^o) = SU(5)$. However, because $\dim SU(5) = 24$ and $\dim S(U(1) \times U(4)) = 16$, this gives a contradiction to the possibilities of dimension of $p(K_2^o)$ as mentioned above.

The argument above establishes that $G = K_1 = G_1'' = SO(2n)$ □ □

Therefore, by this claim,

$$G = K_1 = G_1'' = SO(2n).$$

By Lemma 5.5, we have that

$$K^o = K = SO(2n-1).$$

Using $K \subset K_2 \subset G$, $K_2/K \cong S^{2k_2-2}$, $\dim G/K_2 = 2n - 2k_2 + 1$ and $k_2 > 1$, we also have that

$$SO(2n-1) \subsetneq K_2 \subsetneq SO(2n).$$

Now we have $\sigma_2(K_2^o) = SO(2k_2 - 1)$. In particular, we have $\sigma_2^{-1}(SO(2k_2 - 2)) = K^o = K = SO(2n-1)$. This implies that there is a surjective homomorphism from $SO(2n-1)$ to $SO(2k_2 - 2)$. Let us prove there is no such homomorphism. If there exists a surjective homomorphism from $SO(2n-1)$ to $SO(2k_2-2)$, there exists a transitive $SO(2n-1)$-action on S^{2k_2-3} via this homomorphism. However, by using the classification of transitive actions of spheres (see [**HsHs**, Section 1] or [**AlAl**, Table 1]), the transitive $SO(l)$-action on S^{2k_2-3} is just $l = 2k_2-2$. Therefore, there is no such homomorphism. This gives a contradiction. Consequently, we have

$$\sigma_2((K_2'')^o) \neq SO(2k_2 - 1).$$

In summary, we have the following lemma:

Lemma 6.7. *If $G/K_2 \cap M_1^T = \emptyset$ and G/K_2 is a singular orbit, i.e., $k_1 > 1$, then*

$$K_2 = G_1' \times K_2''$$

and

$$\sigma_2(G_1') = SO(2k_2 - 1).$$

By Lemma 6.7, we have $\sigma_2(G_1') = SO(2k_2 - 1)$. Because G_1' is a product of simple Lie group, we may assume $\sigma_2(SU(l_1 + 1)) = SO(2k_2 - 1)$ or $\sigma_2(SO(2m_1 + 1)) = SO(2k_2 - 1)$ by Lemma 5.3 and 6.2. As is well known, if $\sigma_2(SU(l_1 + 1)) = SO(2k_2 - 1)$ then $l_1 = 1$ and $k_2 = 2$. However, this is a contradiction to the assumption $l_i \geq 2$ (see Remark 5.2). Therefore, we have that $\sigma_2(SO(2m_1 + 1)) = SO(2k_2 - 1)$.

Because $\ker \sigma_2 \cap G_1'$ is finite (see Remark 6.5) and G_1' is connected (see Lemma 5.3), we get

$$G_1' = SO(2m_1 + 1) = SO(2k_2 - 1),$$

i.e., $a = 0$, $b = 1$ and $m_1 = k_2 - 1$. Now we may prove the following proposition:

PROPOSITION 6.8. *Suppose that (M_1, G) is a primitive torus manifold and G/K_2 does not contain a T-fixed point. If $k_2 > 1$, then there are the following four cases:*

(1) $G = SO(2k_2 - 1) \times SU(k_1) \times T^1$, $K_1 = SO(2k_2 - 2) \times SU(k_1) \times T^1$,

$$K_2 = SO(2k_2 - 1) \times \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \Big| \ z = a^{-1} \right\}$$

and

$$K = SO(2k_2 - 2) \times \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \Big| \ z = a^{-1} \right\};$$

(2) $G = SO(2k_2 - 1) \times SU(k_1) \times T^1$, $K_1 = S(O(1) \times O(2k_2 - 2)) \times SU(k_1) \times T^1$,

$$K_2 = SO(2k_2 - 1) \times \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \Big| \ \pm z = a^{-1} \right\}$$

and

$$K = \left\{ \left(\begin{pmatrix} x & 0 \\ 0 & X \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \Big| \ xz = a^{-1} \right\};$$

(3) $G = SO(2k_2-1) \times SO(2k_1)$, $K_1 = SO(2k_2-2) \times SO(2k_1)$, $K_2 = SO(2k_2-1) \times SO(2k_1-1)$ *and* $K = SO(2k_2 - 2) \times SO(2k_1 - 1)$*;*

(4) $G = SO(2k_2 - 1) \times SO(2k_1)$, $K_1 = S(O(1) \times O(2k_2 - 2)) \times SO(2k_1)$, $K_2 = SO(2k_2 - 1) \times S(O(1) \times O(2k_1 - 1))$ *and*

$$K = \left\{ \left(\begin{pmatrix} x & 0 \\ 0 & X \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & Y \end{pmatrix} \right) \Big| \ \det X = x, \det Y = y, \ xy = 1 \right\},$$

where $k_1 \geq 2$.

Here, in both of the cases above, $K \subset K_1 \cap K_2$.

PROOF. Using the argument before this proposition and Lemma 5.3, we have that

$$G = SO(2k_2 - 1) \times G_1''$$

and

$$K_1 = SO(2k_2 - 2) \times G_1''$$

or

$$K_1 = S(O(1) \times O(2k_2 - 2)) \times G_1'',$$

where $G_1'' = SU(k_1) \times T^1$ or $SO(2k_1)$. Moreover, by Lemma 6.1 (also see Lemma 6.7), we have

$$K_2 = SO(2k_2 - 1) \times K_2''.$$

Assume $K_1 = SO(2k_2 - 2) \times G_1''$. If $G_1'' = SU(k_1) \times T^1$, then it follows from Lemma 5.5 that we may regard K as follows up to essential isomorphism:

$$K = SO(2k_2 - 2) \times \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \Big| \ z = a^{-1} \right\}. \tag{6}$$

If $G_1'' = SO(2k_1)$, then it follows from Lemma 5.5 that

$$K = SO(2k_2 - 2) \times SO(2k_1 - 1). \tag{7}$$

Recall that $K_2/K \cong S^{2k_2-2}$ and $G_1' = SO(2k_2 - 1)$ acts on K_2/K transitively. Let $p'' : K_2 = G_1' \times K_2'' \to K_2''$ be the natural projection. Then, one can easily show that $p''(K) = K_2''$ (e.g. see [**Ku2**, Lemma 8.0.2]). Together with (6) and (7), we have the 1st and 3rd cases in the statement.

Assume $K_1 = S(O(1) \times O(2k_2 - 2)) \times G_1''$. If $G_1'' = SU(k_1) \times T^1$, by Lemma 5.5, we have

$$K = \left\{ \left(\begin{pmatrix} x & 0 \\ 0 & X \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, z \right) \Big| \ \sigma_1(x) z = a^{-1} \right\}.$$

Note that $\sigma_1(x) \in \{\pm 1\}$. With the method similar to that demonstrated as above, we also have

$$\sigma_1(x) = x = \pm 1$$

and

$$K_2 = SO(2k_2 - 1) \times \left\{ \left(\left(\begin{array}{cc} a & 0 \\ 0 & A \end{array} \right), z \right) \Big| \pm z = a^{-1} \right\}.$$

This establishes the 2^{nd} case in the statement. Similarly, we have the 4^{th} case in the statement. □ □

REMARK 6.9. Propositions 6.3, 6.6 and 6.8 also say that if we determine the slice representation σ_1 then another slice representation σ_2 is determined automatically. Moreover, σ_1 is uniquely determined once we choose (G, K_1, K_2, K) up to essential isomorphism.

7. Classification of primitive torus manifolds

In this section, we claasify the primitive torus manifolds. The goal of this section is to prove the following theorem:

THEOREM 7.1. *Let (M_1, T) be a primitive torus manifold. Then, a codimension one extended action (M_1, G) is essentially isomorphic to one of the followings:*

	M_1	G	k_1, k_2
(1)	$P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2})$	$S(U(k_1) \times U(k_2))$	$k_1, k_2 \geq 1,\ k_1 + k_2 \geq 3$
(2)	$S(\mathbb{C}^k \oplus \mathbb{R})$	$U(k)$	$k = k_1 = k_2 \geq 1$
(3)	$S(\mathbb{R}^{2k} \oplus \mathbb{R})$	$SO(2k)$	$k = k_1 = k_2 \geq 1$
(4)	$S(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1})$	$U(k_1) \times SO(2k_2 - 1)$	$k_1 \geq 1,\ k_2 \geq 2$
(5)	$S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$	$SO(2k_1) \times SO(2k_2 - 1)$	$k_1 \geq 1,\ k_2 \geq 2$
(6)	$\mathbb{R}P(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1})$	$U(k_1) \times SO(2k_2 - 1)$	$k_1 \geq 1,\ k_2 \geq 1$
(7)	$\mathbb{R}P(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$	$SO(2k_1) \times SO(2k_2 - 1)$	$k_1 \geq 1,\ k_2 \geq 1$

Here, each G in the table acts on each M_1 standardly.

7.1. Attaching maps. We have already seen (G, K_1, K_2, K) and two slice representations σ_1, σ_2 in Section 5 and 6. Moreover, by the slice theorem (Theorem 3.2), we also get the tubular neighborhoods X_1 and X_2 of G/K_1 and G/K_2, respectively. Due to Lemma 3.1, a primitive torus manifold M_1 decomposes into $X_1 \cup X_2$ equivariantly. Therefore, in order to show Theorem 7.1, it is enough to classify the attaching map $f : \partial X_1 \to \partial X_2$ and construct a G-manifold $M(f) = X_1 \cup_f X_2$ attached by f. Note that $\partial X_1 \cong \partial X_2 \cong G/K$; therefore, we may regard ∂X_1 and ∂X_2 as G/K. Moreover, the attaching map f must be a G-equivariant diffeomorphism because G-actions on X_1 and X_2 extends to the G-action on $M(f) = X_1 \cup_f X_2$. This implies that the attaching map $f : G/K \to G/K$ may be regarded as an element in

$$\mathrm{Aut}_G(G/K) \simeq N_G(K)/K,$$

where $N_G(K)$ is the normalizer of K in G (see [**Ka**]).

Let f and f' be two attaching maps. In order to check whether $M(f)$ and $M(f')$ are equivariantly diffeomorphic, the following lemma is useful (see [**Uc**, Lemma 5.3.1]).

LEMMA 7.2 (Uchida's criterion). *Let $f, f' : \partial X_1 \to \partial X_2$ be G-equivariant diffeomorphisms. Then $M(f)$ is equivariantly diffeomorphic to $M(f')$ as G-manifolds, if one of the following conditions are satisfied:*

(1) *f is G-diffeotopic to f';*
(2) *$f^{-1}f'$ is extendable to a G-equivariant diffeomorphism on X_1;*
(3) *$f'f^{-1}$ is extendable to a G-equivariant diffeomorphism on X_2.*

As in [**Ga**], we call this lemma the *Uchida's criterion.* Note that this criterion also holds for non-orientable manifolds.

Because of the Uchida's criterion (1), it is sufficient to compute

$$\mathcal{N} = N_G(K)/N_G^o(K)$$

instead of dealing with the whole $N_G(K)/K$, where $N_G^o(K)$ is a connected component of $N_G(K)$.

