HAMILTONIAN STABILITY OF THE GAUSS IMAGES OF HOMOGENEOUS ISOPARAMETRIC HYPERSURFACES

HUI MA AND YOSHIHIRO OHNITA

Abstract. The image of the Gauss map of any oriented isoparametric hypersurface of the unit standard sphere $S^{n+1}(1)$ is a minimal Lagrangian submanifold in the complex hyperquadric $Q_n(\mathbf{C})$. In this paper we determine the Hamiltonian stability of all compact minimal Lagrangian submanifolds embedded in complex hyperquadrics which are obtained as the images of the Gauss map of homogeneous isoparametric hypersurfaces in the unit spheres.

Introduction

In 1990's Oh initialed the study of Hamiltonian minimality and Hamiltonian stability of Lagrangian submanifolds in Kähler manifolds ([28], [29], [30]). It provides a suitable constrained volume variational problem of Lagrangian submanifolds in Kähler manifolds under Hamiltonian deformations. Recall that a submanifold L is an n-dimensional submanifold of a symplectic manifold (M^{2n}, ω) on which the symplectic form vanishes identically. The Lagrangian property is preserved by Hamiltonian deformations in M. So it is natural to define in a Kähler manifold (M, ω, g, J) that a Lagrangian submanifold is *Hamiltonian minimal* ([28]) or *Hamiltonian stationary* ([38]), if it is the critical point of the volume functional under any Hamiltonian deformation of L. Similarly, a Hamiltonian minimal Lagrangian submanifold in a Kähler manifold is said to be *Hamiltonian stable*, if the second variational of the volume is nonnegative under every Hamiltonian deformation. Minimal Lagrangian submanifolds are apparently Hamiltonian minimal. It

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is a natural and interesting problem what Lagrangian submanifolds in specific Kähler manifolds are Hamiltonian stable. After Oh's pioneer papers, there has been extensive research done on Hamiltonian stabilities of minimal or Hamiltonian minimal Lagrangian submanifolds in various Kähler manifolds, such as complex Euclidean spaces, complex projective spaces, compact Hermitian symmetric spaces, certain toric Kähler manifolds and so on. (See e.g., [2, 7, 33, 34, 37, 42] and references therein.)

It is known that the Gauss map of an oriented hypersurface N^n in the unit standard sphere $S^{n+1}(1)$ is a Lagrangian immersion of N^n into the complex hyperquadric $Q_n(\mathbf{C})$. It follows from Palmer's mean curvature formula [37] that if an oriented hypersurface N^n in $S^{n+1}(1)$ is of constant principal curvatures, the so called "isoparametric hypersurface", then its Gauss map is a minimal Lagrangian immersion of N^n into $Q_n(\mathbf{C})$. Münzner [26], [27] showed that the number g of distinct principal curvatures of N^n in $S^{n+1}(1)$ and their multiplicities $m_1, \cdots, m_g$ must be $g = 1, 2, 3, 4, 6$ and $m_1 = m_3 = \cdots \leq m_2 = m_4 = \cdots$, and N^n is always real algebraic in the sense that N^n is defined by a certain real homogeneous polynomial of degree g which is called the "Cartan-Münzner polynomial". We observe that the image of the Gauss map of any compact oriented isoparametric hypersurface N^n embedded in $S^{n+1}(1)$ is a *smooth* compact minimal Lagrangian submanifold embedded in $Q_n(\mathbf{C})$ and the Gauss map is a covering map with covering transformation group $\mathbb{Z}_g$.

An isoparametric hypersurface N^n in the unit standard sphere $S^{n+1}(1)$ is called *homogeneous* if N^n can be obtained as an orbit of a compact Lie subgroup of $SO(n+2)$. All isoparametric hypersurfaces in the unit standard sphere are classified into homogeneous ones and non-homogeneous ones. Every homogeneous isoparametric hypersurface in a sphere can be obtained as a principal orbit of a linear isotropy representation of a compact Riemannian symmetric pair (U, K) of rank 2, due to Hsiang-Lawson [14] and Takagi-Takahashi [39] (cf. Dadok, Asoh, Ucihda, Yasukura). Only in the case of $g = 4$ there are known to exist non-homogeneous isoparametric hypersurfaces, which were discovered first by Ozeki-Takeuchi [35], [36] and extensively generalized by Ferus-Karcher-Münzner [10].

In this paper we determine the Hamiltonian stability of all compact minimal Lagrangian embedded submanifolds in $Q_n(\mathbf{C})$ which are obtained as the Gauss images of homogeneous isoparametric hypersurfaces in $S^{n+1}(1)$. This paper is a continuation of [21], where we have already treated the cases of $g = 1, 2, 3$.

The main result of this paper is as follows :

Theorem. *Suppose that* (U,K) *is not of type* EIII*, that is,* $(U,K) \neq (E_6, U(1)\cdot Spin(10))$*. Then* $L = \mathcal{G}(N)$ *is not Hamiltonian stable if and only if* $m_2 - m_1 \geq 3$*. Moreover if* (U,K) *is of type* EIII*, that is,* $(U,K) = (E_6, U(1)\cdot Spin(10))$*, then* $(m_1, m_2) = (6,9)$ *but* $L = \mathcal{G}(N)$ *is strictly Hamiltonian stable.*

This paper is organized as follows : In Section 1 we recall the notion and fundamental properties on Hamiltonian minimality, Hamiltonian stability and strictly Hamiltonian stability of Lagrangian submanifolds in Kähler manifolds. In Section 2 we briefly explain properties of minimal Lagrangian submanifolds in complex hyperquadrics as the Gauss image of isoparametric hypersurfaces in spheres, which were discussed in our previous works. In Section 3 we explain the method of eigenvalue computations of our compact homogeneous spaces which are the Gauss images of compact homogeneous isoparametric hypersurfaces in spheres, and the fibrations on homogeneous isoparametric hypersurfaces by homogeneous isoparametric hypersurfaces. The fibrations are very useful for our computation. In Sections 4 and 5, we determine the strictly Hamilonian stability of the Gauss images of compact homogeneous isoparametric hypersurfaces with $g = 6$. In Sections 6-11, we determine the strictly Hamilonian stability of the Gauss images of compact homogeneous isoparametric hypersurfaces with $g = 4$.

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1. Hamiltonian minimality and Hamiltonian stability

Assume that (M, ω, J, g) is a Kähler manifold with the compatible complex structure J and Kähler metric g. Let $\varphi : L \to M$ be a Lagrangian immersion and H denote the mean curvature vector field of φ. The corresponding 1-form $\alpha_{\mathrm{H}} := \omega(\mathrm{H}, \cdot) \in \Omega^1(L)$ is called the *mean curvature form* of φ. For simplicity, throughout this paper we assume that L is compact without boundary.

Definition 1.1. Let M be a Kähler manifold. A Lagrangian immersion $\varphi : L \to M$ is called *Hamiltonian minimal* (shortly, H-minimal) or *Hamiltonian stationary*, if it is the critical point of the volume functional for all Hamiltonian deformations $\{\varphi_t\}$.

The corresponding Euler-Lagrange equation is $\delta\alpha_{\mathrm{H}} = 0$, where δ is the codifferential with the respect to the induced metric on L.

Definition 1.2. An H-minimal Lagrangian immersion φ is called *Hamiltonian stable* (shortly, *H-stable*) if the second variation of the volume is nonnegative under every Hamiltonian deformation $\{\varphi_t\}$.

The second variational formula is given as follows ([30]):

$$\begin{aligned}&\frac{d^2}{dt^2}\mathrm{Vol}\,(L, \varphi_t^* g)|_{t=0}\\ &= \int_L \left(\langle \Delta_L^1 \alpha, \alpha\rangle - \langle \overline{R}(\alpha), \alpha\rangle - 2\langle \alpha \otimes \alpha \otimes \alpha_{\mathrm{H}}, S\rangle + \langle \alpha_{\mathrm{H}}, \alpha\rangle^2\right) dv\end{aligned}$$

where Δ_L^1 denotes the Laplace operator of $(L, \varphi^* g)$ acting on the vector space $\Omega^1(L)$ of smooth 1-forms on L and $\alpha := \omega(V, \cdot) \in B^(L)$ is the exact 1-form corresponding to an infinitesimal Hamiltonian deformation V. Here,

$$\langle \overline{R}(\alpha), \alpha\rangle := \sum_{i,j=1}^{n} \mathrm{Ric}^M(e_i, e_j)\alpha(e_i)\alpha(e_j),$$

for a local orthonormal frame $\{e_i\}$ on L and

$$S(X, Y, Z) := \omega(B(X, Y), Z)$$

for each $X, Y, Z \in C^\infty(TL)$, which is a symmetric 3-tensor field on L defined by the second fundamental form B of L in M.

For an H-minimal Lagrangian immersion $\varphi : L \to M$, denote by $E_0(\varphi)$ the null space of the second variation on $B^1(L)$ and $n(\varphi) := \dim E_0(\varphi)$ is called the *nullity* of φ.

If $H^1(M, \mathbf{R}) = \{0\}$, then any holomorphic Killing vector field on M is a Hamiltonian vector field, and thus it generates a volume-preserving Hamiltonian deformation of φ. Namely,

$$\{\varphi^*\alpha_X \mid X \text{ is a holomorphic Killing vector field on } M\} \subset E_0(\varphi) \subset B^1(L).$$

Set $n_{hk}(\varphi) := \dim\{\varphi^*\alpha_X \mid X \text{ is a holomorphic Killing vector field on } M\}$, which is called the *holomorphic Killing nullity* of φ.

Definition 1.3. An H-minimal Lagrangian immersion φ is called *strictly Hamiltonian stable* (shortly, strictly H-stable) if φ is Hamiltonian stable and $n_{hk}(\varphi) = n(\varphi)$.

Note that if L is strictly Hamiltonian stable, then L has local minimum volume under each Hamiltonian deformation.

In the case when L is a compact minimal Lagrangian submanifold in an Einstein-Käher manifold M with Einstein constant κ, the second variational formula becomes much simpler. It follows that L is H-stable if and only if the first (positive) eigenvalue λ_1 of the Laplacian of L acting on functions satisfies $\lambda_1 \geq \kappa$ ([28]). On the other hand, the upper bound of the first eigenvalue λ_1 of the Laplacian of a compact minimal Lagrangian submanifold L in a compact homogenous Einstein-Kähler manifold with positive Einstein constant κ is given by $\lambda_1 \leq \kappa$ ([32], [33]). Combing with the argument of H-stability, we get in this case, L is H-stable if and only if $\lambda_1 = \kappa$.

Assume that (M, ω, J, g) is a Kähler manifold and G is an analytic subgroup of its automorphism group $\mathrm{A}ut(M, \omega, J, g)$. A Lagrangian orbit $L = G \cdot x \subset M$ of G is called a *homogeneous Lagrangian submanifold* of M. An easy but useful observation can be given as follows.

Proposition 1.1. *Any compact homogeneous Lagrangian submanifold in a Kähler manifold is Hamiltonian minimal.*

Proof. Since α_{H} is an invariant 1-form on L, $\delta\alpha_{\mathrm{H}}$ is a constant function on L. Hence by the divergence theorem we obtain $\delta\alpha_{\mathrm{H}} = 0$. $\square$

Set

$$\tilde{G} := \{a \in \mathrm{Aut}(M, \omega, J, g) \mid a(L) = L\}.$$

Then $G \subset \tilde{G}$ and $\tilde{G}$ is the maximal subgroup of $\mathrm{Aut}(M, \omega, J, g)$ preserving L. Moreover we have $n_{hk}(\varphi) = \dim(\mathrm{Aut}(M, \omega, J, g)) - \dim(\tilde{G})$.

2. Gauss maps of isoparametric hypersurfaces in a sphere

2.1. **Gauss maps of oriented hypersurfaces in spheres.** Let $N^n \subset S^{n+1}(1) \subset \mathbf{R}^{n+2}$ be an oriented hypersurface in the unit sphere. Denote by $\mathbf{x}$ its position vector of a point p of N and $\mathbf{n}$ the unit normal vector field of N in $S^{n+1}(1)$. It is a fundamental fact in symplectic geometry that the *Gauss map* defined by

$$\mathcal{G} : N^n \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \cong [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] \in \widetilde{\mathrm{Gr}}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C})$$

is always a Lagrangian immersion in the complex hyperquadric $Q_n(\mathbf{C})$.