7.2. Construction of primitive torus manifolds. In this subsection, we compute $N_G(K)/N_G^o(K)$ and consturuct the primitive torus manifold (M_1, G) with codimension one extended G-action. Recall that (G, K_1, K_2, K) are classified as in Proposition 6.3 (1), (2), (3), Proposition 6.6 (1), (2) and Proposition 6.8 (1), (2), (3), (4). We call each case CASE I-(1), (2), (3), CASE II-(1), (2) and CASE III-(1), (2), (3), (4), respectively. It is easy to check the following lemma:

LEMMA 7.3. *The following statements hold for each $\mathcal{N} = N_G(K)/N_G^o(K)$:*
- *if (G, K) is one of the pairs in CASE I-(1), (2) and CASE II-(1), then $\mathcal{N} = \{e\}$;*
- *if (G, K) is one of the pairs in CASE I-(3) and CASE II-(2), then $\mathcal{N} \simeq C$;*
- *if (G, K) is the pair in CASE III-(1), (2), then $\mathcal{N} \simeq F$;*
- *if (G, K) is the pair in CASE III-(3), (4), then $\mathcal{N} \simeq F \times C$,*

where

$$F \simeq S(O(1) \times O(2l))/SO(2l)$$

and

$$C \simeq \{\pm I_{2l}\} = S(O(1) \times O(2l-1))/SO(2l-1),$$

i.e., C is the center of G.

We next prove the following lemma:

LEMMA 7.4. *Let f be an element of $\mathcal{N}$ in Lemma 7.3. Then,*

$$M(f) \cong M(e),$$

where $M(g) = X_1 \cup_g X_2$ $(g = e, f)$ and $e \in \mathcal{N}$ is the identity element.

PROOF. We will check Uchida's criterion (2) (Lemma 7.2), i.e., for all $f : G/K \to G/K \in \mathcal{N}$, $f = e \circ f$ extends to a G-equivariant diffeomorphism $X_1 \to X_1$, where $X_1 \cong G \times_{K_1} D^{2k_1}$. Note that the attaching map $f \in \mathcal{N}$ can be regarded as $f : G/K \to G/K$ by $f(gK) = gfK$, i.e., the multiplication from the right-hand side.

We first consider the case where $f \in C \subset \mathcal{N}$, i.e., $f \in \mathcal{N}$ can be taken as an element in the center of G. Because $fg = gf$ for all $g \in G$, the following map is well-defined and commute:

$$\begin{array}{ccc} G \times_{K_1} K_1/K & \xrightarrow{\pi} & G/K \\ \downarrow{\scriptstyle L_f \times \mathrm{Id}} & & \downarrow{\scriptstyle f} \\ G \times_{K_1} K_1/K & \xrightarrow{\pi} & G/K \end{array}$$

where $\partial X_1 = G \times_{K_1} K_1/K$, $\pi([g,\ kK]) = gkK$ and $(L_f \times \mathrm{Id})([g,\ kK]) = [fg,\ kK]$. Note that all maps in the diagram above are G-equivariantly diffeomorphic. The diffeomorphism Id : $K_1/K \cong S^{2k_1-1} \to S^{2k_1-1} \cong K_1/K$ obviously extends to Id : $D^{2k} \to D^{2k}$ as the K_1-equivariant diffeomorphism. Therefore, we have that $L_f \times \mathrm{Id}$ extends to the G-equivariant diffeomorphism $G \times_{K_1} D^{2k_1} = X_1 \to X_1 = G \times_{K_1} D^{2k_1}$. Hence, $M(f) \cong M(e)$ by Uchida's criterion (2).

We next consider the case where $f \in F \subset \mathcal{N}$, i.e., CASE III. By Proposition 6.8, $f \in F$ can be taken as an element (A, I) in $G_1' \times G_1''$ such that $ASO(2k_2-2)A^{-1} = SO(2k_2-2)$, where $A \in G_1' = SO(2k_2-1)$ and $I \in G_1''$ is the identity element. Moreover, using Lemma 5.4 and Proposition 6.8, there are the following three cases:

- if $K_1' = SO(2k_2-2)$, then $G/K \cong G_1' \times_{K_1'} (K_1/K) = S^{2k_2-2} \times S(V)$ and $X_1 \cong S^{2k_2-2} \times D(V)$;
- if $K_1' = S(O(1) \times O(2k_2-2))$ and σ_1 is tirvial, then $G/K \cong G_1' \times_{K_1'} (K_1/K) = \mathbb{R}P^{2k_2-2} \times S(V)$ and $X_1 \cong \mathbb{R}P^{2k_2-2} \times D(V)$;
- if $K_1' = S(O(1) \times O(2k_2-2))$ and σ_1 is non-tirvial, then $G/K \cong G_1' \times_{K_1'} (K_1/K) = S^{2k_2-2} \times_{\mathbb{Z}_2} S(V)$ and $X_1 \cong S^{2k_2-2} \times_{\mathbb{Z}_2} D(V)$,

where, in the final case above, $\mathbb{Z}_2$ acts on S^{2k_2-2} and on $D(V)$, $S(V)$ via the representation to $\{\pm 1\}$. Here, $V = \mathbb{C}^{k_1}$ or $\mathbb{R}^{2k_1}$, $S(V) \cong K_1/K$ is its unit sphere, $D(V)$ is its unit disk. Let $N \times_\Gamma S(V)$ be the manifolds appearing above which are diffeomorphic to G/K, i.e., $N = S^{2k_2-2}$ or $\mathbb{R}P^{2k_2-2}$ and $\Gamma = \{e\}$ or $\mathbb{Z}_2$. Therefore, $f = (A, I) : G/K \to G/K$ may be regarded as the induced equivariant map from

$$(\widetilde{A}, \mathrm{Id}) : N \times_\Gamma S(V) \longrightarrow N \times_\Gamma S(V),$$

where $\widetilde{A}$ is an equivariant involution on N and Id is the identity map on $S(V)$. Namely, there exists the following commutative diagram:

$$\begin{array}{ccccc} N \times_\Gamma S(V) & \xrightarrow{\cong} & G_1' \times_{K_1'} K_1/K & \xrightarrow{\pi} & G/K \\ \downarrow{\scriptstyle (\widetilde{A},\mathrm{Id})} & & \downarrow{\scriptstyle R_A \times \mathrm{Id}} & & \downarrow{\scriptstyle f} \\ N \times_\Gamma S(V) & \xrightarrow{\cong} & G_1' \times_{K_1'} K_1/K & \xrightarrow{\pi} & G/K \end{array}$$

where $(R_A \times \mathrm{Id})([g,\ kK]) = [gA,\ kK]$. Now Id : $S(V) \to S(V)$ extends to Id : $D(V) \to D(V)$ equivariantly. Hence, $(\widetilde{A}, I)$ extends to the G-equivariant diffeomorphism on $X_1 = N \times_\Gamma D(V)$. This establishes $M(f) \cong M(e)$ by Uchida's criterion. □ □

Remark 6.9 and Lemma 7.4 say that the primitive torus manifold (M_1, G) is uniquely determined by (G, K_1, K_2, K) up to essential isomorphism. Hence, in order to classify the primitive torus manifolds, it is enough to find G-manifolds with isotropy groups $K_1,\ K_2,\ K$ appearing in CASE I-(1) to CASE III-(4).

Let us find the manifold with G-action for each case.

7.2.1. *CASE I-(1).* Set (G, K_1, K_2, K) as in Proposition 6.3 (1). Namely, we may find a manifold with $G = SU(k_1) \times SU(k_2) \times T^1$-action whose isotropy subgroups are K_1, K_2, K appearing in Proposition 6.3. Let $M_1 = P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2})$ be the complex projectivization of $\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}$, i.e., $P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}) \cong \mathbb{C}P^{k_1+k_2-1}$. Now we define the G-action on M_1 as follows: $SU(k_1)$ acts on the $\mathbb{C}^{k_1}$-factor standardly; $SU(k_2)$ acts on the $\mathbb{C}^{k_2}$-factor by $w \mapsto \overline{B}w$, where $w \in \mathbb{C}^{k_2}$ and $\overline{B} \in SU(k_2)$ is the complex conjugation of $B \in SU(k_2)$; and T^1 acts on $\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}$ diagonally except the first coordinate of $\mathbb{C}^{k_2}$. Then, the isotropy subgroups are $G_{[0,e_1]} = K_1$ $G_{[e_1,0]} = K_2$ and $G_{[e_1,e_1]} = K$ appearing in Proposition 6.3 (1), where $(e_1, 0)$ represents the first coordinate of $\mathbb{C}^{k_1}$, $(0, e_1)$ represents the first coordinate of $\mathbb{C}^{k_2}$ and $[x, y]$ represents the projective coordinate in $P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2})$. By using the surjective homomorphism

$$\begin{array}{ccc} SU(k_1) \times SU(k_2) \times T^1 & \longrightarrow & S(U(k_1) \times U(k_2)) \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ (A, B, t) & \longmapsto & \begin{pmatrix} At^{k_2} & 0 \\ 0 & \overline{B}t^{-k_1} \end{pmatrix}, \end{array}$$

we have that the G-action defined above is essentially isomorphic to the natural action of $S(U(k_1) \times U(k_2))$ on $P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2})$, where

$$S(U(k_1) \times U(k_2)) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in SU(k_1 + k_2) \middle| \ A \in U(k_1), B \in U(k_2) \right\}.$$

This implies that (M_1, G) of CASE I-(1) is essentially isomorphic to

$$(P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}), S(U(k_1) \times U(k_2))),$$

where $k_1, k_2 \geq 1$ and $k_1 + k_2 - 1 = n$. This establishes Theorem 7.1 (1) if $n \geq 2$.

If $n = 1$, i.e., $k_1 = k_2 = 1$, then we easily obtain that $(P(\mathbb{C} \oplus \mathbb{C}), S(U(1) \times U(1)))$ and $(S(\mathbb{C} \oplus \mathbb{R}), U(1))$ are essentially isomorphic. So we may regard $n \geq 2$, i.e., $k_1 + k_2 \geq 3$, in this case. We shall discuss $(S(\mathbb{C} \oplus \mathbb{R}), U(1))$ in the next CASE I-(2).

7.2.2. *CASE I-(2).* Set (G, K_1, K_2, K) as in Proposition 6.3 (2). Let $M_1 = S(\mathbb{C}^n \oplus \mathbb{R})$ be the unit sphere in $\mathbb{C}^n \oplus \mathbb{R}$, i.e., $S(\mathbb{C}^n \oplus \mathbb{R}) \cong S^{2n}$. Then, M_1 has the natural $G = SU(n) \times T^1$-action on the coordinate of $\mathbb{C}^n$ (where T^1 acts on it by the scaler multiplication) and its isotropy subgroups are $G_{(0,1)} = K_1$ $G_{(0,-1)} = K_2$ and $G_{(e_1,0)} = K$ appearing in Proposition 6.3 (2), where $(z, r) \in \mathbb{C}^n \oplus \mathbb{R}$. Moreover, it is easy to check that this action is essentially isomorphic to the natural action of $U(n)$ on $S(\mathbb{C}^n \oplus \mathbb{R})$. This implies that (M_1, G) of CASE I-(2) is essentially isomorphic to

$$(S(\mathbb{C}^n \oplus \mathbb{R}), U(n)),$$

where $k_1 = k_2 = n$. This establishes Theorem 7.1 (2).

7.2.3. *CASE I-(3).* Set (G, K_1, K_2, K) as in Proposition 6.3 (3). Let $M_1 = S(\mathbb{R}^{2n} \oplus \mathbb{R})$ be the unit sphere in $\mathbb{R}^{2n} \oplus \mathbb{R}$, i.e., $S(\mathbb{R}^{2n} \oplus \mathbb{R}) \cong S^{2n}$. Then, M_1 has the natural $G = SO(2n)$-action on the coordinate of $\mathbb{R}^{2n}$ and its isotropy subgroups are $G_{(0,1)} = K_1$ $G_{(0,-1)} = K_2$ and $G_{(e_1,0)} = K$ appearing in Proposition 6.3 (3). This implies that (M_1, G) of CASE I-(3) is essentially isomorphic to

$$(S(\mathbb{R}^{2n} \oplus \mathbb{R}), SO(2n)),$$

where $k_1 = k_2 = n$. This establishes Theorem 7.1 (3).

7.2.4. *CASE II-(1).* Set (G, K_1, K_2, K) as in Proposition 6.6 (1). Note that (G, K_1, K) of this case coincides with that of CASE I-(2). Moreover, K_2 of this case is the double covering of K. These facts imply that the manifold M_1 of CASE II-(1) can be obtained by a $\mathbb{Z}_2$-quotient of $S(\mathbb{C}^n \oplus \mathbb{R})$ in CASE I-(2). Let $M_1 = \mathbb{R}P(\mathbb{C}^n \oplus \mathbb{R})$ be the quotient of $S(\mathbb{C}^n \oplus \mathbb{R})$ by the antipodal $\mathbb{Z}_2$-action, i.e., $\mathbb{R}P(\mathbb{C}^n \oplus \mathbb{R})$ is the $2n$-dimensional real projective space. Then, M_1 has the natural $G = SU(n) \times T^1$-action (where T^1 acts on $\mathbb{C}^n$ diagonally) and its isotropy subgroups are $G_{[0,1]} = K_1$ $G_{[e_1,0]} = K_2$ and $G_{[e_1,1]} = K$ appearing in Proposition 6.6 (1), where $[z, r]$ ($z \in \mathbb{C}^n$, $r \in \mathbb{R}$) represents the projective coordinate in $\mathbb{R}P(\mathbb{C}^n \oplus \mathbb{R})$. Moreover, this action is essentially isomorphic to the natural action of $U(n)$ on $\mathbb{R}P(\mathbb{C}^n \oplus \mathbb{R})$. This implies that (M_1, G) of CASE II-(1) is essentially isomorphic to

$$(\mathbb{R}P(\mathbb{C}^n \oplus \mathbb{R}), U(n)),$$

where $k_1 = n$ and $k_2 = 1$. This establishes Theorem 7.1 (6) with $k_2 = 1$.

7.2.5. *CASE II-(2).* Set (G, K_1, K_2, K) as in Proposition 6.6 (2). With the method similar to that demonstrated in the CASE II-(1), we have that (M_1, G) of CASE II-(2) is essentially isomorphic to

$$(\mathbb{R}P(\mathbb{R}^{2n} \oplus \mathbb{R}), SO(2n)),$$

where $k_1 = n$ and $k_2 = 1$. This establishes Theorem 7.1 (7) with $k_2 = 1$.

7.2.6. *CASE III-(1).* Set (G, K_1, K_2, K) as in Proposition 6.8 (1). Let $M_1 = S(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1})$ be the unit sphere of $\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1}$. Then, M_1 has the natural $G = SU(k_1) \times T^1 \times SO(2k_2 - 1)$-action, and its isotropy subgroups are $G_{(e_1,0)} = K_1$ $G_{(0,e_1)} = K_2$ and $G_{(e_1,e_1)} = K$ appearing in Proposition 6.8 (1), where $(e_1, 0)$ is the first coordinate in $\mathbb{C}^{k_1}$ and $(0, e_1)$ is that in $\mathbb{R}^{2k_2-1}$. Moreover, this action is essentially isomorphic to the natural action of $U(k_1) \times SO(2k_2 - 1)$ on $S(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1})$. This implies that (M_1, G) of CASE III-(1) is essentially isomorphic to

$$(S(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1}), U(k_1) \times SO(2k_2 - 1)),$$

where $k_1 \geq 1$ and $k_2 \geq 2$. This establishes Theorem 7.1 (4).

7.2.7. *CASE III-(2).* Set (G, K_1, K_2, K) as in Proposition 6.8 (2). Let $M_1 = \mathbb{R}P(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1})$ be the real projective space of $\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1}$. Then, M_1 has the natural $G = SU(k_1) \times T^1 \times SO(2k_2-1)$-action, and its isotropy subgroups are $G_{[e_1,0]} = K_1$ $G_{[0,e_1]} = K_2$ and $G_{[e_1,e_1]} = K$ appearing in Proposition 6.8 (2), where $[z, x]$ is the projective coordinate in $\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1}$. With the method similar to that demonstrated as above, we have that (M_1, G) of CASE III-(2) is essentially isomorphic to

$$(\mathbb{R}P(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1}), U(k_1) \times SO(2k_2 - 1)),$$

where $k_1 \geq 1$ and $k_2 \geq 2$. This establishes Theorem 7.1 (5).

7.2.8. *CASE III-(3).* Set (G, K_1, K_2, K) as in Proposition 6.8 (3). Let $M_1 = S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$ be the unit sphere of $\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1}$. Then, M_1 has the natural $G = SO(2k_1) \times SO(2k_2-1)$-action, and its isotropy subgroups are $G_{(e_1,0)} = K_1$ $G_{(0,e_1)} = K_2$ and $G_{(e_1,e_1)} = K$ appearing in Proposition 6.8 (3). With the method similar to that demonstrated as above, this establishes that (M_1, G) of CASE III-(3) is essentially isomorphic to

$$(S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1}), SO(2k_1) \times SO(2k_2-1)),$$

where $k_1 \geq 1$ and $k_2 \geq 2$. This establishes Theorem 7.1 (6) with $k_2 \geq 2$.

7.2.9. *CASE III-(4).* Set (G, K_1, K_2, K) as in Proposition 6.8 (4). Let $M_1 = \mathbb{R}P(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$ be the real projective space. Then, M_1 has the natural $G = SO(2k_1) \times SO(2k_2-1)$-action, and its isotropy subgroups are $G_{[e_1,0]} = K_1$ $G_{[0,e_1]} = K_2$ and $G_{[e_1,e_1]} = K$ appearing in Proposition 6.8 (4). With the method similar to that demonstrated as above, this establishes that (M_1, G) of CASE III-(4) is essentially isomorphic to

$$(\mathbb{R}P(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1}), SO(2k_1) \times SO(2k_2-1)),$$

where $k_1 \geq 1$ and $k_2 \geq 2$. This establishes Theorem 7.1 (7) with $k_2 \geq 2$.

Consequently, we have Theorem 7.1.

8. Preliminary to classifying non-primitive torus manifolds

In this section, we consider the general structures of non-primitive torus manifolds. Let (M^{2n}, T^n) be a non-primitive torus manifold with codimension one extended action (M, G). Due to Theorem 4.5, such (M, G) is essentially isomorphic to the following manifolds:

$$M \cong G' \times_{H'} M_1;$$
$$G \simeq G' \times G'',$$

where (M_1, G'') is one of the primitive torus manifolds in Theorem 7.1 and

$$G' \simeq \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1);$$
$$H' \simeq \prod_{i=1}^{a} S(U(1) \times U(l_i)) \times \mathcal{S}$$

for some subgroup $\mathcal{S}$ such that $\mathcal{S}^o = \prod_{j=1}^{b} SO(2m_j) \subset \mathcal{S} \subset \prod_{j=1}^{b} S(O(1) \times O(2m_j))$. Note that

$$\mathcal{S}/\mathcal{S}^o \simeq \mathcal{A} \subset (\mathbb{Z}_2)^b.$$

Here, M is the quotient of the H'-action on $G' \times M_1$ defined by the product of the natural action on the G'-factor and on the M_1-factor via

$$\mu : H' \to \mathrm{Diff}_{G''}(M_1),$$

where $\mathrm{Diff}_{G''}(M_1)$ represents the set of all G''-equivariant diffeomorphisms on M_1. By the definition of M, we can define the $G = G' \times G''$-action on it naturally.

Note that there exists the natural surjective homomorphism

$$s : H' \to T^a \times \mathcal{A},$$

because $T^a \times \mathcal{A} \simeq H'/(\prod_{i=1}^{a} SU(l_i) \times \mathcal{S}^o)$. We also note the following remark.

Remark 8.1. Let $q_j : G \to SO(2m_j+1)$ be the natural projection. If $q_j(H') = SO(2)$, i.e., $SO(2m_j+1) \cap \mathcal{A} = \{e\}$ and $m_j = 1$, then we may regard the $SO(2m_j+1)$-factor as the $SU(l_{a+1}+1)$-factor ($l_{a+1} = 1$) up to essential isomorphism, because $(SO(3), SO(2))$ and $(SU(2), S(U(1)\times U(1)))$ are locally isomorphic. Hence, we assume if $m_j = 1$ then $q_j(H') = S(O(1) \times O(2))$.

As we mentioned in Section 4, in order to classify all the torus manifolds with codimension one extended actions, we need to analyze the representation

$$\mu : H' \to \mathrm{Diff}_{G''}(M_1).$$

We first analyze the general property of the representation μ. Because the H'-action on M_1 commutes with the G''-action on M_1, we have that

$$h(G''(x)) = G''(hx),$$

for all $h \in H'$ and G''-orbit $G''(x)$ of $x \in M_1$. Thus, $G''(hx) \cong G''(x)$. This implies that if $G''(x)$ is a principal orbit then $G''(hx)$ is also a principal orbit. On the other hand, if $G''(x) = G''/K_1$ is a non-principal orbit then $G''(hx) = G''/K_1$ or G''/K_2, where K_1 and K_2 are non-principal isotropy subgroups of (M_1, G''). Therefore, we can define the induced H'-action on $M_1/G'' = [-1, 1]$ via a homomorphism

$$\mu_{[-1,1]} : H' \to O(1),$$

where $O(1)$ acts on $[-1, 1] \subset \mathbb{R}$ naturally. Note that if $\mu_{[-1,1]}$ is non-trivial then $G''/K_1 \cong G''/K_2$. Thus, there are the following two cases:

- $\mu_{[-1,1]}$ is trivial;
- $\mu_{[-1,1]}$ is non-trivial and $G''/K_1 \cong G''/K_2$.

In Section 9 and 10, we classify all torus manifolds with codimension one extended actions.

9. The case when $\mu_{[-1,1]}$ is trivial

Let (M_1, G'') be a primitive torus manifolds appearing in Theorem 7.1. Assume $\mu_{[-1,1]}$ is trivial. Let K_1 and K_2 be non-principal isotropy subgroups of (M_1, G''). We first analyze the H'-action on tubular neighborhoods of two non-principal orbits of (M_1, G'').