Let $g^{std}_{Q_n(\mathbf{C})}$ be the standard Kähler metric of $Q_n(\mathbf{C})$ induced from the standard inner product of $\mathbf{R}^{n+2}$. Note that the Einstein constant of $g^{std}_{Q_n(\mathbf{C})}$ is equal to n. Let κ_i $(i = 1, \cdots, n)$ denote the principal curvatures of $N^n \subset S^{n+1}(1)$ and H denote the mean curvature vector field of the Gauss map $\mathcal{G}$. Palmer gave the following mean curvature form formula ([37]):

$$\alpha_{\mathrm{H}} = d\left(\mathrm{Im}\left(\log \prod_{i=1}^{n}(1 + \sqrt{-1}\kappa_i)\right)\right).$$

Hence, if N^n is an oriented austere hypersurface in $S^{n+1}(1)$, introduced by Harvey-Lawson [13], then its Gauss map $\mathcal{G} : N^n \to Q_n(\mathbf{C})$ is a minimal Lagrangian immersion. In particular, since any minimal surface in $S^3(1)$ is austere, its Gauss map is a minimal Lagrangian immersion in $Q_2(\mathbf{C}) \cong S^2 \times S^2$ ([7]). Note that more minimal Lagrangian submanifolds of complex hyperquadrics can be obtained from Gauss maps of certain oriented hypersurfaces in spheres through Palmer's formula.

2.2. **Gauss maps of isoparametric hypersurfaces in spheres.** Now suppose that N^n is a compact oriented hypersurface in $S^{n+1}(1)$ with constant principal curvatures, i.e., *isoparametric hypersurface*. By Müzner's result ([26, 27]), the number g of distinct principal curvatures must be $1, 2, 3, 4$ or 6. It follows from Palmer's mean curvature form formula that its Gauss map $\mathcal{G} : N^n \to Q_n(\mathbf{C})$ is a minimal Lagrangian immersion. Moreover, the "Gauss images "of $\mathcal{G}$ gives a compact embedded minimal Lagrangian submanifold $L^n = \mathcal{G}(N^n) \cong N^n/\mathbf{Z}_g$ in $Q_n(\mathbf{C})$.

Münzner showed that the distinct principal curvatures of an isoparametric hypersurface in a sphere have at most two different multiplicities m_1, m_2. In the following, we assume that $m_1 \leq m_2$.

All isoparametric hypersurfaces in spheres are classified into homogeneous one and non-homogeneous ones. A hypersurface N^n in $S^{n+1}(1)$ is *homogeneous* if it is obtained as an orbit of a compact connected subgroup G of $SO(n+2)$. Obviously a homogeneous hypersurface in $S^{n+1}(1)$ is an isoparametric hypersurface. It turns out that N^n is homogeneous if and only if its Gauss image $\mathcal{G}(N^n)$ is homogeneous ([21]). Due to Hsiang-Lawson ([13]) and Takagi-Takahashi ([39]), any homogeneous isoparametric hypersurface in a sphere can be obtained as a principal orbit of the isotropy representation of a Riemannian symmetric pair (U, K) of rank 2 (Table 1). The construction of homogeneous isoparametric hypersurfaces are obtained in the following way (cf. [21]).

Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{u}$ as a symmetric Lie algebra of a symmetric pair (U, K) of rank 2 and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Define an $\mathrm{Ad}U$-invariant inner product $\langle\, ,\, \rangle_{\mathfrak{u}}$ of $\mathfrak{u}$ from the Killing-Cartan form of $\mathfrak{u}$. Then the vector space $\mathfrak{p}$ can be identified with the Euclidean space $\mathbf{R}^{n+2}$ with respect to the inner product $\langle\, ,\, \rangle_{\mathfrak{u}}$ and the $(n+1)$-dimensional unit sphere $S^{n+1}(1)$ can be defined in $\mathfrak{p}$ naturally. The linear isotropy action $\mathrm{Ad}_{\mathfrak{p}}$ of K on $\mathfrak{p}$ and thus $S^{n+1}(1)$ induces the group action of K on $\widetilde{\mathrm{Gr}}_2(\mathfrak{p}) \cong Q_n(\mathbf{C})$. For each *regular* element H of $\mathfrak{a} \cap S^{n+1}(1)$, we get a homogeneous isoparametric hypersurface in the unit sphere

$$N^n = (\mathrm{Ad}_{\mathfrak{p}} K)H \subset S^{n+1}(1) \subset \mathfrak{p} \cong \mathbf{R}^{n+2}.$$

Its Gauss image is

$$\mathcal{G}(N^n) = K \cdot [\mathfrak{a}] = [(\mathrm{Ad}_{\mathfrak{p}} K)\mathfrak{a}] \subset \widetilde{\mathrm{Gr}}_2(\mathfrak{p}) \cong Q_n(\mathbf{C}).$$

Here N and $\mathcal{G}(N^n)$ have homogeneous space expressions $N \cong K/K_0$ and $\mathcal{G}(N^n) \cong K/K_{[\mathfrak{a}]}$, where

$$\begin{aligned} K_0 &:= \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(H) = H\} \\ &= \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(H) = H \text{ for each } H \in \mathfrak{a}\}, \\ K_{\mathfrak{a}} &:= \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(\mathfrak{a}) = \mathfrak{a}\}, \\ K_{[\mathfrak{a}]} &:= \{k \in K_{\mathfrak{a}} \mid \mathrm{Ad}_{\mathfrak{p}}(k) : \mathfrak{a} \longrightarrow \mathfrak{a} \text{ preserves the orientation of } \mathfrak{a}\}. \end{aligned}$$

The deck transformation group of the covering map $\mathcal{G} : N \to \mathcal{G}(N^n)$ is equal to $K_{[\mathfrak{a}]}/K_0 = W(U,K)/\mathbf{Z}_2 \cong \mathbf{Z}_g$, where $W(U,K) = K_{\mathfrak{a}}/K_0$ is the Weyl group of (U,K).

TABLE 1. Homogeneous isoparametric hypersurfaces in spheres

g	*Type*	(U,K)	dimN	m_1, m_2	K/K_0
1	$S^1\times$ BDII	$(S^1 \times SO(n+2), SO(n+1))$ $(n \geq 1)$ $[\mathbf{R} \oplus A_1]$	n	n	S^n
2	BDII$\times$ BDII	$(SO(p+2) \times SO(n+2-p),$ $SO(p+1) \times SO(n+1-p))$ $(1 \leq p \leq n-1)$ $[A_1 \oplus A_1]$	n	$p, n-p$	$S^p \times S^{n-p}$
3	AI_2	$(SU(3), SO(3))\,[A_2]$	3	$1, 1$	$\frac{SO(3)}{\mathbf{Z}_2+\mathbf{Z}_2}$
3	$\mathfrak{a}_2$	$(SU(3) \times SU(3), SU(3))\,[A_2]$	6	$2, 2$	$\frac{SU(3)}{T^2}$
3	AII_2	$(SU(6), Sp(3))\,[A_2]$	12	$4, 4$	$\frac{Sp(3)}{Sp(1)^3}$
3	EIV	$(E_6, F_4)\,[A_2]$	24	$8, 8$	$\frac{F_4}{Spin(8)}$
4	$\mathfrak{b}_2$	$(SO(5) \times SO(5), SO(5))\,[B_2]$	8	$2, 2$	$\frac{SO(5)}{T^2}$
4	AIII_2	$(SU(m+2), S(U(2) \times U(m)))$ $(m \geq 2)$ $[BC_2](m \geq 3), [B_2](m = 2)$	$4m-2$	$2,$ $2m-3$	$\frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))}$
4	BDI_2	$(SO(m+2), SO(2) \times SO(m))$ $(m \geq 3)$ $[B_2]$	$2m-2$	$1,$ $m-2$	$\frac{SO(2)\times SO(m)}{\mathbf{Z}_2\times SO(m-2)}$
4	CII_2	$(Sp(m+2), Sp(2) \times Sp(m))$ $(m \geq 2)$ $[BC_2](m \geq 3), [B_2](m = 2)$	$8m-2$	$4,$ $4m-5$	$\frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)}$
4	DIII_2	$(SO(10), U(5))\,[BC_2]$	18	$4, 5$	$\frac{U(5)}{SU(2)\times SU(2)\times U(1)}$
4	EIII	$(E_6, U(1) \cdot Spin(10))\,[BC_2]$	30	$6, 9$	$\frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)}$
6	$\mathfrak{g}_2$	$(G_2 \times G_2, G_2)\,[G_2]$	12	$2, 2$	$\frac{G_2}{T^2}$
6	G	$(G_2, SO(4))\,[G_2]$	6	$1, 1$	$\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}$

2.3. **A remark on strictly Hamiltonian stability and Hamiltonian rigidiy of Gauss images of isoparametic hypersurfaces.** Consider

$$\mathcal{G} : N^n \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \in \widetilde{Gr}_2(\mathbf{R}^{n+2}) \subset \bigwedge^2 \mathbf{R}^{n+2}.$$

Here $\bigwedge^2 \mathbf{R}^{n+2} \cong \mathfrak{o}(n+2)$ can be identified with the Lie algebra of all (holomorphic) Killing vector fields on $S^{n+1}(1)$ or $\widetilde{Gr}_2(\mathbf{R}^{n+2})$. Let $\tilde{\mathfrak{k}}$ be the Lie subalgebra of $\mathfrak{o}(n+2)$ consisting of all Killing vector fields tangent to N^n or $\mathcal{G}(N^n)$ and $\tilde{K}$ be an analytic subgroup of $SO(n+2)$ generated by $\tilde{\mathfrak{k}}$. Take the orthogonal direct sum

$$\bigwedge^2 \mathbf{R}^{n+2} = \tilde{\mathfrak{k}} + \mathcal{V},$$

where $\mathcal{V}$ is a vector subspace of $\mathfrak{o}(n+2)$. The linear map

$$\mathcal{V} \ni X \longmapsto \alpha_X|_{\mathcal{G}(N^n)} \in E_0(\mathcal{G}) \subset B^1(\mathcal{G}(N^n))$$

is injective and $n_{hk}(\mathcal{G}) = \dim \mathcal{V}$. Then $\mathcal{G}(N^n) \subset \mathcal{V}$ and thus $\mathcal{G}(N^n) \subset \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V}$. Indeed, for each $X \in \tilde{\mathfrak{k}}$ and each $p \in N^n$, $\langle X, \mathbf{x}(p) \wedge \mathbf{n}(p) \rangle = \langle X\mathbf{x}(p), \mathbf{n}(p) \rangle - \langle \mathbf{x}(p), X\mathbf{n}(p) \rangle = 2\langle X\mathbf{x}(p), \mathbf{n}(p) \rangle = 0$.

Note that $\mathcal{G}(N^n)$ is a compact minimal submanifold embedded in the unit hypersphere of $\mathcal{V}$ and by the theorem of Tsunero Takahashi each coordinate function of $\mathcal{V}$ restricted to $\mathcal{G}(N^n)$ is an eigenfunction of the Laplace operator with eigenvalue n. Then we observe

Lemma 2.1. *n is just the first (positive) eigenvalue of $\mathcal{G}(N^n)$ if and only if $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is Hamiltonian stable. Moreover the dimension of the vector space $\mathcal{V}$ is equal to the multiplicity of the (resp. first) eigenvalue n if and only if $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is Hamilitonian rigid (resp. strictly Hamiltonian stable).*

Next we discuss a relationship between the Gauss images $\mathcal{G}(N^n)$of isoparametric hypersurfaces and the intersection $\widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V}$.

Proposition 2.1 ([22])**.** *If N^n is homogeneous, then*

$$\mathcal{G}(N^n) = \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V}.$$

Define $\mu : \widetilde{Gr}_2(\mathbf{R}^{n+2}) \to \bigwedge^2 \mathbf{R}^{n+2}$ in the following way:

$$\mu : \widetilde{Gr}_2(\mathbf{R}^{n+2}) \ni [W] \longmapsto \mathbf{a} \wedge \mathbf{b} \in \bigwedge^2 \mathbf{R}^{n+2} \cong \mathfrak{o}(n+2) = \tilde{\mathfrak{k}} + \mathcal{V}.$$

The moment map of the action $\tilde{K}$ on $\widetilde{Gr}_2(\mathbf{R}^{n+2})$ is given by $\mu_{\tilde{\mathfrak{k}}} := \pi_{\tilde{\mathfrak{k}}} \circ \mu : \widetilde{Gr}_2(\mathbf{R}^{n+2}) \to \tilde{\mathfrak{k}}$. For any $p \in N^n$,

$$\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) \subset \mathcal{G}(N^n) \subset \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V} = \mu_{\tilde{\mathfrak{k}}}^{-1}(0).$$

It is obvious that N^n is homogeneous if and only if $\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) = \mathcal{G}(N^n)$.