9.1. Two tubular neighborhoods $\widehat{X}_i$. Because $\mu_{[-1,1]}$ is trivial, we see that H' acts on G''/K_1 and G''/K_2 via G''-equivariant automorphisms. Namely, using the argument in Section 7.1, the representation $\mu : H' \to \mathrm{Diff}_{G''}(M_1)$ induces the representations

$$\mu_i : H' \longrightarrow \mathrm{Aut}_{G''}(G''/K_i) \simeq N_{G''}(K_i)/K_i,$$

for $i = 1, 2$. Let $\widehat{K}_1$ and $\widehat{K}_2$ be non-principal isotropy subgroups of (M, G). Then, two non-principal orbits $G/\widehat{K}_i$ of (M, G) are denoted by

$$G' \times_{H'} (G''/K_i) \tag{8}$$

such that H' acts on G''/K_i via μ_i $(i = 1, 2)$. Therefore, we have the following lemma:

Lemma 9.1. *Two singular isotropy subgroups $\widehat{K}_i$, $i = 1, 2$, of (M, G) is isomorphic to the following group:*

$$\widehat{K}_i \simeq \{(h, k) \in H' \times N_{G''}(K_i) \mid \mu_i(h) = [k] \in N_{G''}(K_i)/K_i\}$$

for some representation $\mu_i : H' \to N_{G''}(K_i)/K_i$ $(i = 1, 2)$.

In particular, if μ_i is the trivial representation, then we have

$$\widehat{K}_i = H' \times K_i.$$

Moreover, tubular neighborhoods of two non-principal orbits of (M, G) can be denoted by

$$G \times_{\widehat{K}_i} D^{l_i} \cong G' \times_{H'} X_i \cong G' \times_{H'} (G'' \times_{K_i} D^{l_i})$$

for $i = 1, 2$. We now analyze how H' acts on $X_i \cong G'' \times_{K_i} D^{l_i}$.

Using (8) and the slice theorem, we have that the H'-action on X_i preserves the bundle structure of X_i. Therefore, it follows from $\partial X_i = G''/K$ that the following commutative diagram:

$$\begin{array}{ccc} G''/K & \xrightarrow{\widetilde{\mu}_i(h)} & G''/K \\ \downarrow & & \downarrow \\ G''/K_1 & \xrightarrow{\mu_i(h)} & G''/K_1 \end{array}$$

for $i = 1, 2$ and $h \in H'$, where $\widetilde{\mu}_i(h) \in N_{G''}(K)/K$ is the induced automorphism on the principal orbit G''/K from $\mu : H' \to \mathrm{Diff}_{G''}(M_1)$. This implies that $\mu_i(h) \in N_{G''}(K_i)/K_i$ is induced from

$\widetilde{\mu}_i(h) \in N_{G''}(K)/K$. Namely, we may regard $\widetilde{\mu}_i(h)$ as an element of the following subgroup of $N_{G''}(K)/K$:

$$(N_{G''}(K_i) \cap N_{G''}(K))/K,$$

and $\mu_i(h)$ as the image of $\widetilde{\mu}_i(h)$ of the natural projection

$$p : N_{G''}(K_i)/K \longrightarrow N_{G''}(K_i)/K_i,$$

where $(N_{G''}(K_i) \cap N_{G''}(K))/K \subset N_{G''}(K_i)/K$, i.e.,

$$p(\widetilde{\mu}_i(h)) = \mu_i(h).$$

Let M_1 be a manifold appearing in Theorem 7.1. If $M_1 = P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2})$, then

$$N_{G''}(K) \cap N_{G''}(K_i) = S(U(1) \times U(k_1 - 1) \times U(1) \times U(k_2 - 1)) \subset K_i.$$

Therefore, in this case, $\mu_i(h)$ is the identity element in $N_{G''}(K_i)/K_i$ for $i = 1, 2$. Otherwise, i.e., if $M_1 \neq P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2})$, it is easy to check that

$$N_{G''}(K) \subset N_{G''}(K_i)$$

for $i = 1, 2$. Moreover, by using this relation, we have that the following homomorphism

$$\begin{array}{ccc} N_{G''}(K)/K & \longrightarrow & N_{G''}(K_i)/K_i \\ \cup & & \cup \\ [g] & \longmapsto & [g] \end{array}$$

is well-defined and surjective when $M_1 \neq P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2})$. This implies that $\mu_i(h)$ can be taken as any element in $N_{G''}(K_i)/K_i$. Hence, we have the following lemma:

LEMMA 9.2. *The induced automorphism $\mu_i(h) : G''/K_i \to G''/K_i$ for $h \in H'$ can be regarded as an element of the following groups:*

	M_1	$\mu_1(h)$	$\mu_2(h)$
CASE (1)	$P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2})$	$\{e\}$	$\{e\}$
CASE (2)	$S(\mathbb{C}^{k} \oplus \mathbb{R})$	$\{e\}$	$\{e\}$
CASE (3)	$S(\mathbb{R}^{2k} \oplus \mathbb{R})$	$\{e\}$	$\{e\}$
CASE (4)	$S(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1})$	$F \simeq \mathbb{Z}_2$	S^1
CASE (5)	$S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$	$F \simeq \mathbb{Z}_2$	$C \simeq \mathbb{Z}_2$
CASE (6)	$\mathbb{R}P(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1})$	$\{e\}$	$S^1/\{\pm 1\}$
CASE (7)	$\mathbb{R}P(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$	$\{e\}$	$\{e\}$

where S^1 is the diagonal subgroup of $U(k_1)$, $F = N_{SO(2k_2-1)}(SO(2k_2 - 2))/SO(2k_2 - 2)$ and $C = N_{SO(2k_1)}(SO(2k_1 - 1))/SO(2k_1 - 1)$. Here, the numbers of CASE (1)–(7) in the list coincide with those of Theorem 7.1.

Now we may prove the following lemma which tells us how H' acts on X_i:

LEMMA 9.3. *Let $\widehat{X}_i$ be a tubular neighborhood of non-principal orbit in (M, G). Then, $\widehat{X}_i$ is equivariantly diffeomorphic to*

$$G' \times_{H'} (G'' \times_{K_i} D^{l_i})$$

such that $h \in H'$ acts on $[g'', x] \in G'' \times_{K_i} D^{l_i}$ by

$$[g'', x] \mapsto [g''\mu_i(h)^{-1}, s_i(h)x],$$

where μ_i is the representation appearing in Lemma 9.2 and

$$s_i : H' \xrightarrow{s} T^a \times \mathcal{A} \xrightarrow{\rho_i} Z(\sigma_i(K_i); O(l_i))$$

for some representation ρ_i. Here, $\mathcal{A}$ is a subgroup of $(\mathbb{Z}_2)^b$ appearing in Section 8 and the representation $s : H' \to T^a \times \mathcal{A}$ is the natural surjective homomorphism defined in Section 8.

PROOF. We first assume that (M_1, G'') is one of CASE (1)–(3) in Lemma 9.2. In this case, $l_i = 2k_i$ and $\mu_i(h)$ is identity. Therefore, by Lemma 9.1, the isotropy subgroup is $H' \times K_i$. This implies that an element of H' commutes with that of K_i. Hence, there exists a representation $s_i = \sigma_i|_{H'} : H' \to Z(\sigma_i(K_i); O(2k_i))$ such that its slice representation is denoted by

$$\widehat{\sigma}_i : H' \times K_i \xrightarrow{s_i \times \sigma_i} O(2k_i),$$

where $\sigma_i : K_i \to O(2k_i)$ is the slice representation of K_i in (M_1, G''). By the arguments in Sections 5 and 6, we already know how to embed $\sigma_i(K_i)$ into $O(2k_i)$; by using this, it is easy to check that $Z(\sigma_i(K_i); O(l_i))$ is a commutative group for all CASE (1)–(7). Using the notations in Section 5.2 together with Remark 8.1, we can put the elements of $H' = \prod_{i=1}^{a} S(U(1) \times U(m_i)) \times \mathcal{S}$ as follows:

$$\begin{aligned} \tau_I &= \left(\begin{pmatrix} t_1 & 0 \\ 0 & A_1 \end{pmatrix}, \ \cdots, \ \begin{pmatrix} t_a & 0 \\ 0 & A_a \end{pmatrix} \right) \in \prod_{i=1}^{a} S(U(1) \times U(l_i)), \\ \nu_J &= \left(\begin{pmatrix} x_1 & 0 \\ 0 & X_1 \end{pmatrix}, \ \cdots, \ \begin{pmatrix} x_b & 0 \\ 0 & X_b \end{pmatrix} \right) \in \mathcal{S} \subset \prod_{j=1}^{b} S(O(1) \times O(2m_j)). \end{aligned}$$

Recall $\mathcal{S}/\mathcal{S}^o = \mathcal{A} \subset (\mathbb{Z}_2)^b$. Now we can define the natural surjective homomorphism $s : H' \to T^a \times \mathcal{A}$ as follows:

$$s(\tau_I, \nu_J) = ((t_1, \ldots, t_a), (x_1, \ldots, x_b)) \in T^a \times \mathcal{A} \subset T^a \times (\mathbb{Z}_2)^b.$$

By abuse of notation, we often denote $((t_1, \ldots, t_a), (x_1, \ldots, x_b))$ by (τ_I, ν_J) simply. Because $Z(\sigma_i(K_i); O(2k_i))$ is a commutative group, there exists the following decomposition:

$$s_i : H' \xrightarrow{s} T^a \times \mathcal{A} \xrightarrow{\rho_i} Z(\sigma_i(K_i); O(2k_i)),$$

for some ρ_i. Now recall that H' acts on G''/K_i trivially in CASE (1)–(3). Therefore, one can easily check that the map

$$\begin{array}{ccc} (G' \times G'') \times_{(H' \times K_i)} D^{l_i} & \longrightarrow & G' \times_{H'} (G'' \times_{K_i} D^{l_i}) \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ [(g', g''), x] & \longmapsto & [g', [g'', x]] \end{array}$$

induces the equivariant diffeomorphism from the tubular neighborhood

$$\widehat{X}_i = G \times_{\widehat{K}_i} D^{l_i} \cong (G' \times G'') \times_{(H' \times K_i)} D^{l_i}$$

to

$$G' \times_{H'} X_i \cong G' \times_{H'} (G'' \times_{K_i} D^{l_i}),$$

where $h \in H'$ acts on $[g'', x] \in G'' \times_{K_i} D^{l_i}$ by

$$[g'', x] \mapsto [g'', s_i(h)(x)].$$

We next assume that (M_1, G'') is one of CASE (4)–(7) in Lemma 9.2. In this case, $l_1 = 2k_1$ and $l_2 = 2k_2 - 1$. Moreover, by Theorem 7.1, we may put

$$\begin{aligned} G'' &= G''_1 \times G''_2; \\ K_1 &= G''_1 \times \ker \sigma_1; \\ K_2 &= \ker \sigma_2 \times G''_2, \end{aligned} \tag{9}$$

where $\sigma_i : K_i \to O(l_i)$ is the slice representation of K_i in (M_1, G''); for example, if (M_1, G'') is the manifold in CASE (4), then $G'' = U(k_1) \times SO(2k_2 - 1) = G''_1 \times G''_2$, $K_1 = G''_1 \times \ker \sigma_1$ (where $\ker \sigma_1 = SO(2k_2 - 2)$), and $K_2 = \ker \sigma_2 \times G''_2$ (where $\ker \sigma_2 = U(k_1 - 1)$). This implies that $\ker \sigma_i \subset G''_r$ and

$$N_{G''_r}(\ker \sigma_i)/\ker \sigma_i = N_{G''}(K_i)/K_i, \tag{10}$$

where $(i, r) = (1, 2)$ or $(2, 1)$. Now we may regard $K_i \subset \widehat{K}_i$ as the subset

$$\{(e, k) \in \widehat{K}_i \mid k \in K_i\},$$

where $e \in G'$ is the identity element. Then, it is easy to check that $\widehat{\sigma}_i|_{K_i} = \sigma_i$, where $\widehat{\sigma}_i : \widehat{K}_i \to O(l_i)$ is the slice representation of $\widehat{K}_i$ in (M, G). Hence, $\ker \sigma_i$ can be regarded as the normal subgroup of $\widehat{K}_i$. Therefore, it follows from the relations above (9), (10) that

$$\widehat{K}_i / \ker \sigma_i = \{(h, A, \mu_i(h)) \in H' \times G''_i \times N_{G''}(K_i)/K_i\}.$$