Proposition 2.2 ([22])**.** *Assume that $\mathcal{G}(N^n) = \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V}$. Then $\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) = \mathcal{G}(N^n)$, that is, N^n is homogeneous.*

Corollary 2.1 ([22])**.** *N^n is not homogeneous if and only if*

$$\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) \subsetneqq \mathcal{G}(N^n) \subsetneqq \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V} = \mu_{\tilde{\mathfrak{k}}}^{-1}(0).$$

3. The method of eigenvalue computations for our compact homogeneous spaces

3.1. **Basic results from harmonic analysis on compact homogeneous spaces.** First we recall the basic theory of harmonic analysis on general compact homogeneous spaces. Let $\mathcal{D}(G)$ be the complete set of all inequivalent irreducible unitary representations of a compact connected Lie group G. For a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, let $\Sigma(G)$ be the set of all roots of $\mathfrak{k}$ and $\Sigma^+(G)$ be its subset of all positive root $\alpha \in \Sigma(G)$ relative to a linear order fixed on $\mathfrak{g}$. Set

$$\Gamma(G) := \{\xi \in \mathfrak{h} \mid \exp(\xi) = e\},$$
$$Z(G) := \{\Lambda \in \mathfrak{h}^* \mid \Lambda(\xi) \in \mathbf{Z} \text{ for each } \xi \in \Gamma(K)\},$$
$$D(G) := \{\Lambda \in Z(G) \mid \langle \Lambda, \alpha\rangle \geq 0 \text{ for each } \alpha \in \Sigma^+(G)\}.$$

Then there is a bijective correspondence between $D(G) \ni \Lambda \longmapsto (V_\lambda, \rho_\Lambda) \in \mathcal{D}(G)$, where $(V_\Lambda, \rho_\Lambda)$ denotes an irreducible unitary representation of G with the highest weight Λ equipped with a $\rho_\Lambda(K)$-invariant Hermitian inner product $\langle\ ,\ \rangle_{V_\Lambda}$ (which is unique up to a multiplication by a positive constant). Let $\langle\ ,\ \rangle_{\mathfrak{g}}$ be an AdG-invariant inner product of $\mathfrak{g}$. For a compact Lie subgroup H of G with Lie algebra $\mathfrak{H}$, we take the orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ relative to $\langle\ ,\ \rangle_{\mathfrak{g}}$. Set

$$D(G, H) := \{\Lambda \in D(G) \mid (V_\Lambda)_S \neq \{0\}\}, \tag{3.1}$$

where

$$(V_\Lambda)_H := \{w \in V_\lambda \mid \rho_\lambda(a)w = w\ (\forall a \in H)\}. \tag{3.2}$$

Let $\Lambda \in D(G, H)$. For each $\bar{w} \otimes v \in (V_\Lambda)^*_H \otimes V_\Lambda$, we define a real analytic function $f_{\bar{w}\otimes v}$ on G/H by

$$(f_{\bar{w}\otimes v})(aH) := \langle\langle v, \rho_\Lambda(a)w\rangle\rangle_{V_\Lambda} \tag{3.3}$$

for all $aH \in G/H$. By virtue of the Peter-Weyl's theorem and the Frobenius reciprocity law, we have a linear injection

$$(V_\Lambda)^*_H \otimes V_\Lambda \ni \bar{w} \otimes v \longmapsto f_{\bar{w}\otimes v} \in C^\infty(G/H, \mathbf{C}) \tag{3.4}$$

and the decomposition

$$C^\infty(G/H, \mathbf{C}) = \bigoplus_{\Lambda \in D(G,H)} (V_\Lambda)^*_H \otimes V_\Lambda. \tag{3.5}$$

in the sense of C^∞-topology. Via the natural homogeneous projection $\pi : G \to G/H$, the vector space $C^\infty(G/H, \mathbf{C})$ of all complex valued smooth functions on G/H can be identified with the vector space $C^\infty(K, \mathbf{C})_H$ of all complex valued smooth functions on G invariant under the right action of H. Let $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra of Lie algebra $\mathfrak{g}$, which is identified to the algebra of all left-invariant linear differential operators on $C^\infty(G, \mathbf{C})$. Let

$$\mathrm{U}(\mathfrak{g})_H := \{D \in \mathrm{U}(\mathfrak{g}) \mid \mathrm{Ad}(k)D = R_k \circ D \circ R_{k^{-1}} = D \text{ for each } k \in H\}$$

be a subalgebra of $\mathrm{U}(\mathfrak{g})$ consisting of elements fixed by the adjoint action of H. Here define $(R_k\tilde{f})(u) := \tilde{f}(uk)$ for $\tilde{f} \in C^\infty(G, \mathbf{C})$. For each $D \in \mathrm{U}(\mathfrak{g})_{K_0}$, we have $D(C^\infty(G, \mathbf{C})_H) \subset C^\infty(G, \mathbf{C})_H$. The Casimir operator $\mathcal{C}_{G/K, \langle\ ,\ \rangle_\mathfrak{g}}$ of (G, K) relative to $\langle\ ,\ \rangle_\mathfrak{g}$ is defined by $\mathcal{C} := \sum_{i=1}^n (X_i)^2$ where $\{X_i \mid i = 1, \cdots, n\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle_\mathfrak{g}$. Then $\mathcal{C}_{G/K, \langle\ ,\ \rangle_\mathfrak{g}} \in \mathrm{U}(\mathfrak{g})_H$ and by the $\mathrm{Ad}G$-invariance of $\langle\ ,\ \rangle_\mathfrak{g}$ and Schur's Lemma there is a non-positive real constant $c(\Lambda, \langle\ ,\ \rangle_\mathfrak{g})$ such that

$$\mathcal{C}_{G/K, \langle\ ,\ \rangle_\mathfrak{g}}(f_{\bar{w}\otimes v}) = c(\Lambda, \langle\ ,\ \rangle_\mathfrak{g}) f_{\bar{w}\otimes v} \tag{3.6}$$

for each $\bar{w} \otimes v \in (V_\Lambda)^*_H \otimes V_\Lambda$. The eigenvalue $c(\Lambda, \langle\ ,\ \rangle_\mathfrak{g})$ is given by the Freudenthal's formula

$$c(\Lambda, \langle\ ,\ \rangle_\mathfrak{g}) = -\langle \Lambda, \Lambda + 2\delta \rangle_\mathfrak{g}, \tag{3.7}$$

where $2\delta = \sum_{\alpha \in \Sigma^+(G)} \alpha$.

The Laplace operator $\Delta^0_{L^n} = \delta d$ acting on $C^\infty(K/K_0, \mathbf{C})$ with respect to the induced metric $\mathcal{G}^* g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ corresponds to the linear differential operator $-\mathcal{C}_{L^n}$ on $C^\infty(K, \mathbf{C})_{K_0}$, where $\mathcal{C}_{L^n} \in \mathrm{U}(\mathfrak{k})$ is the Casimir operator relative to the $\mathrm{Ad}_\mathfrak{m}(K_0)$-invariant inner product $\langle\ ,\ \rangle$ of $\mathfrak{m}$ defined by

$$\mathcal{C}_{L^n} := \sum_{\gamma \in \Sigma^+(U,K)} \sum_{i=1}^{m(\gamma)} \frac{1}{||\gamma||^2_\mathfrak{u}} (X_{\gamma,i})^2. \tag{3.8}$$

Notice that $\mathcal{C}_{L^n} \in \mathrm{U}(\mathfrak{k})_{K_0}$ because of the $\mathrm{Ad}_\mathfrak{m}(K_0)$-invariance of $\langle\ ,\ \rangle$.

Suppose that $\Sigma(U,K)$ is irreducible. Let γ_0 denote the highest root of $\Sigma(U,K)$. For $g = 3, 4$, or 6, the restricted root systems $\Sigma(U,K)$ is of type $\mathfrak{a}_2$, B_2, BC_2 or G_2. Then we know that for each $\gamma \in \Sigma^+(U,K)$,

$$\frac{\|\gamma\|_{\mathfrak{u}}^2}{\|\gamma_0\|_{\mathfrak{u}}^2} = \begin{cases} 1 & \text{if } \Sigma(U,K) \text{ is of type } A_2, \\ 1 \text{ or } 1/3 & \text{if } \Sigma(U,K) \text{ is of type } G_2, \\ 1 \text{ or } 1/2 & \text{if } \Sigma(U,K) \text{ is of type } B_2, \\ 1, 1/2 \text{ or } 1/4 & \text{if } \Sigma(U,K) \text{ is of type } BC_2. \end{cases}$$

Define

$$\Sigma_1^+(U,K) := \{\gamma \in \Sigma_1^+(U,K) \mid \|\gamma\|_{\mathfrak{u}}^2 = \|\gamma_0\|_{\mathfrak{u}}^2\}. \tag{3.9}$$

Define a symmetric Lie subalgebra $(\mathfrak{u}_1, \mathfrak{k}_1)$ by

$$\begin{aligned} &\mathfrak{k}_1 := \mathfrak{k}_0 + \sum_{\gamma \in \Sigma_1^+(U,K)} \mathfrak{k}_\gamma, \quad \mathfrak{p}_1 := \mathfrak{a} + \sum_{\gamma \in \Sigma_1^+(U,K)} \mathfrak{p}_\gamma, \\ &\mathfrak{u}_1 := \mathfrak{k}_1 + \mathfrak{p}_1. \end{aligned}$$

Let K_1 and U_1 denote connected compact Lie subgroups of K and U generated by $\mathfrak{k}_1$ and $\mathfrak{u}_1$.

Suppose that $\Sigma^+(U,K)$ is of type BC_2. Define

$$\Sigma_2^+(U,K) := \{\gamma \in \Sigma_1^+(U,K) \mid \|\gamma\|_{\mathfrak{u}}^2 = \|\gamma_0\|_{\mathfrak{u}}^2 \text{ or } \|\gamma_0\|_{\mathfrak{u}}^2/2\}. \tag{3.10}$$

Define a symmetric Lie subalgebra $(\mathfrak{u}_2, \mathfrak{k}_2)$ by

$$\begin{aligned} &\mathfrak{k}_2 := \mathfrak{k}_0 + \sum_{\gamma \in \Sigma_2^+(U,K)} \mathfrak{k}_\gamma, \quad \mathfrak{p}_2 := \mathfrak{a} + \sum_{\gamma \in \Sigma_2^+(U,K)} \mathfrak{p}_\gamma, \\ &\mathfrak{u}_2 := \mathfrak{k}_2 + \mathfrak{p}_2. \end{aligned}$$

Let K_2 and U_2 denote connected compact Lie subgroups of K and U generated by $\mathfrak{k}_2$ and $\mathfrak{u}_2$. We have the following subgroups of K in each case:

$$\begin{array}{ll} K_0 \subset K, & \text{if } \Sigma(U,K) \text{ is of type } \mathfrak{a}_2; \\ K_0 \subset K_1 \subset K, & \text{if } \Sigma(U,K) \text{ is of type } \mathfrak{b}_2 \text{ or } \mathfrak{g}_2; \\ K_0 \subset K_1 \subset K_2 \subset K, & \text{if } \Sigma(U,K) \text{ is of type } BC_2. \end{array}$$

Set

$$\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} := \sum_{\gamma\in\Sigma^+(U,K)} \sum_{i=1}^{m(\gamma)} (X_{\gamma,i})^2,$$

$$\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} := \sum_{\gamma\in\Sigma_1^+(U,K)} \sum_{i=1}^{m(\gamma)} (X_{\gamma,i})^2, \tag{3.11}$$

$$\mathcal{C}_{K_2/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} := \sum_{\gamma\in\Sigma_2^+(U,K)} \sum_{i=1}^{m(\gamma)} (X_{\gamma,i})^2.$$

Then $\mathcal{C}_{K/K_0}, \mathcal{C}_{K_1/K_0}, \mathcal{C}_{K_2/K_0} \in \mathrm{U}(\mathfrak{k})_{K_0}$ and the Casimir operator $\mathcal{C}_{L^n}$ can be decomposed as follows:

Lemma 3.1.

$$\mathcal{C}_{L^n} = \begin{cases} \frac{1}{\|\gamma_0\|_{\mathfrak{u}}}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} & \text{if } \Sigma(U,K) \text{ is of type } A_2; \\ \frac{3}{\|\gamma_0\|_{\mathfrak{u}}}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \frac{2}{\|\gamma_0\|_{\mathfrak{u}}}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} & \text{if } \Sigma(U,K) \text{ is of type } G_2; \\ \frac{2}{\|\gamma_0\|_{\mathfrak{u}}}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \frac{1}{\|\gamma_0\|_{\mathfrak{u}}}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} & \text{if } \Sigma(U,K) \text{ is of type } B_2; \\ \frac{4}{\|\gamma_0\|_{\mathfrak{u}}}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \frac{1}{\|\gamma_0\|_{\mathfrak{u}}}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \frac{2}{\|\gamma_0\|_{\mathfrak{u}}}\mathcal{C}_{K_2/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} & \\ \qquad \text{if } \Sigma(U,K) \text{ is of type } BC_2. & \end{cases}$$

3.2. **Fibrations on homogeneous isoparametric hypersurfaces by homogeneous isoparametric hypersurfaces.** For $g = 4, 6$, (U, K) are of G_2, B_2 or BC_2 type. See the 3rd column of Table 1.