This also implies that

$$\widehat{K}_i / \ker \sigma_i \simeq H' \times G''_i.$$

Hence, the slice representation $\widehat{\sigma}_i$ can be decomposed into as the following diagram:

$$\begin{array}{ccc} \widehat{K}_i & & \\ \downarrow & \overset{\widehat{\sigma}_i}{\searrow} & \\ H' \times G''_i & \xrightarrow{s_i \times \sigma_i} & O(l_i) \end{array}$$

for some representation $s_i : H' \to Z(\sigma_i(K_i); O(l_i))$. With the method similar to that demonstrated as above in CASE (1)–(3), there exists a representation

$$\rho_i : T^a \times \mathcal{A} \to Z(\sigma_i(K_i); O(l_i))$$

such that

$$s_i = \rho_i \circ s.$$

Therefore, by using Lemma 9.1, we have that the following map is well-defined:

$$\begin{array}{ccc} (G' \times G'') \times_{\widehat{K}_i} D^{l_i} & \longrightarrow & G' \times_{H'} (G'' \times_{K_i} D^{l_i}) \\ \cup & & \cup \\ [(g', g''), x] & \longmapsto & [g', [g'', x]] \end{array}$$

where $h \in H'$ acts on $[g'', x] \in G'' \times_{K_i} D^{l_i}$ by

$$[g'', x] \mapsto [g'' \mu_i(h)^{-1}, s_i(h)(x)].$$

It is easy to check that this map gives the equivariant diffeomorphism. This establishes Lemma 9.3. □ □

9.2. Torus manifolds with trivial $\mu_{[-1,1]}$. In this subsection, we classify (M, G) with trivial $\mu_{[-1,1]}$. Let $M = G' \times_{H'} M_1$ be a torus manifold with codimension one extended $G' \times G''$-action and the H'-action on M_1 preserves the orbits of (M_1, G''), i.e., $\mu_{[-1,1]}$ is trivial. We will analyze in each case; CASE (1)–(7).

9.2.1. *CASE (1).* Let (M_1, G'') be CASE (1), i.e.,

$$(P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}), S(U(k_1) \times U(k_2))).$$

Because G'' acts on M_1 by the standard multiplication, its two non-principal orbits are

$$\begin{aligned} G''/K_1 &\cong \{[0 : w] \in P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2})\}, \\ G''/K_2 &\cong \{[z : 0] \in P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2})\}, \end{aligned}$$

and two tubular neighborhoods are

$$\begin{aligned} G'' \times_{K_1} D^{2k_1} &\cong X_1 = \{[z : w] \in P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}) \mid w \neq 0,\ z \in D^{2k_1} \subset \mathbb{C}^{k_1}\}, \\ G'' \times_{K_2} D^{2k_2} &\cong X_2 = \{[z : w] \in P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}) \mid z \neq 0,\ w \in D^{2k_2} \subset \mathbb{C}^{k_2}\}. \end{aligned}$$

Using Lemma 9.2 and 9.3, we may define H'-action on X_i as follows:

$$\begin{aligned} [z, w] &\mapsto [s_1(h) z, w] \quad \text{for } X_1; \\ [z, w] &\mapsto [z, s_2(h) w] \quad \text{for } X_2, \end{aligned}$$

where $h \in H'$ and $s_i : H' \to Z(\sigma_i(K_i); O(2k_i)) \simeq S^1$ (scaler multiplication). Due to Lemma 7.4, we may regard the attaching map between X_1 and X_2 in M_1 as the identity map. Therefore, the restricted H'-action on X_1 to ∂X_1 coincides with that of ∂X_2. This implies that the following relation:

$$s_1(h) = s_2(h)^{-1}.$$

Hence, M is equivariantly diffeomorphic to

$$G' \times_{H'} P(\mathbb{C}^{k_1}_{s_1} \oplus \mathbb{C}^{k_2}),$$

where H' acts on $\mathbb{C}^{k_1}_{s_1}$ by the scaler multiplication via a representation $s_1 : H' \to S^1$ and on $\mathbb{C}^{k_2}$ trivially.

Put

$$\mathcal{S}(a; b) = \prod_{i=1}^{a} S^{2l_i+1} \times \prod_{j=1}^{b} S^{2m_j}.$$

Because there exists a decomposition $s_1 : H' \xrightarrow{s} T^a \times \mathcal{A} \xrightarrow{\rho} S^1$ for some representation ρ (see Section 9.1), we have the follow proposition:

Proposition 9.4. *Let $M = G' \times_{H'} M_1$ be a non-principal torus manifold with the trivial $\mu_{[-1,1]}$. If $M_1 = P(\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2})$, then there exists a representation $\rho : T^a \times \mathcal{A} \to S^1$ such that (M, G) is essentially isomorphic to the following manifold:*

$$M = \mathcal{S}(a; b) \times_{T^a \times \mathcal{A}} P(\mathbb{C}^{k_1}_{\rho} \oplus \mathbb{C}^{k_2}),$$
$$G = \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1) \times S(U(k_1) \times U(k_2)),$$

where G acts on M standardly and M is defined by the following $T^a \times \mathcal{A}$ quotient: $T^a \times \mathcal{A} \subset T^a \times (\mathbb{Z}_2)^b$ acts on $\mathcal{S}(a; b)$ via the natural action; on $\mathbb{C}^{k_1}_{\rho}$ by the scaler multiplication via ρ; and on $\mathbb{C}^{k_2}$ trivially.

9.2.2. *CASE (2).* Let (M_1, G'') be CASE (2), i.e.,

$$(S(\mathbb{C}^k \oplus \mathbb{R}), U(k)).$$

Then, its two tubular neighborhoods are

$$G'' \times_{K_1} D^{2k} \cong X_1 = \{(z, r) \in S(\mathbb{C}^k \oplus \mathbb{R}) \mid 0 \le r \le 1\};$$
$$G'' \times_{K_2} D^{2k} \cong X_2 = \{(z, r) \in S(\mathbb{C}^k \oplus \mathbb{R}) \mid -1 \le r \le 0\},$$

where $z \in \mathbb{C}^k$ and $r \in \mathbb{R}$ such that $|z|^2 + r^2 = 1$. Using Lemma 9.2 and 9.3, we may define H'-action on X_i as follows:

$$(z, r) \mapsto (s_1(h)z, r) \quad \text{for } X_1;$$
$$(z, r) \mapsto (s_2(h)z, r) \quad \text{for } X_2,$$

where $h \in H'$ and $s_i : H' \to Z(\sigma_i(K_i); O(2k)) \simeq S^1$ (scaler multiplication). With the method similar to that demonstrated in the proof of Proposition 9.4, we have that

$$s_1(h) = s_2(h)$$

and the follow proposition:

Proposition 9.5. *Let $M = G' \times_{H'} M_1$ be a non-principal torus manifold with the trivial $\mu_{[-1,1]}$. If $M_1 = S(\mathbb{C}^k \oplus \mathbb{R})$, then there exists a representation $\rho : T^a \times \mathcal{A} \to S^1$ such that (M, G) is essentially isomorphic to the following manifold:*

$$M = \mathcal{S}(a; b) \times_{T^a \times \mathcal{A}} S(\mathbb{C}^k_{\rho} \oplus \mathbb{R}),$$
$$G = \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1) \times U(k),$$

where $T^a \times \mathcal{A}$ acts on $\mathbb{C}^k_{\rho}$ by the scaler multiplication via ρ and on $\mathbb{R}$ trivially.

9.2.3. *CASE (3).* Let (M_1, G'') be CASE (3), i.e.,

$$(S(\mathbb{R}^{2k} \oplus \mathbb{R}), SO(2k)).$$

Similarly as in CASE (2), its two tubular neighborhoods are

$$\begin{aligned} G'' \times_{K_1} D^{2k} &\cong X_1 = \{(x, r) \in S(\mathbb{R}^{2k} \oplus \mathbb{R}) \mid 0 \leq r \leq 1\}; \\ G'' \times_{K_2} D^{2k} &\cong X_2 = \{(x, r) \in S(\mathbb{R}^{2k} \oplus \mathbb{R}) \mid -1 \leq r \leq 0\}, \end{aligned}$$

where $x \in \mathbb{R}^{2k}$ and $r \in \mathbb{R}$ such that $|x|^2 + r^2 = 1$. Using Lemma 9.2 and 9.3, we may define H'-action on X_i as follows:

$$\begin{aligned} (x, r) &\mapsto (s_1(h)x, r) \quad \text{for } X_1; \\ (x, r) &\mapsto (s_2(h)x, r) \quad \text{for } X_2, \end{aligned}$$

where $h \in H'$ and $s_i : H' \to Z(\sigma_i(K_i); O(2k)) \simeq \mathbb{Z}_2$ (scaler multiplication). Similarly as in CASE (2), we also have that

$$s_1(h) = s_2(h)$$

and the follow proposition:

PROPOSITION 9.6. *Let $M = G' \times_{H'} M_1$ be a non-principal torus manifold with the trivial $\mu_{[-1,1]}$. If $M_1 = S(\mathbb{R}^{2k} \oplus \mathbb{R})$, then there exists a representation $\rho : T^a \times \mathcal{A} \to \mathbb{Z}_2$ such that (M, G) is essentially isomorphic to the following manifold:*

$$\begin{aligned} M &= \mathcal{S}(a; b) \times_{T^a \times \mathcal{A}} S(\mathbb{R}^{2k}_\rho \oplus \mathbb{R}), \\ G &= \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1) \times SO(k), \end{aligned}$$

where $T^a \times \mathcal{A}$ acts on $\mathbb{R}^{2k}_\rho$ by the scaler multiplication via ρ and on $\mathbb{R}$ trivially.

Note that in Proposition 9.6 M is equivariantly diffeomorphic to the following manifold:

$$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j+1} \times_{\mathcal{A}} S(\mathbb{R}^{2k}_\rho \oplus \mathbb{R}) \right),$$

because the restricted representation $\rho|_{T^a}$ is trivial.