In the case when (U, K) is of type B_2 or G_2, we have one fibration as follows:

$$\begin{array}{c} N^n = K/K_0 \\ \Big\downarrow {\scriptstyle K_1/K_0} \\ K/K_1 \end{array}$$

In the case when (U,K) is of type BC_2, we have the following two fibrations:

$$\begin{tikzcd} N^n = K/K_0 \arrow[r, "="] \arrow[d, "K_1/K_0"] & K/K_0 \arrow[d, "K_2/K_0"] \\ K/K_1 \arrow[r, "K_2/K_1"] & K/K_2 \end{tikzcd}$$

3.2.1. *In case* $g = 6$ *and* $(U,K) = (G_2, SO(4))$, $(m_1, m_2) = (1,1)$.

$$\begin{tikzcd} N^n = K/K_0 = SO(4)/(\mathbf{Z}_2 + \mathbf{Z}_2) \arrow[d, "K_1/K_0 = SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2)"] \\ K/K_1 = SO(4)/SO(3) \cong S^3 \end{tikzcd}$$

$$U/K = G_2/SO(4) \supset U_1/K_1 = SU(3)/SO(3)$$

$$\begin{aligned} K/K_0 &= SO(4)/(\mathbf{Z}_2 + \mathbf{Z}_2) : g = 6, m_1 = m_2 = 1, \\ K_1/K_0 &= SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2) : g = 3, m_1 = m_2 = 1 \end{aligned}$$

3.3. **In case** $g = 6$ **and** $(U,K) = (G_2 \times G_2, G_2)$**,** $(m_1, m_2) = (2,2)$**.**

$$\begin{tikzcd} N^n = K/K_0 = G_2/T^2 \arrow[d, "K_1/K_0 = SU(3)/T^2"] \\ K/K_1 = G_2/SU(3) \cong S^6 \end{tikzcd}$$

$$U/K = (G_2 \times G_2)/G_2 \supset U_1/K_1 = (SU(3) \times SU(3))/SO(3)$$

$$\begin{aligned} K/K_0 &= G_2/T^2 : g = 6, m_1 = m_2 = 2, \\ K_1/K_0 &= SU(3)/T^2 : g = 3, m_1 = m_2 = 2 \end{aligned}$$

3.4. **In case** $g = 4$ **and** $(U, K) = (SO(5) \times SO(5), SO(5))$**,** $(m_1, m_2) = (2, 2)$**.**

$$\begin{array}{l} N^n = K/K_0 = SO(5)/T^2 \\ \quad \Big\downarrow\, K_1/K_0 = SO(4)/T^2 \\ K/K_1 = SO(5)/SO(4) \cong S^4 \end{array}$$

$$\begin{aligned} & U/K = (SO(5) \times SO(5))/SO(5) \cong SO(5) \\ & \supset U_1/K_1 = (SO(4) \times SO(4))/SO(4) \cong SO(4) \cong S^3 \cdot S^3 \\ & \qquad \text{maximal totally geodesic submanifold} \end{aligned}$$

$$SO(4) \cong Spin(4)/\mathbf{Z}_2 \cong (SU(2) \times SU(2))/\mathbf{Z}_2 \cong (S^3 \times S^3)/\mathbf{Z}_2 \cong S^3 \cdot S^3$$

Remark ([20]). The classification of maximal totally geodesic submanifolds embedded in $Sp(2) \cong Spin(5)$:

$$\widetilde{Gr}_2(\mathbf{R}^5),\ S^1 \cdot S^3,\ S^3 \times S^3,\ S^4\,.$$

$$\begin{aligned} & \tilde{U}/\tilde{K} = (Sp(2) \times Sp(2))/Sp(2) \cong Sp(2) \\ & \supset \tilde{U}_1/\tilde{K}_1 = ((Sp(1) \times Sp(1)) \times (Sp(1) \times Sp(1)))/(Sp(1) \times Sp(1)) \\ & \quad \cong Sp(1) \times Sp(1) \cong S^3 \times S^3 \end{aligned}$$

$$\tilde{K}/\tilde{K}_0 = Sp(2)/\tilde{T}^2 : g = 4, (m_1, m_2) = (2, 2)$$

$$\tilde{K}_1/\tilde{K}_0 = (Sp(1) \times Sp(1))/\tilde{T}^2 \cong S^2 \times S^2 : g = 2, (m_1, m_2) = (2, 2)$$

$$
\begin{aligned}
& U/K = (SO(5) \times SO(5))/SO(5) \cong SO(5) \cong Spin(5)/\mathbf{Z}_2 \\
\supset\, & U_1/K_1 = (SO(4) \times SO(4))/SO(4) \cong SO(4) \cong S^3 \cdot S^3 \cong (S^3 \times S^3)/\mathbf{Z}_2
\end{aligned}
$$

(max. tot. geod. submfd.)

$$
\begin{aligned}
& K/K_0 = SO(5)/T^2 : g = 4, m_1 = m_2 = 2, \\
& K_1/K_0 = SO(4)/T^2 \cong S^2 \times S^2 : g = 2, m_1 = m_2 = 2
\end{aligned}
$$

3.5. **In case** $g = 4$ **and** $(U, K) = (SO(10), U(5))$**,** $(m_1, m_2) = (4, 5)$**.**

$$
\begin{tikzcd}
N^n = K/K_0 = \dfrac{U(5)}{SU(2) \times SU(2) \times U(1)} \arrow[r, "="] \arrow[d, "K_1/K_0 = \frac{U(2)\times U(2)\times U(1)}{SU(2)\times SU(2)\times U(1)}"] & K/K_0 = \dfrac{U(5)}{SU(2) \times SU(2) \times U(1)} \arrow[d, "K_2/K_0 = \frac{U(4)\times U(1)}{SU(2)\times SU(2)\times U(1)}"] \\
K/K_1 = \dfrac{U(5)}{U(2) \times U(2) \times U(1)} \arrow[r, "K_2/K_1 = \frac{U(4)\times U(1)}{U(2)\times U(2)\times U(1)}"] & K/K_2 = \dfrac{U(5)}{U(4) \times U(1)}
\end{tikzcd}
$$

$$
\begin{aligned}
& U/K = SO(10)/U(5) \\
\supset\, & U_2/K_2 = \frac{SO(8) \times SO(2)}{U(4) \times U(1)} \\
& \qquad \cong \frac{SO(8)}{U(4)} \cong \frac{SO(8)}{SO(2) \times SO(6)} \cong \widetilde{Gr}_2(\mathbf{R}^8) \quad \text{(DIII(4)=BDI(2,6))}
\end{aligned}
$$

(maximal totally geodesic submanifold)

$$
\begin{aligned}
\supset\, & U_1/K_1 = \frac{SO(4) \times SO(4) \times SO(2)}{U(2) \times U(2) \times U(1)} \\
& \qquad \cong \frac{SO(4) \times SO(4)}{U(2) \times U(2)} \cong \frac{SO(4)}{U(2)} \times \frac{SO(4)}{U(2)} \\
& \qquad \cong S^2 \times S^2 \cong \widetilde{Gr}_2(\mathbf{R}^4) \quad \left(\frac{SO(4)}{U(2)} \cong S^2 \right)
\end{aligned}
$$

(not maximal totally geodesic submanifold)

Remark ([20]). The classification of maximal totally geodesic submanifolds embedded in $\dfrac{SO(10)}{U(5)}$ (DIII(5)):

$$
\widetilde{Gr}_2(\mathbf{R}^8),\ Gr_2(\mathbf{C}^5), SO(5),\ S^2 \times \mathbf{C}P^3,\ \mathbf{C}P^4.
$$

The classification of maximal totally geodesic submanifolds embedded in $\widetilde{Gr}_2(\mathbf{R}^8)$:

$$\widetilde{Gr}_2(\mathbf{R}^7),\ S^p \cdot S^q (p+q=6),\ \mathbf{C}P^3.$$

Here $U_2/K_2 = \widetilde{Gr}_2(\mathbf{R}^8) \supset \widetilde{Gr}_2(\mathbf{R}^7) \supset \widetilde{Gr}_2(\mathbf{R}^4) = U_1/K_1$.

Remark.

$$\frac{SO(8)}{U(4)} \cong \frac{SO(8)}{SO(2)\times SO(6)} \cong \widetilde{Gr}_2(\mathbf{R}^8) \text{ (DIII(4)=BDI(2,6)) :}$$

$$\begin{aligned}
K/K_0 =& \frac{U(5)}{SU(2)\times SU(2)\times U(1)},\ g=4, (m_1,m_2)=(4,5)\\
K_2/K_0 =& \frac{U(4)\times U(1)}{SU(2)\times SU(2)\times U(1)} \cong \frac{SO(2)\times SO(6)}{\mathbf{Z}_2\times SO(4)},\ g=4, (m_1,m_2)=(1,4)\\
K_1/K_0 =& \frac{U(2)\times U(2)\times U(1)}{SU(2)\times SU(2)\times U(1)}\\
\cong& \frac{U(2)}{SU(2)} \times \frac{U(2)}{SU(2)} \cong S^1\times S^1,\ g=2, (m_1,m_2)=(1,1)
\end{aligned}$$

3.6. **In case** $g=4$ **and** $(U,K)=(SO(m+2), SO(2)\times SO(m))\,(m\geq 3)$**,** $(m_1,m_2)=(1,m-2)$**.**

$$N^n = K/K_0 = \tfrac{SO(2)\times SO(m)}{\mathbf{Z}_2\times SO(m-2)}$$

$$\Big\downarrow \quad K_1/K_0 = \tfrac{SO(2)\times SO(2)\times SO(m-2)}{\mathbf{Z}_2\times SO(m-2)} \cong \tfrac{SO(2)\times SO(2)}{\mathbf{Z}_2} \cong S^1\times S^1$$

$$K/K_1 = \tfrac{SO(2)\times SO(m)}{SO(2)\times SO(2)\times SO(m-2)} \cong \tfrac{SO(m)}{SO(2)\times SO(m-2)} \cong \widetilde{Gr}_2(\mathbf{R}^m)$$

$$\begin{aligned}
U/K =& \frac{SO(m+2)}{SO(2)\times SO(m)} \cong \widetilde{Gr}_2(\mathbf{R}^{m+2})\\
&\Big(\supset \frac{SO(m+1)}{SO(2)\times SO(m-1)} \cong \widetilde{Gr}_2(\mathbf{R}^{m+1}) \text{ max. tot. geod. submfd.}\Big)\\
&\supset U_1/K_1 = \frac{SO(4)\times SO(m-2)}{SO(2)\times SO(2)\times SO(m-2)} \cong \widetilde{Gr}_2(\mathbf{R}^4) \cong S^2\times S^2\\
&\quad \text{(not maximal totally geodesic submanifold)}
\end{aligned}$$

The classification of maximal totally geodesic submanifolds embedded in $\widetilde{Gr}_2(\mathbf{R}^{m+2})$ $(m \geq 3)$:

$$\widetilde{Gr}_2(\mathbf{R}^{m+1}),\ S^p \cdot S^q (p+q=m),\ \mathbf{C}P^{[\frac{m}{2}]}.$$

$$\begin{aligned} K/K_0 &= \frac{SO(2)\times SO(m)}{\mathbf{Z}_2 \times SO(m-2)} :\ g=4, (m_1,m_2)=(1,m-2) \\ K_1/K_0 &= \frac{SO(2)\times SO(2)\times SO(m-2)}{\mathbf{Z}_2\times SO(m-2)} \cong \frac{SO(2)\times SO(2)}{\mathbf{Z}_2} \\ &\cong S^1 \times S^1 :\ g=2, (m_1,m_2)=(1,1) \end{aligned}$$

3.7. **In case** $g=4$ **and** $(U,K)=(SU(m+2), S(U(2)\times U(m))$ $(m \geq 2)$**,** $(m_1,m_2)=(2,2m-3)$**.**

Remark. The classification of maximal totally geodesic submanifolds embedded in $Gr_2(\mathbf{C}^{m+2})$ $(m \geq 3)$:

$$Gr_2(\mathbf{C}^{m+1}),\ \widetilde{Gr}_2(\mathbf{R}^{m+2}),\ \mathbf{C}P^p \times \mathbf{C}P^q (p+q=m),\ \mathbf{H}P^{[\frac{m}{2}]}.$$

3.7.1. $m=2$. $(U,K)=(SU(4), S(U(2)\times U(2))$, $(m_1,m_2)=(2,1)$

$$\begin{aligned} N^n = K/K_0 &= \tfrac{S(U(2)\times U(2))}{S(U(1)\times U(1))} \\ &\downarrow \quad K_1/K_0 = \tfrac{S(U(1)\times U(1)\times U(1)\times U(1))}{S(U(1)\times U(1))} \cong S^1\times S^1 \\ K/K_1 &= \tfrac{S(U(2)\times U(2))}{S(U(1)\times U(1)\times U(1)\times U(1))} \cong S^2 \times S^2 \end{aligned}$$

$$\begin{aligned} U/K &= \frac{SU(4)}{S(U(2)\times U(2))} \cong Gr_2(\mathbf{C}^4) \\ \supset U_1/K_1 &= \frac{S(U(2)\times U(2))}{S(U(1)\times U(1)\times U(1)\times U(1))} \cong S^2\times S^2 \end{aligned}$$

$$\begin{aligned} K/K_0 &= \frac{S(U(2)\times U(2))}{S(U(1)\times U(1))},\ g=4, (m_1,m_2)=(2,1) \\ K_1/K_0 &= \frac{S(U(1)\times U(1)\times U(1)\times U(1))}{S(U(1)\times U(1))} \cong S^1\times S^1,\ g=2, (m_1,m_2)=(1,1) \end{aligned}$$

3.7.2. $m \geq 3$.

$$
\begin{tikzcd}
N^n = \frac{K}{K_0} = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))} \arrow[r, "="] \arrow[d, "\frac{K_1}{K_0} = \frac{S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))}{S(U(1)\times U(1)\times U(m-2))}"] & \frac{K}{K_0} = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))} \arrow[d, "\frac{K_2}{K_0} = \frac{S(U(2)\times U(2)\times U(m-2))}{S(U(1)\times U(1)\times U(m-2))}"] \\
\frac{K}{K_1} = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))} \arrow[r, "\frac{K_2}{K_1} \cong \mathbf{C}P^1 \times \mathbf{C}P^1"] & \frac{K}{K_2} = \frac{S(U(2)\times U(m))}{S(U(2)\times U(2)\times U(m-2))}
\end{tikzcd}
$$

$$
\begin{aligned}
& U/K = \frac{SU(m+2)}{S(U(2)\times U(m))} \cong Gr_2(\mathbf{C}^{m+2}) \\
& \supset U_2/K_2 = \frac{S(U(4)\times U(m-2))}{S(U(2)\times U(2)\times U(m-2))} \cong \frac{SU(4)}{S(U(2)\times U(2))} \cong Gr_2(\mathbf{C}^4) \\
& \qquad \text{(not maximal tot. geod. submfd.)} \\
& \supset U_1/K_1 = \frac{S(U(2)\times U(2)\times U(m-2))}{S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))} \\
& \qquad \cong \frac{S(U(2)\times U(2))}{S(U(1)\times U(1)\times U(1)\times U(1))} \cong \mathbf{C}P^1 \times \mathbf{C}P^1 \\
& \qquad \text{(maximal tot. geod. submfd.)}
\end{aligned}
$$

$$
\begin{aligned}
K/K_0 &= \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))},\ g=4, (m_1, m_2) = (2, 2m-3) \\
K_2/K_0 &= \frac{S(U(2)\times U(2)\times U(m-2))}{S(U(1)\times U(1)\times U(m-2))},\ g=4, (m_1, m_2) = (2, 1) \\
K_1/K_0 &= \frac{S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))}{S(U(1)\times U(1)\times U(m-2))} \cong S^1 \times S^1, \\
& \qquad g = 2, (m_1, m_2) = (1, 1).
\end{aligned}
$$

3.8. **In case** $g = 4$ **and** $(U, K) = (Sp(m+2), Sp(2)\times Sp(m))$ $(m \geq 2)$**,** $(m_1, m_2) = (4, 4m-5)$**.**

Remark. The classification of maximal totally geodesic submanifolds embedded in $Gr_2(\mathbf{H}^{m+2})$ $(m \geq 3)$:

$$Gr_2(\mathbf{H}^{m+1}),\ Gr_2(\mathbf{C}^{m+2}),\ \mathbf{H}P^p \times \mathbf{H}P^q (p+q=m).$$

3.8.1. *In case* $g=4$ *and* $(U,K)=(Sp(4),Sp(2)\times Sp(2))$ $(m=2)$, $(m_1,m_2)=(4,3)$.

$$\begin{array}{l}
N^n = K/K_0 = \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)} \\
\quad\Big\downarrow \quad K_1/K_0 = \frac{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)}{Sp(1)\times Sp(1)} \cong S^3\times S^3 \\
K/K_1 = \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)} \cong \mathbf{H}P^1\times\mathbf{H}P^1 \cong S^4\times S^4
\end{array}$$

$$\begin{aligned}
U/K &= \frac{Sp(4)}{Sp(2)\times Sp(2)} \cong Gr_2(\mathbf{H}^4) \\
\supset U_1/K_1 &= \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)} \cong \mathbf{H}P^1\times\mathbf{H}P^1 \\
&\text{(maximal totally geodesic submanifold)}
\end{aligned}$$

$$\begin{aligned}
K/K_0 &= \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)},\ g=4, (m_1,m_2)=(4,3) \\
K_1/K_0 &= \frac{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)}{Sp(1)\times Sp(1)} \\
&\cong Sp(1)\times Sp(1)\cong S^3\times S^3,\ g=2, (m_1,m_2)=(3,3)
\end{aligned}$$

3.8.2. $m\geq 3$.

$$\begin{array}{ccc}
N^n = \frac{K}{K_0} = \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)} & \xrightarrow{=} & \frac{K}{K_0} = \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)} \\
\Big\downarrow\ \frac{K_1}{K_0} = \frac{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)}{Sp(1)\times Sp(1)\times Sp(m-2)} & & \Big\downarrow\ \frac{K_2}{K_0} = \frac{Sp(2)\times Sp(2)\times Sp(m-2)}{Sp(1)\times Sp(1)\times Sp(m-2)} \\
\frac{K}{K_1} = \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)} & \xrightarrow{\frac{K_2}{K_1}\cong \mathbf{H}P^1\times\mathbf{H}P^1} & \frac{K}{K_2} = \frac{Sp(2)\times Sp(m))}{Sp(2)\times Sp(2)\times Sp(m-2)}
\end{array}$$

$$
\begin{aligned}
U/K &= \frac{Sp(m+2)}{Sp(2)\times Sp(m)} \cong Gr_2(\mathbf{H}^{m+2}) \\
\supset U_2/K_2 &= \frac{Sp(4)\times Sp(m-2)}{Sp(2)\times Sp(2)\times Sp(m-2)} \cong \frac{Sp(4)}{Sp(2)\times Sp(2)} \cong Gr_2(\mathbf{H}^4) \\
&\quad \text{(not maximal tot. geod. submfd.)} \\
\supset U_1/K_1 &= \frac{Sp(2)\times Sp(2)\times Sp(m-2)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)} \\
&\cong \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)} \cong \mathbf{H}P^1\times\mathbf{H}P^1 \\
&\quad \text{(maximal tot. geod. submfd.)}
\end{aligned}
$$

$$
\begin{aligned}
K/K_0 &= \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)} :\ g=4, (m_1,m_2)=(4,4m-5) \\
K_2/K_0 &= \frac{Sp(2)\times Sp(2)\times Sp(m-2)}{Sp(1)\times Sp(1)\times Sp(m-2)} \\
&\cong \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)} :\ g=4, (m_1,m_2)=(4,3) \\
K_1/K_0 &= \frac{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)}{Sp(1)\times Sp(1)\times Sp(m-2)} \\
&\cong \frac{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)}{Sp(1)\times Sp(1)} \\
&\cong Sp(1)\times Sp(1) \cong S^3\times S^3 :\ g=2, (m_1,m_2)=(3,3).
\end{aligned}
$$

3.9. **In case** $g=4$ **and** $(U,K)=(E_6, U(1)\cdot Spin(10))$**,** $(m_1,m_2)=(6,9)$**.**

Remark. The classification of maximal totally geodesic submanifolds embedded in $E_6/U(1)\cdot Spin(10)\,(EIII)$:

$Gr_2(\mathbf{H}^4)/\mathbf{Z}_2$, $\mathbf{O}P^2$, $S^2\times\mathbf{C}P^2$, $SO(10)/U(5)\,(DIII(5))$, $Gr_2(\mathbf{C}^6)$, $\widetilde{Gr}_2(\mathbf{R}^{10})$.

$$
\begin{tikzcd}
N^n = \frac{K}{K_0} = \frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)} \arrow[r, "="] \arrow[d, "\frac{K_1}{K_0} = \frac{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))}{S^1\cdot Spin(6)}"'] & \frac{K}{K_0} = \frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)} \arrow[d, "\frac{K_2}{K_0} = \frac{U(1)\cdot(Spin(2)\cdot Spin(8))}{S^1\cdot Spin(6)}"] \\
\frac{K}{K_1} = \frac{U(1)\cdot Spin(10)}{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))} \arrow[r, "\frac{K_2}{K_1} = \frac{U(1)\cdot(Spin(2)\cdot Spin(8))}{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))}"] & \frac{K}{K_2} = \frac{U(1)\cdot Spin(10)}{U(1)\cdot(Spin(2)\cdot Spin(8))}
\end{tikzcd}
$$

$$
\begin{aligned}
& U/K = \frac{E_6}{U(1)\cdot Spin(10)} \\
& \supset U_2/K_2 = \frac{U(1)\cdot Spin(10)}{U(1)\cdot(Spin(2)\cdot Spin(8))} \cong \frac{Spin(10)}{Spin(2)\cdot Spin(8)} \cong \widetilde{Gr}_2(\mathbf{R}^{10}) \\
& \qquad \text{(maximal tot. geod. submfd.)} \\
& \supset U_1/K_1 = \frac{S^1\cdot Spin(4)\cdot Spin(6)}{S^1\cdot(Spin(2)\cdot Spin(2)\cdot Spin(6))} \\
& \qquad \cong \frac{Spin(4)}{Spin(2)\cdot Spin(2)} \cong \widetilde{Gr}_2(\mathbf{R}^4) \cong S^2\times S^2 \\
& \qquad \text{(not maximal tot. geod. submfd.)}
\end{aligned}
$$

$$
\begin{aligned}
K/K_0 &= \frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)} : \ g=4, (m_1,m_2) = (6,9) \\
K_2/K_0 &= \frac{U(1)\cdot(Spin(2)\cdot Spin(8))}{S^1\cdot Spin(6)} \\
&\cong \frac{Spin(2)\cdot Spin(8)}{Spin(6)} \\
&\cong \frac{SO(2)\times SO(8)}{\mathbf{Z}_2\times SO(6)} : \ g=4, (m_1,m_2) = (1,6) \\
K_1/K_0 &= \frac{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))}{S^1\cdot Spin(6)} \\
&\cong S^1\times S^1 : \ g=2, (m_1,m_2) = (1,1).
\end{aligned}
$$

4. The case $(U,K) = (G_2\times G_2, G_2)$

(U,K) is of type G_2.

In this case, $U = G_2\times G_2$, $K = \{(x,x)\in U \mid x\in G_2\}$ and (U,K) is the corresponding symmetric pair. Let $\mathfrak{u} = \mathfrak{k}+\mathfrak{p}$ be the canonical decomposition and $\mathfrak{a}\subset\mathfrak{p}$ be the maximal abelian subspace of $\mathfrak{p}$. Then

$K_0 = \{k \in K \mid \mathrm{Ad}(k)H = H \text{ for each } H \in \mathfrak{a}\} \cong T^2$ is a maximal torus of G_2 and $N = K/K_0 \cong G_2/T^2$ is a flag manifold of dimension $n = 12$. Thus the Gauss image $L^{12} = \mathcal{G}(N^{12})(\cong N^{12}/\mathbb{Z}_6) = K \cdot [\mathfrak{a}] \cong (K/K_{[\mathfrak{a}]}) \subset Q_{12}(\mathbf{C})$.

Note that $T_{eK_0}(K/K_0) \cong T_{eK_{[\mathfrak{a}]}}(K/K_{[\mathfrak{a}]}) \cong \mathfrak{m}$. Set $\langle\ ,\ \rangle_{\mathfrak{u}} = -B_{\mathfrak{u}}(\ ,\ )$, where $B_{\mathfrak{u}}(\ ,\ )$ denotes the Killing-Cartan form of $\mathfrak{u}$. Let $\langle\ ,\ \rangle$ be the inner product corresponding to the induced metric on L^n from $(Q_n(\mathbf{C}), g^{\mathrm{std}}_{Q_n(\mathbf{C})})$.