9.2.4. *CASE (4).* Let (M_1, G'') be CASE (4), i.e.,

$$(S(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1}), U(k_1) \times SO(2k_2 - 1)).$$

Similarly as in CASE (1), its two tubular neighborhoods are

$$\begin{aligned} G'' \times_{K_1} D^{2k_1} &\cong X_1 = \{(z, x) \in S(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1}) \mid 0 \leq |x| \leq 1/\sqrt{2}\}; \\ G'' \times_{K_2} D^{2k_2-1} &\cong X_2 = \{(z, x) \in S(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1}) \mid 0 \leq |z| \leq 1/\sqrt{2}\}, \end{aligned}$$

where $z \in \mathbb{C}^{k_1}$ and $x \in \mathbb{R}^{2k_2-1}$ such that $|z|^2 + |x|^2 = 1$. Using Lemma 9.2 and 9.3, we may define H'-action on X_i as follows:

$$\begin{aligned} (z, x) &\mapsto (s_1(h)z, \mu_1(h)^{-1}x) \quad \text{for } X_1; \\ (z, x) &\mapsto (\mu_2(h)^{-1}z, s_2(h)x) \quad \text{for } X_2, \end{aligned}$$

for some scaler representations

$$\begin{aligned} s_1 &: H' \to Z(\sigma_1(K_1); O(2k_1)) \simeq S^1, \\ s_2 &: H' \to Z(\sigma_2(K_2); O(2k_2 - 1)) \simeq \mathbb{Z}_2, \\ \mu_1 &: H' \to N_{G''}(K_1)/K_1 \simeq \mathbb{Z}_2, \\ \mu_2 &: H' \to N_{G''}(K_2)/K_2 \simeq S^1. \end{aligned}$$

With the method similar to that demonstrated in the proof of Proposition 9.4, we have that

$$\begin{aligned} s_1(h) &= \mu_2(h)^{-1} \in S^1, \\ s_2(h) &= \mu_1(h)^{-1} \in \mathbb{Z}_2, \end{aligned}$$

and the follow proposition:

Proposition 9.7. *Let $M = G' \times_{H'} M_1$ be a non-principal torus manifold with the trivial $\mu_{[-1,1]}$. If $M_1 = S(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1})$, then there exist representations $\rho_1 : T^a \times \mathcal{A} \to S^1$ and $\rho_2 : T^a \times \mathcal{A} \to \mathbb{Z}_2$ such that (M, G) is essentially isomorphic to the following manifold:*

$$M = \mathcal{S}(a; b) \times_{T^a \times \mathcal{A}} S(\mathbb{C}^{k_1}_{\rho_1} \oplus \mathbb{R}^{2k_2-1}_{\rho_2}),$$
$$G = \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1) \times U(k_1) \times SO(2k_2 - 1),$$

where $T^a \times \mathcal{A}$ acts on $\mathbb{C}^{k_1}_{\rho_1}$ by the scaler multiplication via ρ_1 and on $\mathbb{R}^{2k_2-1}_{\rho_2}$ by the scaler multiplication via ρ_2.

9.2.5. *CASE (5).* Let (M_1, G'') be CASE (5), i.e.,

$$(S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1}), SO(2k_1) \times SO(2k_2 - 1)).$$

Similarly, its two tubular neighborhoods are

$$G'' \times_{K_1} D^{2k_1} \cong X_1 = \{(x, y) \in S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1}) \mid 0 \leq |y| \leq 1/\sqrt{2}\};$$
$$G'' \times_{K_2} D^{2k_2-1} \cong X_2 = \{(x, y) \in S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1}) \mid 0 \leq |x| \leq 1/\sqrt{2}\},$$

where $x \in \mathbb{R}^{2k_1}$ and $y \in \mathbb{R}^{2k_2-1}$ such that $|x|^2 + |y|^2 = 1$. Using Lemma 9.2 and 9.3, we may define H'-action on X_i as follows:

$$(x, y) \mapsto (s_1(h)x, \mu_1(h)^{-1}y) \quad \text{for } X_1;$$
$$(x, y) \mapsto (\mu_2(h)^{-1}x, s_2(h)y) \quad \text{for } X_2,$$

for some scaler representations

$$s_1 : H' \to Z(\sigma_1(K_1); O(2k_1)) \simeq \mathbb{Z}_2,$$
$$s_2 : H' \to Z(\sigma_2(K_2); O(2k_2 - 1)) \simeq \mathbb{Z}_2,$$
$$\mu_1 : H' \to N_{G''}(K_1)/K_1 \simeq \mathbb{Z}_2,$$
$$\mu_2 : H' \to N_{G''}(K_2)/K_2 \simeq \mathbb{Z}_2.$$

With the method similar to that demonstrated in the proof of Proposition 9.4, we have that

$$s_1(h) = \mu_2(h)^{-1} \in \mathbb{Z}_2,$$
$$s_2(h) = \mu_1(h)^{-1} \in \mathbb{Z}_2,$$

and the follow proposition:

Proposition 9.8. *Let $M = G' \times_{H'} M_1$ be a non-principal torus manifold with the trivial $\mu_{[-1,1]}$. If $M_1 = S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$, then there exist representations $\rho_1 : T^a \times \mathcal{A} \to \mathbb{Z}_2$ and $\rho_2 : T^a \times \mathcal{A} \to \mathbb{Z}_2$ such that (M, G) is essentially isomorphic to the following manifold:*

$$M = \mathcal{S}(a; b) \times_{T^a \times \mathcal{A}} S(\mathbb{R}^{2k_1}_{\rho_1} \oplus \mathbb{R}^{2k_2-1}_{\rho_2}),$$
$$G = \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1) \times SO(2k_1) \times SO(2k_2 - 1),$$

where $T^a \times \mathcal{A}$ acts on $\mathbb{R}^{2k_1}_{\rho_1}$ by the scaler multiplication via ρ_1 and on $\mathbb{R}^{2k_2-1}_{\rho_2}$ by the scaler multiplication via ρ_2.

Note that in Proposition 9.8 M is equivariantly diffeomorphic to the following manifold:

$$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j+1} \times_{\mathcal{A}} S(\mathbb{R}^{2k_1}_{\rho_1} \oplus \mathbb{R}^{2k_2-1}_{\rho_2}) \right),$$

because the restricted representation $\rho_1|_{T^a}$ is trivial.

9.2.6. *CASE (6).* Let (M_1, G'') be CASE (6), i.e.,

$$(\mathbb{R}P(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1}), U(k_1) \times SO(2k_2 - 1)).$$

Then, its two tubular neighborhoods are

$$\begin{aligned} G'' \times_{K_1} D^{2k_1} &\cong X_1 = \{[z : x] \in \mathbb{R}P(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1}) \mid x \neq 0,\ z \in D^{2k_1} \subset \mathbb{C}^{k_1}\}; \\ G'' \times_{K_2} D^{2k_2-1} &\cong X_2 = \{[z : x] \in \mathbb{R}P(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1}) \mid z \neq 0,\ x \in D^{2k_2-1} \subset \mathbb{R}^{2k_2-1}\}. \end{aligned}$$

Using Lemma 9.2 and 9.3, we may define H'-action on X_i as follows:

$$\begin{aligned} &[z : x] \mapsto [s_1(h)z : x] \quad \text{for } X_1; \\ &[z : x] \mapsto [\mu_2(h)^{-1}z : s_2(h)x] \quad \text{for } X_2, \end{aligned}$$

for some scaler representations

$$\begin{aligned} &s_1 : H' \to Z(\sigma_1(K_1); O(2k_1)) \simeq S^1, \\ &s_2 : H' \to Z(\sigma_2(K_2); O(2k_2 - 1)) \simeq \mathbb{Z}_2, \\ &\mu_2 : H' \to N_{G''}(K_2)/K_2 \simeq S^1. \end{aligned}$$

With the method similar to that demonstrated in the proof of Proposition 9.4, we have that

$$s_1(h) = s_2(h)\mu_2(h)^{-1} \in S^1,$$

where $s_2(h) \in \mathbb{Z}_2 = \{\pm 1\} \subset S^1$. Therefore, we have the follow proposition:

Proposition 9.9. *Let $M = G' \times_{H'} M_1$ be a non-principal torus manifold with the trivial $\mu_{[-1,1]}$. If $M_1 = \mathbb{R}P(\mathbb{C}^{k_1} \oplus \mathbb{R}^{2k_2-1})$, then there exists a representation $\rho : T^a \times \mathcal{A} \to S^1$ such that (M, G) is essentially isomorphic to the following manifold:*

$$\begin{aligned} M &= \mathcal{S}(a; b) \times_{T^a \times \mathcal{A}} \mathbb{R}P(\mathbb{C}^{k_1}_\rho \oplus \mathbb{R}^{2k_2-1}), \\ G &= \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1) \times U(k_1) \times SO(2k_2 - 1), \end{aligned}$$

where $T^a \times \mathcal{A}$ acts on $\mathbb{C}^{k_1}_\rho$ by the scaler multiplication via ρ and on $\mathbb{R}^{2k_2-1}$ trivially.

9.2.7. *CASE (7).* Let (M_1, G'') be CASE (7), i.e.,

$$(\mathbb{R}P(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1}), SO(2k_1) \times SO(2k_2 - 1)).$$

Similarly, its two tubular neighborhoods are

$$\begin{aligned} G'' \times_{K_1} D^{2k_1} &\cong X_1 = \{[x : y] \in \mathbb{R}P(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1}) \mid y \neq 0,\ x \in D^{2k_1} \subset \mathbb{R}^{2k_1}\}; \\ G'' \times_{K_2} D^{2k_2-1} &\cong X_2 = \{[x : y] \in \mathbb{R}P(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1}) \mid x \neq 0,\ y \in D^{2k_2-1} \subset \mathbb{R}^{2k_2-1}\}. \end{aligned}$$

Using Lemma 9.2 and 9.3, we may define H'-action on X_i as follows:

$$\begin{aligned} &[x : y] \mapsto [s_1(h)x : y] \quad \text{for } X_1; \\ &[x : y] \mapsto [x : s_2(h)y] \quad \text{for } X_2, \end{aligned}$$

for some scaler representations

$$\begin{aligned} &s_1 : H' \to Z(\sigma_1(K_1); O(2k_1)) \simeq \mathbb{Z}_2, \\ &s_2 : H' \to Z(\sigma_2(K_2); O(2k_2 - 1)) \simeq \mathbb{Z}_2. \end{aligned}$$

With the method similar to that demonstrated in the proof of Proposition 9.4, we have that

$$s_1(h) = s_2(h)^{-1} \in \mathbb{Z}_2,$$

and the follow proposition:

Proposition 9.10. *Let $M = G' \times_{H'} M_1$ be a non-principal torus manifold with the trivial $\mu_{[-1,1]}$. If $M_1 = \mathbb{R}P(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$, then there exists a representation $\rho : T^a \times \mathcal{A} \to \mathbb{Z}_2$ such that (M, G) is essentially isomorphic to the following manifold:*

$$M = \mathcal{S}(a; b) \times_{T^a \times \mathcal{A}} \mathbb{R}P(\mathbb{R}^{2k_1}_\rho \oplus \mathbb{R}^{2k_2-1}),$$
$$G = \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1) \times SO(2k_1) \times SO(2k_2 - 1),$$

where $T^a \times \mathcal{A}$ acts on $\mathbb{R}^{2k_1}_\rho$ by the scaler multiplication via ρ and on $\mathbb{R}^{2k_2-1}$ trivially.

Note that in Proposition 9.10 M is equivariantly diffeomorphic to the following manifold:

$$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j+1} \times_{\mathcal{A}} \mathbb{R}P(\mathbb{R}^{2k_1}_\rho \oplus \mathbb{R}^{2k_2-1}) \right),$$

because the restricted representation $\rho|_{T^a}$ is trivial.