The restricted root system $\Sigma(U,K)$ is of G_2 type, which can be given as follows ([6]):

$$\begin{aligned}\Sigma(U,K) =\{&\pm(\varepsilon_1-\varepsilon_2)=\pm\alpha_1, \pm(\varepsilon_3-\varepsilon_1)=\pm(\alpha_1+\alpha_2),\\ &\pm(\varepsilon_3-\varepsilon_2)=\pm(2\alpha_1+\alpha_2), \pm(-2\varepsilon_1+\varepsilon_2+\varepsilon_3)=\pm\alpha_2,\\ &\pm(\varepsilon_1-2\varepsilon_2+\varepsilon_3)=\pm(3\alpha_1+\alpha_2),\\ &\pm(2\varepsilon_3-\varepsilon_1-\varepsilon_2)=\pm(3\alpha_1+2\alpha_2)=\tilde{\alpha}\},\end{aligned}$$

where $\Pi(U,K) = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$ is its fundamental root system. Since

$$\left\{\frac{1}{\|\gamma\|_{\mathfrak{u}}} X_{\gamma,i} \mid \gamma \in \Sigma^+(U,K), i = 1, \cdots, m(\gamma)\right\}$$

is an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle$, where $\{X_{\gamma,i}\}$ is the standard orthonormal basis with respect to $\langle\ ,\ \rangle_{\mathfrak{u}}$, the Laplace operator Δ_{L^n} of L^n, or equivalently, the Casimir operator $\mathcal{C}_L$ of L^n, with respect to the induce metric $\langle\ ,\ \rangle$ can be expressed as follows:

$$\mathcal{C}_L = \sum_{\gamma:\text{short}} \frac{1}{\|\gamma\|^2_{\mathfrak{u}}}(X_{\gamma,i})^2 + \sum_{\gamma:\text{long}} \frac{1}{\|\gamma\|^2_{\mathfrak{u}}}(X_{\gamma,i})^2,$$

where

$$\|\gamma\|^2_{\mathfrak{u}} = \begin{cases} \dfrac{1}{24} & \text{if } \gamma \text{ is short,} \\[2ex] \dfrac{1}{8} & \text{if } \gamma \text{ is long.} \end{cases}$$

Let $K_1 = SU(3)$. Then we have $K_0 = T^2 \subset K_1 = SU(3) \subset K = G_2$. Therefore,

$$\begin{aligned}
\mathcal{C}_L = &\sum_{\gamma:\text{short}} 24(X_{\gamma,i})^2 + \sum_{\gamma:\text{long}} 8(X_{\gamma,i})^2 \\
=&24 \sum_{\gamma\in\Sigma(U,K)} (X_{\gamma,i})^2 - 16 \sum_{\gamma:\text{long}} (X_{\gamma,i})^2 \\
=&12\mathcal{C}^{\mathfrak{k}}_{K/K_0} - 8\mathcal{C}^{\mathfrak{k}}_{K_1/K_0} \\
=&12\mathcal{C}^{\mathfrak{k}}_{K/K_0} - 6\mathcal{C}^{\mathfrak{k}_1}_{K_1/K_0},
\end{aligned}$$

where $\mathcal{C}^{\mathfrak{k}}_{K/K_0}$ denotes the Casimir operator of K/K_0 with respect to the K_0-invariant metric induced from the standard Killing-Cartan metric of K, and similarly, $\mathcal{C}^{\mathfrak{k}_1}_{K_1/K_0}$ denotes the Casimir operator of K_1/K_0 with respect to the K_0-invariant metric induced from the standard Killing-Cartan metric of K_1.

Let $\{\alpha_1, \alpha_2\}$ be the fundamental root system of G_2 defined above and $\{\Lambda_1, \Lambda_2\}$ be the fundamental weight system of G_2. By S. Yamaguchi [43], $D(G_2, T^2) = D(G_2)$ and each $\Lambda \in D(G_2, T^2) = D(G_2)$ can be uniquely expressed as

$$\Lambda = m_1\Lambda_1 + m_2\Lambda_2 = p_1\alpha_1 + p_1\alpha_2,$$

where

$$\begin{aligned}
&m_1, m_2 \in \mathbf{Z}, m_1 \geq 0, m_2 \geq 0, \\
&p_1, p_2 \in \mathbf{Z}, p_1 \geq 1, p_2 \geq 1
\end{aligned}$$

and

$$\begin{cases} m_1 = 2p_1 - 3p_2 \geq 0, \\ m_2 = -p_1 + 2p_2 \geq 0, \end{cases}$$

$$\begin{cases} p_1 = 2m_1 + 3m_2 \geq 1, \\ p_2 = m_1 + 2m_2 \geq 1. \end{cases}$$

The eigenvalue formula of the Casimir operator $\mathcal{C}_{K/K_0}$ with respect to the inner produce induced from the standard Killing-Cartan metric of G_2 is given by

$$-c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}_2}) = \frac{1}{24}(m_1p_1 + 3m_1p_2 + 2p_1 + 6p_2)$$

for any $\Lambda \in D(G_2, T^2)$. Since

$$\mathcal{C}_L = 4\,\mathcal{C}^{\mathfrak{g}_2}_{K/K_0} + \sum_{\gamma:\text{short}} 16(X_{\gamma,i})^2 \geq 4\,\mathcal{C}^{\mathfrak{g}_2}_{K/K_0}\,,$$

the first eigenvalue of $\mathcal{C}_L$, i.e., $-c_L \leq n = 12$ implies $-c_\Lambda \leq 3$. Therefore we get

$$\begin{aligned}&\{\Lambda \in D(G_2, T^2) \mid -c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}_2}) \leq 3\}\\=&\{0, \Lambda_1((p_1,p_2)=(2,1)), 2\Lambda_1((p_1,p_2)=(4,2)), 3\Lambda_1((p_1,p_2)=(6,3)),\\&\Lambda_2((p_1,p_2)=(3,2)), 2\Lambda_2((p_1,p_2)=(6,4)), \Lambda_1+\Lambda_2((p_1,p_2)=(5,3)),\\&2\Lambda_1+\Lambda_2((p_1,p_2)=(7,4))\}.\end{aligned}$$

Let α'_1, α'_2 be the fundamental root system of $SU(3)$ and Λ'_1, Λ'_2 be the fundamental weight system of $SU(3)$. By the branching laws of $(G_2, SU(3))$ in [23], we can obtain that for each $\Lambda \in D(G_2, T^2)$ such that $-c(\Lambda, \langle\,,\rangle_{\mathfrak{g}_2}) \leq 3$, the irreducible G_2-module V_Λ with the heighest weight Λ contains an irreducible $SU(3)$-submodule $V_{\Lambda'}$ with the heighest weight $\Lambda' = m'_1\Lambda'_1 + m_2\Lambda'_2$ given in the following table:

(m_1, m_2)	(p_1, p_2)	$-c$	$\dim_{\mathbf{C}} V_\Lambda$	irred.$SU(3)$-submodules (m'_1, m'_2)
$(1,0)$	$(2,1)$	$\frac{1}{2}$	7	$(1,0), (0,1), (0,0)$
$(2,0)$	$(4,2)$	$\frac{7}{6}$	27	$(2,0), (1,1), (0,2), (1,0), (0,1), (0,0)$
$(3,0)$	$(6,3)$	2	77	$(3,0), (2,1), (1,2), (0,3), (2,0), (1,1), (0,2), (1,0), (0,1), (0,0)$
$(0,1)$	$(3,2)$	1	14	$(1,1), (1,0), (0,1)$
$(0,2)$	$(6,4)$	$\frac{5}{2}$	77	$(2,2), (2,1), (1,2), (2,0), (1,1), (0,2)$
$(1,1)$	$(5,3)$	$\frac{7}{4}$	64	$(2,1), (1,2), (2,0), 2(1,1), (0,2), (1,0), (0,1)$
$(2,1)$	$(7,4)$	$\frac{8}{3}$	189	$(3,1), (2,2), (1,3), (3,0), 2(2,1), 2(1,2), (0,3), (2,0), 2(1,1), (0,2), (1,0), (0,1)$

Since

$$\begin{aligned}\mathfrak{g}_2^{\mathbf{C}} &= \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha\in\Sigma(\mathfrak{g}_2)} \mathfrak{g}^\alpha = \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha:\text{short}} \mathfrak{g}^\alpha + \sum_{\alpha:\text{long}} \mathfrak{g}^\alpha,\\ \mathfrak{s}u(3)^{\mathbf{C}} &= \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha:\text{long}} \mathfrak{g}^\alpha,\end{aligned}$$

we know that

$$\begin{aligned}T^2\cdot\mathbb{Z}_6 =&\{a \in G_2 \mid \mathrm{Ad}(a)(\mathfrak{t}) = \mathfrak{t} \text{ preserving the orientation of } \mathfrak{t}\}\\ \supset&\{a \in SU(3) \mid \mathrm{Ad}(a)(\mathfrak{t}) = \mathfrak{t} \text{ preserving the orientation of } \mathfrak{t}\}\\ =&T^2\cdot\mathbb{Z}_3.\end{aligned}$$

By the results of Yamaguchi [43], each $\Lambda' \in D(SU(3),T^2)$ can be uniquely expressed as

$$\Lambda' = m_1'\Lambda_1' + m_2'\Lambda_2' = p_1'\alpha_1' + p_2'\alpha_2',$$

where $m_i' \in \mathbf{Z}$, $m_i' \geq 0$, $p_i' \in \mathbf{Z}$, $p_i' \geq 1$ and

$$m_1' = 2p_1' - p_2' \geq 0, \quad m_2' = -p_1' + 2p_2' \geq 0\,.$$

The eigenvalue formula is given by

$$-c(\Lambda', \langle\ ,\ \rangle_{\mathfrak{su}(3)}) = \frac{1}{6}(m_1'p_1' + m_1'p_2' + 2p_1' + 2p_2')$$

for any $\Lambda' \in D(SU(3),T^2)$. Therefore, it is easy to check that $\Lambda' = m_1'\Lambda_1' + m_2'\Lambda_2' \in D(SU(3),T^2)$ such that $V_{\Lambda'} \subset V_\Lambda$ for some $\Lambda \in D(G_2,T^2)$ with $-c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}_2}) \leq 3$ satisfies

$$(m_1', m_2') \in \{(1,1),(3,0),(0,3),(2,2)\}.$$

Then the corresponding eigenvalues of $\mathcal{C}_{K_1/K_0}$ are given in the following:

(p_1',p_2')	(m_1',m_2')	$-c' = -c(\Lambda', \langle\ ,\ \rangle_{\mathfrak{su}(3)})$
$(1,1)$	$(1,1)$	1
$(2,1)$	$(3,0)$	2
$(1,2)$	$(0,3)$	2
$(2,2)$	$(2,2)$	$\frac{8}{3}$

So for $\Lambda \in D(G_2,T^2)$, $\Lambda' \in D(SU(3),T^2)$ such that $V_{\Lambda'} \subset V_\Lambda$ and $-c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}_2}) \leq 3$, the eigenvalues of $-\mathcal{C}_L = -12c + 6c'$ are given in the following table:

(m_1,m_2)	(p_1,p_2)	$\dim_{\mathbf{C}} V_\Lambda$	$-c$	(m_1',m_2')	$-c'$	$-12c+6c'$
$(2,0)$	$(4,2)$	27	$\frac{7}{6}$	$(1,1)$	1	8
$(3,0)$	$(6,3)$	77	2	$(1,1)$	1	18
$(3,0)$	$(6,3)$	77	2	$(3,0)$	2	12
$(3,0)$	$(6,3)$	77	2	$(0,3)$	2	12
$(0,1)$	$(3,2)$	14	1	$(1,1)$	1	6
$(0,2)$	$(6,4)$	77	$\frac{5}{2}$	$(1,1)$	1	24
$(0,2)$	$(6,4)$	77	$\frac{5}{2}$	$(2,2)$	$\frac{8}{3}$	14
$(1,1)$	$(5,3)$	64	$\frac{7}{4}$	$2(1,1)$	1	15
$(2,1)$	$(7,4)$	189	$\frac{8}{3}$	$2(1,1)$	1	26
$(2,1)$	$(7,4)$	189	$\frac{8}{3}$	$(3,0)$	2	20
$(2,1)$	$(7,4)$	189	$\frac{8}{3}$	$(0,3)$	2	20
$(2,1)$	$(7,4)$	189	$\frac{8}{3}$	$(2,2)$	$\frac{8}{3}$	16

Since $\Lambda_1' + \Lambda_2'$ ($(m_1',m_2') = (1,1)$) corresponds to the adjoint representation of $SU(3)$, $(V'_{\Lambda_1'+\Lambda_2'})_{T^2} \cong \mathfrak{t}^2$ and $(V'_{\Lambda_1'+\Lambda_2'})_{T^2\cdot\mathbf{Z}_3} = \{0\}$. Then

$$\Lambda_1' + \Lambda_2' \notin D(SU(3), T^2 \cdot \mathbf{Z}_3).$$

Thus we have

$$2\Lambda_1, \Lambda_2 \notin D(G_2, T^2 \cdot \mathbf{Z}_6).$$

We only need to check if $3\Lambda_1 \in D(K, K_{[\mathfrak{a}]}) = D(G_2, T^2 \cdot \mathbf{Z}_6)$. Consider

$$(V_{3\Lambda_1})_{T^2} = (V'_{3\Lambda'_1})_{T^2} \oplus (V'_{3\Lambda'_2})_{T^2} \oplus (V'_{\Lambda'_1+\Lambda'_2})_{T^2}.$$

Since

$$V'_{3\Lambda'_1} \cong \mathrm{Sym}^3(\mathbf{C}^3) = \mathrm{span}_{\mathbf{C}}\{e_{i_1} \cdot e_{i_2} \cdot e_{i_3} \mid 1 \leq i_1 \leq i_2 \leq i_3 \leq 3\},$$

where $\{e_1, e_2, e_3\}$ is a basis of $\mathbf{C}^3$, we know that

$$(V'_{3\Lambda'_1})_{T^2} = (V'_{3\Lambda'_1})_{T^2 \cdot \mathbf{Z}_3} = \mathrm{span}_{\mathbf{C}}\{e_1 \cdot e_2 \cdot e_3\}.$$

Similarly, $V'_{3\Lambda'_2} \cong \mathrm{Sym}^3(\bar{\mathbf{C}}^3)$ and $(V'_{3\Lambda'_2})_{T^2} = (V'_{3\Lambda'_2})_{T^2 \cdot \mathbf{Z}_3}$ with dimension 1. On the other hand, $(V'_{\Lambda'_1+\Lambda'_2})_{T^2} \cong \mathfrak{t}$ and $(V'_{\Lambda'_1+\Lambda'_2})_{T^2 \cdot \mathbb{Z}_3} = \{0\}$. Hence, $\dim_{\mathbf{C}}(V_{3\Lambda_1})_{T^2} = 4$. But $\dim_{\mathbf{C}}(V_{3\Lambda_1})_{T^2 \cdot \mathbf{Z}_6} = 1$. In fact, $T^2 \cdot \mathbf{Z}_6 \subset G_2$, $T^2 \cdot \mathbf{Z}_6 \not\subset SU(3)$, $T^2 \cdot \mathbf{Z}_3 \subset SU(3)$ and $(T^2 \cdot \mathbf{Z}_6)/(T^2 \cdot \mathbf{Z}_3) \cong \mathbf{Z}_2$. Hence, there exists an element $u \in T^2 \cdot \mathbf{Z}_6$ which satisfies $\mathrm{Ad}(u)(SU(3)) \subset SU(3)$ and provides generators of $(T^2 \cdot \mathbf{Z}_6)/T^2 \cong \mathbf{Z}_6$ and $(T^2 \cdot \mathbf{Z}_6)/(T^2 \cdot \mathbf{Z}_3) \cong \mathbf{Z}_2$. Then we observe that $\rho_{3\Lambda'_1} \circ \mathrm{Ad}(u)|_{SU(3)} \cong \rho_{3\Lambda'_2}$ and $\rho_{3\Lambda_1}(u)(V'_{3\Lambda'_1}) = V'_{3\Lambda'_2}$. Thus $\rho_{3\Lambda_1}(u)(V'_{3\Lambda'_1})_{T^2 \cdot \mathbf{Z}_3} = (V'_{3\Lambda'_2})_{T^2 \cdot \mathbf{Z}_3}$ and

$$\begin{aligned} &(\rho_{3\Lambda_1}(u))^2|_{(V'_{3\Lambda'_1})_{T^2 \cdot \mathbf{Z}_3}} \\ =&(\rho_{3\Lambda_1}(u^2))|_{(V'_{3\Lambda'_1})_{T^2 \cdot \mathbf{Z}_3}} \\ =&\mathrm{Id}, \end{aligned}$$

because $u^2 \in T^2 \cdot \mathbf{Z}_3$. Hence

$$\begin{aligned} &(V_{3\Lambda_1})_{T^2 \cdot \mathbf{Z}_6} \\ \subset&(V'_{3\Lambda'_1})_{T^2 \cdot \mathbf{Z}_3} \oplus (V'_{3\Lambda'_2})_{T^2 \cdot \mathbf{Z}_3} \end{aligned}$$

and

$$\dim(V_{3\Lambda_1})_{T^2 \cdot \mathbf{Z}_6} = 1.$$

Therefor we obtain that $3\Lambda_1 \in D(G_2, T^2 \cdot \mathbf{Z}_6)$ and its multiplicity is equal to 1. Moreover,

$$n(L^{12}) = \dim_{\mathbf{C}}(V_{3\Lambda_1}) = 77 = 91-14 = \dim(SO(14)) - \dim(G_2) = n_{hk}(L^{12}),$$

that is, $\mathcal{G}(G_2/T^2) \subset Q_{12}(\mathbf{C})$ is Hamiltonian rigid.

Let $\bigwedge^2 \mathbf{R}^{14} = \mathfrak{o}(n+2) = \mathrm{ad}_{\mathfrak{p}}(\mathfrak{g}_2) + \mathcal{V} \cong \mathfrak{g}_2 + \mathcal{V}$. Then

$$\bigwedge^2 \mathbf{C}^{14} = (\bigwedge^2 \mathbf{R}^{14})^{\mathbf{C}} = \mathfrak{o}(n+2)^{\mathbf{C}} = \mathfrak{o}(n+2, \mathbf{C}) = \mathrm{ad}_{\mathfrak{p}}(\mathfrak{g}_2^{\mathbf{C}}) + \mathcal{V}^{\mathbf{C}} \cong \mathfrak{g}_2^{\mathbf{C}} + \mathcal{V}^{\mathbf{C}},$$

where $\dim \mathcal{V} = 77$ and $\dim_{\mathbf{C}} \mathcal{V}^{\mathbf{C}} = 77$. More precisely, we observe that $\mathcal{V}$ is a real 77-dimensional irreducible G_2-module with $(\mathcal{V})_{T^2 \cdot \mathbf{Z}_6} \neq \{0\}$, and $\mathcal{V}^{\mathbf{C}}$ is a complex 77-dimensional G_2-module with $(\mathcal{V})^{\mathbf{C}}_{T^2\cdot\mathbf{Z}_6} \neq \{0\}$. Moreover, we have $\mathcal{V}^{\mathbf{C}} \cong V_{3\Lambda_1}$ with $\dim_{\mathbf{C}}(\mathcal{V})^{\mathbf{C}}_{T^2\cdot\mathbf{Z}_6} = 1$.

We conclude that

Theorem 4.1. *The Gauss image* $L^{12} = \frac{G_2}{T^2\cdot\mathbf{Z}_6} \to Q_{12}(\mathbf{C})$ *is strictly Hamiltonian stable.*

5. The case $(U,K) = (G_2, SO(4))$

(U,K) is of type G_2.

Let $\mathfrak{u} = \mathfrak{k}+\mathfrak{p}$ be the orthogonal symmetric Lie algebra of $(G_2, SO(4))$ and $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$. Here $\mathfrak{u} = \mathfrak{g}_2$, $\mathfrak{k} = \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, Let

$$p : \widetilde{K} = Spin(4) = SU(2) \times SU(2) \longrightarrow K = SO(4)$$

be the universal covering Lie group homomorphism with Deck transformation group $\mathbf{Z}_2$. The isotropy representation of $(G_2, SO(4))$ is explicitly described as follows (cf. [12]) :

Let $(V_l \otimes V_m, \rho_l \boxtimes \rho_m)$ denote an irreducible unitary representaion of $SU(2)\times SU(2)$ of complex dimension $(l+1)(m+1)$ obtained by taking the exterior tensor product of V_l and V_m and then

$$\{(V_l \otimes V_m, \rho_l \times \rho_m) \mid l, m \in \mathbf{Z},\ l, m \geq 0\}$$

is the complete set of all inequivalent irreducible unitary representations of $SU(2) \times SU(2)$.

Suppose that $(l,m) = (3,1)$. The real 8-dimensional vector subspace W of $V_3 \otimes V_1$ spanned over $\mathbf{R}$ by

$$\begin{aligned} \{\ & v_0^{(3)} \otimes v_0^{(1)} + v_3^{(3)} \otimes v_1^{(1)},\ \sqrt{-1}\ (v_0^{(3)} \otimes v_0^{(1)} - v_3^{(3)} \otimes v_1^{(1)}), \\ & v_1^{(3)} \otimes v_0^{(1)} - v_2^{(3)} \otimes v_1^{(1)},\ \sqrt{-1}\ (v_1^{(3)} \otimes v_0^{(1)} + v_2^{(3)} \otimes v_1^{(1)}), \\ & v_0^{(3)} \otimes v_1^{(1)} - v_3^{(3)} \otimes v_0^{(1)},\ \sqrt{-1}\ (v_0^{(3)} \otimes v_1^{(1)} + v_3^{(3)} \otimes v_0^{(1)}), \\ & v_2^{(3)} \otimes v_0^{(1)} + v_1^{(3)} \otimes v_1^{(1)},\ \sqrt{-1}\ (v_2^{(3)} \otimes v_0^{(1)} - v_1^{(3)} \otimes v_1^{(1)})\ \}. \end{aligned} \tag{5.1}$$

gives an irreducible orthogonal representation of $SU(2)\times SU(2)$ whose complexification is $V_3 \otimes V_1$, i.e. which is the *real form* of $V_3 \otimes V_1$. Then the vector space $\mathfrak{p}$ is isomorphic to W and $\mathfrak{a}$ corresponds to

$$\mathbf{R}(v_0^{(3)} \otimes v_0^{(1)} + v_3^{(3)} \otimes v_1^{(1)}) + \mathbf{R}(v_2^{(3)} \otimes v_0^{(1)} + v_1^{(3)} \otimes v_1^{(1)})\ .$$

For each

$$X = \begin{pmatrix} \sqrt{-1}x & u \\ -\bar{u} & -\sqrt{-1}x \end{pmatrix}, Y = \begin{pmatrix} \sqrt{-1}y & w \\ -\bar{w} & -\sqrt{-1}y \end{pmatrix} \in \mathfrak{su}(2), \tag{5.2}$$

the following useful formula holds :

Lemma 5.1.

$$
\begin{aligned}
&[d(\rho_l \boxtimes \rho_m)(X,Y)] \left(v_i^{(l)} \otimes v_j^{(m)} \pm v_{l-i}^{(l)} \otimes v_{m-j}^{(m)} \right) \\
=&\{(2i-l)x + (2j-m)y\} \sqrt{-1}(v_i^{(l)} \otimes v_j^{(m)} \mp v_{l-i}^{(l)} \otimes v_{m-j}^{(m)}) \\
&- \sqrt{i(l-i+1)}\ \mathrm{Re}(u)\ (v_{i-1}^{(l)} \otimes v_j^{(m)} \mp v_{l-i+1}^{(l)} \otimes v_{m-j}^{(m)}) \\
&+ \sqrt{i(l-i+1)}\ \mathrm{Im}(u)\ \sqrt{-1}(v_{i-1}^{(l)} \otimes v_j^{(m)} \pm v_{l-i+1}^{(l)} \otimes v_{m-j}^{(m)}) \\
&- \sqrt{j(m-j+1)}\ \mathrm{Re}(w)\ (v_i^{(l)} \otimes v_{j-1}^{(m)} \mp v_{l-i}^{(l)} \otimes v_{m-j+1}^{(m)}) \\
&+ \sqrt{j(m-j+1)}\ \mathrm{Im}(w)\ \sqrt{-1}(v_i^{(l)} \otimes v_{j-1}^{(m)} \mp v_{l-i}^{(l)} \otimes v_{m-j+1}^{(m)}) \\
&+ \sqrt{(l-i)(i+1)}\ \mathrm{Re}(u)\ (v_{i+1}^{(l)} \otimes v_j^{(m)} \mp v_{l-i-1}^{(l)} \otimes v_{m-j}^{(m)}) \\
&+ \sqrt{(l-i)(i+1)}\ \mathrm{Im}(u)\ \sqrt{-1}(v_{i+1}^{(l)} \otimes v_j^{(m)} \pm v_{l-i-1}^{(l)} \otimes v_{m-j}^{(m)}) \\
&+ \sqrt{(m-j)(j+1)}\ \mathrm{Re}(w)\ (v_i^{(l)} \otimes v_{j+1}^{(m)} \mp v_{l-i}^{(l)} \otimes v_{m-j-1}^{(m)}) \\
&+ \sqrt{(m-j)(j+1)}\ \mathrm{Im}(w)\ (v_i^{(l)} \otimes v_{j+1}^{(m)} \pm v_{l-i}^{(l)} \otimes v_{m-j-1}^{(m)})\ .
\end{aligned}
\tag{5.3}
$$

Remark. By using the formula (5.3) we can show that the real vector subspace W is invariant under the action of $SU(2) \times SU(2)$ via $\rho_3 \boxtimes \rho_1$.