10. The case when $\mu_{[-1,1]}$ is non-trivial

Assume $\mu_{[-1,1]}$ is non-trivial. In this case, $K_1 \simeq K_2$ (isomorphism), i.e., $k_1 = k_2$. Therefore, there exist three possibilities; Theorem 7.1 (1), (2), (3). We first prove that the case when Theorem 7.1 (1) does not occur.

Lemma 10.1. *Let (M_1, G'') be $(P(\mathbb{C}^k \oplus \mathbb{C}^k), S(U(k) \times U(k)))$ with $k \geq 2$. Then, $\mu_{[-1,1]}$ is trivial.*

Proof. Let $\pi : M_1 \to M_1/G'' = [-1, 1]$ be the projection to the orbit space. Put $\pi^{-1}(-1) = G''/K_1$ and $\pi^{-1}(1) = G''/K_2$. Because $(P(\mathbb{C}^k \oplus \mathbb{C}^k), S(U(k) \times U(k)))$ is the standard action, we may put that $G''/K_1 = \{[z : 0] \in P(\mathbb{C}^k \oplus \mathbb{C}^k) \mid z \in \mathbb{C}^k\}$ and $G''/K_2 = \{[0 : w] \in P(\mathbb{C}^k \oplus \mathbb{C}^k) \mid w \in \mathbb{C}^k\}$.

Assume there exists a G''-involution f on M_1 induced from the non-trivial $\mu_{[-1,1]}$. Because Im $\mu_{[-1,1]} = O(1)$ acts non-trivially on $M_1/G'' = [-1, 1]$, such f induces the G''-equivariant diffeomorphism between G''/K_1 and G''/K_2. Put $f([z : 0]) = [0 : w]$ for some $z, w \in \mathbb{C}^k$. Because f is G''-equivariant, we have the following equations:

$$f([Az : 0]) = (A, B)f([z : 0]) = (A, B)[0 : w] = [0 : Bw]$$

for all $(A, B) \in S(U(k) \times U(k))$. However, this also implies that $[0 : w] = [0 : Bw]$ for all $(I_k, B) \in S(U(k) \times U(k))$, where I_k is the identity element of $U(k)$. This gives a contradiction to that G''/K_2 is $2k$-dimensional orbit and $k \geq 2$. Therefore, there is no G''-involution f on M_1 induced from the non-trivial $\mu_{[-1,1]}$. □ □

By Lemma 10.1, we may assume that (M_1, G'') is one of the followings:

$$(S(\mathbb{C}^k \oplus \mathbb{R}), U(k));$$
$$(S(\mathbb{R}^{2k} \oplus \mathbb{R}), SO(2k)),$$

where $k \in \mathbb{N}$. We often denote these manifolds by $S(V \oplus \mathbb{R})$, where V represents the complex k-dimensional manifold or the real $2k$-dimensional manifold.

We first analyze the induced H'-action on M_1 by non-trivial $\mu_{[-1,1]} : H' \to O(1)$. Now we may regard the orbit projection $\pi : M_1 \to M_1/G''$ as follows:

$$\begin{array}{ccc} S(V \oplus \mathbb{R}) & \xrightarrow{\pi} & [-1, 1] \\ \cup & & \cup \\ (z, r) & \longmapsto & r \end{array}$$

where $z \in V$ and $r \in \mathbb{R}$ such that $|z|^2 + r^2 = 1$. Therefore, the non-trivial $\mu_{[-1,1]}(h)$ acts on the element $(z, r) \in S(V \oplus \mathbb{R})$ as follows:

$$(z, r) \xmapsto{\mu_{[-1,1]}(h)} (w, -r)$$

for some $w \in V$ such that $|z| = |w|$. It follows from $|z| = |w|$ that there is an element $X \in G''$ such that

$$(z, r) \xmapsto{\mu_{[-1,1]}(h)} (Xz, -r).$$

Because $\mu_{[-1,1]}(h)$ is G''-equivariant, we see that $X \in Z(G'')$, i.e., the center of G''. Therefore, it follows from $\mu_{[-1,1]}(h)^2 = 1$ that

$$X = \pm 1.$$

Hence, we have the following lemma:

LEMMA 10.2. *Let $h \in H'$ be an element such that $\mu_{[-1,1]}(h) \in O(1)$ is non-trivial. Then, $\mu_{[-1,1]}(h)$ induces one of the following maps on $S(V \oplus \mathbb{R})$:*

$$(z, r) \mapsto (z, -r)$$

or

$$(z, r) \mapsto (-z, -r).$$

Let $H'' \subset H'$ be the kernel of $\mu_{[-1,1]} : H' \to O(1)$. Then, there exists the double covering

$$G' \times_{H''} M_1 \longrightarrow G' \times_{H'} M_1,$$

and $G' \times_{H''} M_1$ becomes a torus manifold with the trivial $\mu_{[-1,1]}$. Because M_1 is $S(V \oplus \mathbb{R})$, it follows from Proposition 9.5 and 9.6 that

$$G' \times_{H''} M_1 \cong \mathcal{S}(a; b) \times_{T^a \times \mathcal{A}'} S(V_{\rho'} \oplus \mathbb{R})$$

where $V_{\rho'}$ is the vector space V with a scaler representation ρ' and $\mathcal{A}' \subset (\mathbb{Z}_2)^b$. Here, we note that $\mathcal{A}'$ is a proper subgroup of $(\mathbb{Z}_2)^b$, because H'' is the kernel of $\mu_{[-1,1]}$. It follows from Lemma 10.2 that there exists the subgroup $\mathcal{A} \subset (\mathbb{Z}_2)^b$ such that $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{A}/\mathcal{A}' \simeq O(1) \simeq \mathbb{Z}_2$ and $G' \times_{H'} M_1$ is equivariantly diffeomorphic to

$$\mathcal{S}(a; b) \times_{T^a \times \mathcal{A}} S(V_\rho \oplus \mathbb{R}_\epsilon),$$

where ρ is a scaler representation whose restricted representation to $\mathcal{A}'$ coincides with ρ' and $\epsilon : \mathcal{A} \to O(1)$ is the surjective homomorphism whose kernel coincides with $\mathcal{A}'$. This establishes the following proposition:

PROPOSITION 10.3. *Let (M, G) be the torus manifold such that $M \cong G' \times_{H'} M_1$. If $\mu_{[-1,1]}$ is non-trivial, then there exist the following two cases up to essential isomorphism:*

M	G
$\mathcal{S}(a; b) \times_{T^a \times \mathcal{A}} S(\mathbb{C}^k_\rho \oplus \mathbb{R}_\epsilon)$	$G' \times U(k)$
$\mathcal{S}(a; b) \times_{T^a \times \mathcal{A}} S(\mathbb{R}^{2k}_\rho \oplus \mathbb{R}_\epsilon)$	$G' \times SO(2k)$

where $G' = \prod_{i=1}^a SU(l_i + 1) \times \prod_{j=1}^b SO(2m_j + 1)$, and G acts on M naturally.

Here, in Proposition 10.3, M is defined by the following quotient manifold: $T^a \times \mathcal{A} \subset T^a \times (\mathbb{Z}_2)^b$ acts on the product of spheres $\mathcal{S}(a; b)$ naturally, on $\mathbb{C}^k_\rho$ by the representation $\rho : T^a \times \mathcal{A} \to S^1$, and on $\mathbb{R}^{2k}_\rho$ by the representation $\rho : T^a \times \mathcal{A} \to \{\pm 1\}$; furthermore, $\mathcal{A}$ acts on $\mathbb{R}_\epsilon$ by some surjective homomorphism $\epsilon : \mathcal{A} \to O(1)$.

Consequently, by Propositions 9.4–9.10 and 10.3, we have the classification list in Theorem 1.1. Note that in Theorem 1.1 the representation ϵ might be trivial, i.e., the case when ϵ is trivial corresponds to Propositions 9.5 and 9.6, the case when ϵ is non-trivial corresponds to Proposition 10.3. As a corollary of the list in Theorem 1.1, we have Corollary 1.2.

11. Moment-angle manifolds and the orientability

In closing this paper, we prove the orientability of our manifolds in Proposition 9.4–9.10 and 10.3 by using similar objects with *moment-angle manifolds.* We first recall the moment-angle manifold (see [**BoMe, BuPa, DaJa**] for detail).

11.1. Moment-angle manifolds. Let P be a simple convex polytope with the set of facets $\mathcal{F} = \{F_1, \ldots, F_m\}$. For each facet $F_i \in \mathcal{F}$, the 1-dimensional coordinate subgroup of the m-torus $T^{\mathcal{F}} \simeq T^m$ corresponding to F_i is denoted by T_{F_i}, i.e.,

$$T_{F_i} = \{(1, \ldots, 1, t_i, 1 \ldots, 1) \in T^m \mid t_i \in S^1\},$$

where t_i is the i^{th} coordinate in T^m. Then assign to every face L the coordinate subtorus

$$T_L = \prod_{F_i \supset L} T_{F_i} \subset T^{\mathcal{F}}.$$

For every point $q \in P$, $L(q)$ denotes the unique face containing q in its relative interior. Then a *moment-angle manifold* $\mathcal{Z}_P$ *over* P is defined by the identification space

$$\mathcal{Z}_P = (T^{\mathcal{F}} \times P)/\sim,$$

such that $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1^{-1} t_2 \in T_{L(p)}$. Note that moment-angle manifolds $\mathcal{Z}_P$ have natural T^m-actions on their $T^{\mathcal{F}}$ factors.

Moreover, we have the following relations between quasitoric manifolds M over P and the moment-angle manifold $\mathcal{Z}_P$ over P (see [**BuPa**, Proposition 6.5]):

PROPOSITION 11.1. *Let M be the quasitoric manifold whose orbit space is a simple polytope P. Let m be the number of facets of P, and n be the dimension of P. Then, there is the subtorus $H \subset T^{\mathcal{F}}$ such that $H \simeq T^{m-n}$ and H acts freely on $\mathcal{Z}_P$. Furthermore, this freely H-action induces the principal T^{m-n}-bundle $\mathcal{Z}_P \to M$ as the orbit projection.*

Next we shall show the moment-angle manifold over the quasitoric manifold in Corollary 1.2, i.e., $M = \prod_{i=1}^a S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2})$. The orbit space of M becomes the product of simplices $\prod_{i=1}^a \Delta^{l_i} \times \Delta^{k_1+k_2-1}$. Recall the following two formulas:

(11) $$\mathcal{Z}_{P_1 \times P_2} = \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2};$$

(12) $$\mathcal{Z}_{\Delta^n} = S^{2n+1}.$$

Here, the formula for the product of polytopes (11) is due to [**BuPa**, Proposition 6.4] and the moment-angle manifold over the simplex (12) is due to [**BuPa**, Example 6.7]. By using these formulas (11) and (12), the moment-angle manifold over $P = \prod_{i=1}^a \Delta^{l_i} \times \Delta^{k_1+k_2-1}$ is as follows:

$$\mathcal{Z}_P = \prod_{i=1}^a S^{2l_i+1} \times S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}),$$

where $S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}) \cong S^{2k_1+2k_2-1}$.

Note that the number of facets of $\prod_{i=1}^a \Delta^{l_i} \times \Delta^{k_1+k_2-1}$ and its dimension are

$$m = \sum_{i=1}^a (l_i + 1) + k_1 + k_2 \quad \text{and} \quad n = \sum_{i=1}^a l_i + k_1 + k_2 - 1,$$

respectively. Therefore, by using Corollary 1.2 or Proposition 9.4, the subgroup which acts on $\mathcal{Z}_P$ freely is

$$H = T^a \times S^1.$$

By definitions of M and $\mathcal{Z}_P$, this group $H = T^a \times S^1$ acts on $\mathcal{Z}_P$ as follows:

(1) $T^a \subset H$ acts naturally on the $\prod_{i=1}^a S^{2l_i+1}$ factor, and acts on the $S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}) \cap \mathbb{C}_\rho^{k_1}$ factor via the representation $\rho : T^a \to S^1$;
(2) $S^1 \subset H$ acts only on the $S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}) = S^{2k_1+2k_2-1} \subset \mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}$ factor naturally as the scaler multiplication.

One can easily show that $\mathcal{Z}_P$ has the natural action of $G = \prod_{i=1}^a SU(l_i+1) \times S(U(k_1) \times U(k_2))$, with codimension one principal orbits $\prod_{i=1}^a S^{2l_i+1} \times S^{2k_1-1} \times S^{2k_2-1}$, and two singular orbits $\prod_{i=1}^a S^{2l_i+1} \times S^{2k_1-1}$ and $\prod_{i=1}^a S^{2l_i+1} \times S^{2k_2-1}$. Furthermore, this G-action on $\mathcal{Z}_P$ commutes with the $H = T^a \times S^1$-action and induces the codimension one action on M. Similarly, we have this fact for quasitoric manifolds with codimension 0 extended G-actions (such quasitoric manifolds are only products of complex projective spaces, see [**Ku3**]), i.e., all transitive actions on quasitoric manifolds can be induced from transitive actions on moment-angle manifolds. Hence, we have the following theorem by using the argument as above and our classification results.

THEOREM 11.2. *Assume a (quasi)toric manifold M^{2n} has a codimension* 0 *or* 1 *extended G-actions. Then, there exists the principal T^{a+1}-bundle*

$$\mathcal{Z}_P = \prod_{i=1}^{a+1} S^{2l_i+1} \to M^{2n}$$

such that we can lift the codimension 0 *(resp.* 1*) extended G-actions on M to the G-action on $\mathcal{Z}_P$ with codimension* 0 *(resp.* 1*) principal orbits. In other wards, all of codimension* 0 *and* 1 *extended G-actions on M are induced from G-actions on $\mathcal{Z}_P$ with codimension* 0 *and* 1 *principal orbits, respectively.*

REMARK 11.3. We can easily show that two singular orbits of $(\mathcal{Z}_P, G)$ are moment-angle manifolds of two singular orbits of (M, G), respectively.

11.2. Orientability. We next analyze orientabilities of torus manifolds with codimension one extended actions. We first show the following general property:

PROPOSITION 11.4. *Let E be the total space of fibre bundle, and F its fibre. If the manifold F is non-orientable, then the manifold E is also non-orientable.*

PROOF. Assume F is non-orientable. As is well known, the 1^{st} Stiefel-Whitney class $w_1(M) = 0$ if and only if the manifold M is orientable (see e.g. [**MiSt**]). Therefore, $w_1(F) \neq 0$.

Let ι be an embedding of F into E. Then, its pull-back of the tangent bundle $\iota^*\tau_E$ can be decomposed into $\tau_F \oplus \nu_F$, where τ_F is the tangent bundle of F and ν_F is its normal bundle in E. Because of the local triviality condition of the fibre bundle, we see that ν_F is the trivial bundle. This implies that the total Stiefel-Whitney class satisfies

$$\iota^* w(E) = w(\tau_F \oplus \nu_F) = w(F)w(\nu_F) = w(F).$$

It follows from $w_1(F) \neq 0$ that $w_1(E) \neq 0$. This establishes the statement of proposition. □ □

It follows from Proposition 11.4 that manifolds appearing in Proposition 9.9 and 9.10 never become orientable.

Due to Proposition 9.4–9.8 and 10.3, we can easily show that there is a similar principal $(T^a \times \mathcal{A})$-bundle such as the moment-angle manifold in Theorem 11.2, i.e., we can define the following principal $(T^a \times \mathcal{A})$-bundle such as Theorem 11.2:

$$\mathcal{Z} = \mathcal{S}(a; b) \times S(V_{\rho_V} \oplus W_{\rho_W}) \longrightarrow M$$

where M is a torus manifold with codimension one extended action, the symbols V_{ρ_V}, W_{ρ_W} represent the representation spaces appearing in Proposition 9.4–9.8 and 10.3, and ρ_V, ρ_W represent the scaler representation of $T^a \times \mathcal{A}$. Here, $T^a \times \mathcal{A}$ acts on $\mathcal{S}(a; b)$-factor naturally and on $S(V_{\rho_V} \oplus W_{\rho_W})$-factor by the representation $\rho_V \oplus \rho_W$. By definition, the T^a-action on $\mathcal{Z}$ preserves its orientation. Hence, M is orientable if and only if $\mathcal{A}$ preserves the orientation of $\mathcal{Z}$.

We analyze when $\mathcal{A}$-action preserves the orientation. Because of the definition of $\mathcal{A}$-action, we have that this action is induced from the following representation:

$$\chi : \mathcal{A} \xrightarrow{\iota \oplus \rho_V \oplus \rho_W} (\mathbb{Z}_2)^{b+2} \subset \prod_{j=1}^{b} O(2m_j + 1) \times O(\dim V_{\rho_V}) \times O(\dim W_{\rho_W})$$

$$\subset O(\sum_{j=1}^{b} 2m_j + b + \dim V_{\rho_V} + \dim W_{\rho_W}),$$

where $\prod_{j=1}^{b} O(2m_j+1) \times O(\dim V_{\rho_V}) \times O(\dim W_{\rho_W})$ acts naturally on $\prod_{j=1}^{b} S^{2m_j} \times S(V_{\rho_V} \oplus W_{\rho_W})$, the group $(\mathbb{Z}_2)^{b+2}$ is the diagonal group $\{\pm I\}^{b+2}$, and $\iota : \mathcal{A} \to (\mathbb{Z}_2)^b$ is the embedding. Therefore, one can easily show that the determinant of all elements in $\chi(\mathcal{A})$ is 1 if and only if $\mathcal{A}$-action preserves the orientation of $\mathcal{Z}$. Because V_{ρ_V} is an even dimensional vector space, we have

$$\rho_V(\mathcal{A}) = \chi(\mathcal{A}) \cap O(\dim V_{\rho_V}) \subset SO(\dim V_{\rho_V}).$$

Hence, it is easy to check the following proposition:

Proposition 11.5. *Let M be a torus manifolds with codimension one extended action. Then, for the orientability of M, the following statements hold:*

(1) *if M is one of manifolds appearing in Proposition 9.4–9.6, then M is orientable if and only if*

$$\mathcal{A} \subset SO(\sum_{j=1}^{b} 2m_j + b);$$

(2) *if M is one of manifolds appearing in Proposition 9.7 and 9.8, then M is orientable if and only if*

$$\{(a, \rho_2(a)) \in \prod_{j=1}^{b} O(2m_j + 1) \times O(2k_2 - 1) \mid a \in \mathcal{A}\} \subset SO(\sum_{j=1}^{b} 2m_j + b + 2k_2 - 1);$$

(3) *if M is one of manifolds appearing in Proposition 9.9 and 9.10, then M is non-orientable;*

(4) *if M is one of manifolds appearing in Proposition 10.3, then M is orientable if and only if*

$$\{(a, \epsilon(a)) \in \prod_{j=1}^{b} O(2m_j + 1) \times O(1) \mid a \in \mathcal{A}\} \subset SO(\sum_{j=1}^{b} 2m_j + b + 1).$$

Using Proposition 9.4–9.10, 10.3 and 11.5, we get Theorem 1.1.

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References

[AlAl] A. V. Alekseevskii and D. V. Alekseevskii, *G-manifolds with one-dimensional orbit space*, Adv. in Soviet Math., **8** (1992), 1–31.

[BoMe] F. Bosio and L. Meersseman, *Real quadrics in C^n, complex manifolds and convex polytopes*, Acta Math., **197** (2006), 53–127.

[Br] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, 1972.

[BuPa] V. M. Buchstaber and T. E. Panov, *Torus actions and their applications in topology and combinatorics*, Amer. Math. Soc., 2002.

[DaJa] M. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus action*, Duke. Math. J., **62** (1991), 417–451.

[Fu] W. Fulton, *An Introduction to Toric Varieties*, Ann. of Math. Studies 113, Princeton Univ. Press, Princeton, N.J., 1993.

[Ga] A. Gambioli, *$SU(3)$-manifolds of cohomogeneity one*, Ann. Global Anal. Geom. **34** (2008), no. 1, 77–100.

[GHZ] V. Guillemin, T. S. Holm and C. Zara, *A GKM description of the equivariant cohomology ring of a homogeneous space*, J. Algebraic Combin., **23** (2006), no. 1, 21–41.

[HaMa] A. Hattori and M. Masuda, *Theory of multi-fans*, Osaka. J. Math., **40** (2003), 1–68.

[HsHs] W. C. Hsiang and W. Y. Hsiang, *Classification of differentiable actions on S^n, R^n and D^n with S^k as the principal orbit type*, Ann. of Math., **82** (1965), 421–433.

[Ka] K. Kawakubo, *The theory of transformation groups*, Oxford Univ. Press, London, 1991.

[Ku1] S. Kuroki, *On transformation groups which act on torus manifolds*, in Proceedings of 34th Symposium on Transformation Groups, (T. Kawakami ed.); Wing Co., Wakayama (2007), 10–26.

[Ku2] S. Kuroki, *Classification of compact transformation groups on complex quadrics with codimension one orbits*, Osaka J. Math., **46** (2009), 21–85.

[Ku3] S. Kuroki, *Characterization of homogeneous torus manifolds*, Osaka J. Math., **47** (2010) 285–299.

[Ku4] S. Kuroki, *Classification of quasitoric manifolds with codimension one extended actions*, OCAMI preprint series 09-4 (2009).

[Ku5] S. Kuroki, *Classification of torus manifolds with codimension one extended actions*, OCAMI preprint series 09-5 (2009).

[Ma] M. Masuda, *Unitary toric manifolds, multi-fans and Equivariant index*, Tôhoku Math. J., **51** (1999), 237–265.

[MaPa] M. Masuda and T. Panov, *On the cohomology of torus manifolds*, Osaka J. Math., **43** (2006), 711–746.

[MiSt] J.W. Milnor, J.D. Stasheff, Characteristic classes, Princeton Univ. Press, 1974.

[MiTo] M. Mimura and H. Toda, *Topology of Lie Groups, I and II*, Amer. Math. Soc., 1991.

[MoSa] D. Montgomery and H. Samelson, *Transformation groups of spheres*, Ann. of Math., **44** (1943), 454–470.

[Od] T. Oda, *Convex bodies and algebraic geometry: An introduction to the theory of toric varieties*, Springer, New York (1988).

[Uc] F. Uchida, *Classification of compact transformation groups on cohomology complex projective spaces with codimension one orbits*, Japan. J. Math. Vol. 3, No. 1, (1977), 141–189.

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