Define an orthonormal basis of the real vector space $W \cong \mathfrak{p}$ as

$$
\begin{aligned}
H_1 &:= \frac{1}{\sqrt{2}}(v_0^{(3)} \otimes v_0^{(1)} + v_3^{(3)} \otimes v_1^{(1)}), \\
H_2 &:= \frac{1}{\sqrt{2}}(v_2^{(3)} \otimes v_0^{(1)} + v_1^{(3)} \otimes v_1^{(1)}), \\
E_1 &:= \frac{1}{\sqrt{2}}\sqrt{-1}\ (v_0^{(3)} \otimes v_0^{(1)} - v_3^{(3)} \otimes v_1^{(1)}), \\
E_2 &:= \frac{1}{\sqrt{2}}(v_1^{(3)} \otimes v_0^{(1)} - v_2^{(3)} \otimes v_1^{(1)}), \\
E_3 &:= \frac{1}{\sqrt{2}}\sqrt{-1}\ (v_1^{(3)} \otimes v_0^{(1)} + v_2^{(3)} \otimes v_1^{(1)}), \\
E_4 &:= \frac{1}{\sqrt{2}}(v_0^{(3)} \otimes v_1^{(1)} - v_3^{(3)} \otimes v_0^{(1)}), \\
E_5 &:= \frac{1}{\sqrt{2}}\sqrt{-1}\ (v_0^{(3)} \otimes v_1^{(1)} + v_3^{(3)} \otimes v_0^{(1)}), \\
E_6 &:= \frac{1}{\sqrt{2}}\sqrt{-1}\ (v_2^{(3)} \otimes v_0^{(1)} - v_1^{(3)} \otimes v_1^{(1)})\ .
\end{aligned}
\tag{5.4}
$$

Then we have the matrix expression as follows :

$$
\begin{aligned}
&[d(\rho_3 \boxtimes \rho_1)(X,Y)]\,(H_1,H_2)\\
&= (E_1,E_2,E_3,E_4,E_5,E_6,) \begin{pmatrix} -(3x+y) & 0 \\ \sqrt{3}\ \mathrm{Re}(u) & -(2\ \mathrm{Re}(u)+\mathrm{Re}(w)) \\ \sqrt{3}\ \mathrm{Im}(u) & 2\ \mathrm{Im}(u)+\mathrm{Im}(w) \\ \mathrm{Re}(w) & -\sqrt{3}\ \mathrm{Re}(u) \\ \mathrm{Im}(w) & \sqrt{3}\ \mathrm{Im}(u) \\ 0 & x-y \end{pmatrix}.
\end{aligned}
\tag{5.5}
$$

The inner product $\langle\ ,\ \rangle$ corresponding to the metric induced from $g^{std}_{Q_6(\mathbf{C})}$ is given as follows : For $(X,X'),(Y.Y') \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$,

$$
\begin{aligned}
&\langle (X,X'),(Y,Y')\rangle\\
:=\ & (3x+x')(3y+y')\\
&+3\ \mathrm{Re}(u)\mathrm{Re}(w) + (2\ \mathrm{Re}(u)+\mathrm{Re}(u'))(2\ \mathrm{Re}(w)+\mathrm{Re}(w'))\\
&+3\ \mathrm{Im}(u)\mathrm{Im}(w) + (2\ \mathrm{Im}(u)+\mathrm{Im}(u'))(2\ \mathrm{Im}(w)+\mathrm{Im}(w'))\\
&+\mathrm{Re}(u')\mathrm{Re}(w') + 3\mathrm{Re}(u)\mathrm{Re}(w)\\
&+\mathrm{Im}(u')\mathrm{Im}(w') + 3\mathrm{Im}(u)\mathrm{Re}(w)\\
&+(x-x')(y-y')\\
=\ & 10xy + 2x'y + 2xy' + 2x'y'\\
&+10\ \mathrm{Re}(u)\mathrm{Re}(w) + 2\ \mathrm{Re}(u')\mathrm{Re}(w) + 2\ \mathrm{Re}(u)\mathrm{Re}(w') + 2\mathrm{Re}(u')\mathrm{Re}(w')\\
&+10\ \mathrm{Im}(u)\mathrm{Im}(w) + 2\ \mathrm{Im}(u')\mathrm{Im}(w) + 2\ \mathrm{Im}(u)\mathrm{Im}(w') + 2\mathrm{Im}(u')\mathrm{Im}(w')
\end{aligned}
\tag{5.6}
$$

Thus the Casimir operator of $(\widetilde{K}, \widetilde{K}_{[\mathfrak{a}]})$ with respect to the inner product $\langle\ ,\ \rangle$ is given as follows :

$$
\begin{aligned}
\mathcal{C}_L =\ & \frac{1}{2}\,(X_1,0)\cdot(X_1,0) + \frac{1}{2}\,(X_2,0)\cdot(X_2,0) + \frac{1}{2}\,(X_3,0)\cdot(X_3,0)\\
&+\frac{5}{2}\,(0,X_1)\cdot(0,X_1) + \frac{5}{2}\,(0,X_2)\cdot(0,X_2) + \frac{5}{2}\,(0,X_3)\cdot(0,X_3)\\
&-\ (X_1,0)\cdot(0,X_1) -\ (X_2,0)\cdot(0,X_2) -\ (X_3,0)\cdot(0,X_3).
\end{aligned}
$$

Set

$$
\widetilde{K}_0 := \{(A,B) \in \widetilde{K} \mid \mathrm{A}d(p(A,B))H = H \text{ for each } H \in \mathfrak{a}\,\}. \tag{5.7}
$$

Then Therefore

$$
\begin{aligned}
&\widetilde{K}_0\\
=&\Big\{\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix}\right),\ \left(\begin{pmatrix}\sqrt{-1}&0\\0&-\sqrt{-1}\end{pmatrix},\begin{pmatrix}\sqrt{-1}&0\\0&-\sqrt{-1}\end{pmatrix}\right),\\
&\left(\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right),\ \left(\begin{pmatrix}-\sqrt{-1}&0\\0&\sqrt{-1}\end{pmatrix},\begin{pmatrix}-\sqrt{-1}&0\\0&\sqrt{-1}\end{pmatrix}\right),\\
&\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right),\ \left(\begin{pmatrix}0&\sqrt{-1}\\\sqrt{-1}&0\end{pmatrix},\begin{pmatrix}0&\sqrt{-1}\\\sqrt{-1}&0\end{pmatrix}\right),\\
&\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix},\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right),\ \left(\begin{pmatrix}0&-\sqrt{-1}\\-\sqrt{-1}&0\end{pmatrix},\begin{pmatrix}0&-\sqrt{-1}\\-\sqrt{-1}&0\end{pmatrix}\right)\Big\}.
\end{aligned}
\tag{5.8}
$$

In particular, the order of $\widetilde{K}_0$ is 8. This result is consistent with the results of [4, p.611], [5, p.651], [41, p.573] in Topology of Transformation Group Theory. Moreover we obtain

$$
\begin{aligned}
&K_0\\
=&\Big\{p\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)=p\left(\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right),\\
&p\left(\begin{pmatrix}\sqrt{-1}&0\\0&-\sqrt{-1}\end{pmatrix},\begin{pmatrix}\sqrt{-1}&0\\0&-\sqrt{-1}\end{pmatrix}\right)=p\left(\begin{pmatrix}-\sqrt{-1}&0\\0&\sqrt{-1}\end{pmatrix},\begin{pmatrix}-\sqrt{-1}&0\\0&\sqrt{-1}\end{pmatrix}\right),\\
&p\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)=p\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix},\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right),\\
&p\left(\begin{pmatrix}0&\sqrt{-1}\\\sqrt{-1}&0\end{pmatrix},\begin{pmatrix}0&\sqrt{-1}\\\sqrt{-1}&0\end{pmatrix}\right)=p\left(\begin{pmatrix}0&-\sqrt{-1}\\-\sqrt{-1}&0\end{pmatrix},\begin{pmatrix}0&-\sqrt{-1}\\-\sqrt{-1}&0\end{pmatrix}\right)\Big\}\\
\cong\ &\mathbf{Z}_2+\mathbf{Z}_2.
\end{aligned}
\tag{5.9}
$$

Hence the order of group K_0 is equal to 4 and

$$
\widetilde{K}/\widetilde{K}_0 \cong K/K_0 = SO(4)/\mathbf{Z}_2+\mathbf{Z}_2. \tag{5.10}
$$

Next we describe the subgroups of $\widetilde{K}$ defined as

$$
\begin{aligned}
\widetilde{K}_{\mathfrak{a}}\ :=&\{(A,B)\in\widetilde{K}\mid[(\rho_3\boxtimes\rho_1)(A,B)](\mathfrak{a})=\mathfrak{a}\},\\
\widetilde{K}_{[\mathfrak{a}]}:=&\{(A,B)\in\widetilde{K}\mid[(\rho_3\boxtimes\rho_1)(A,B)](\mathfrak{a})=\mathfrak{a}\\
&\qquad\text{preserving the orientation of }\mathfrak{a}\}\subset\widetilde{K}_{\mathfrak{a}}.
\end{aligned}
\tag{5.11}
$$

$(A, B) \in \widetilde{K}_{\mathfrak{a}}$ if and only if (A, B) is one of the following elements :

$$
\begin{aligned}
&\left(\begin{pmatrix} e^{\sqrt{-1}\theta_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} e^{\sqrt{-1}\theta'_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta'_1} \end{pmatrix} \right), \\
&\text{where } \theta_1 = \frac{\pi}{4}k_1, \theta'_1 = \frac{\pi}{4}k'_1, k_1, k'_1 \in \mathbf{Z}, k_1 - k'_1 \in 4\mathbf{Z}, \\
&\left(\begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2} \\ e^{\sqrt{-1}\theta_2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta'_2} \\ e^{\sqrt{-1}\theta'_2} & 0 \end{pmatrix} \right), \\
&\text{where } \theta_2 = \frac{\pi}{4}k_2, \theta'_2 = \frac{\pi}{4}k'_2, k_2, k'_2 \in \mathbf{Z}, k_2 - k'_2 \in 4\mathbf{Z}, \\
&\left(\begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta'_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta'_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta'_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta'_1} \end{pmatrix} \right) \\
&\text{where } \theta_1 = \frac{\pi}{4}k_1, \theta_2 = \frac{\pi}{4}k_2, \theta'_1 = \frac{\pi}{4}k'_1, \theta'_2 = \frac{\pi}{4}k'_2, \\
&\quad k_1, k_2, k'_1, k'_2 \in \mathbf{Z}, \\
&\quad k_1 + k_2, k_1 - k_2, k'_1 + k'_2, k'_1 - k'_2 \in 2\mathbf{Z}, \\
&\quad k_1 - k'_1, k_2 - k'_2 \in 4\mathbf{Z}, \\
&\quad k_1 + k_2 - k'_1 - k'_2, k_1 - k_2 - k'_1 + k'_2, \in 8\mathbf{Z}.
\end{aligned}
\tag{5.12}
$$

The order of $\widetilde{K}_{\mathfrak{a}}$ is equal to $16 + 16 + 32 + 32 = 96$.

$(A,B) \in \widetilde{K}_{[\mathfrak{a}]}$ if and only if (A,B) is one of the following elements :

$$\left(\begin{pmatrix} e^{\sqrt{-1}\theta_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} e^{\sqrt{-1}\theta'_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta'_1} \end{pmatrix} \right),$$
$$\text{where } \theta_1 = \frac{\pi}{4}k_1, \theta'_1 = \frac{\pi}{4}k'_1,\ k_1, k'_1 \in 2\mathbf{Z}, k_1 - k'_1 \in 4\mathbf{Z},$$
$$\left(\begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2} \\ e^{\sqrt{-1}\theta_2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta'_2} \\ e^{\sqrt{-1}\theta'_2} & 0 \end{pmatrix} \right),$$
$$\text{where } \theta_2 = \frac{\pi}{4}k_2, \theta'_2 = \frac{\pi}{4}k'_2,\ k_2, k'_2 \in 2\mathbf{Z}, k_2 - k'_2 \in 4\mathbf{Z},$$
$$\left(\begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta'_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta'_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta'_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta'_1} \end{pmatrix} \right) \tag{5.13}$$
$$\text{where } \theta_1 = \frac{\pi}{4}k_1, \theta_2 = \frac{\pi}{4}k_2, \theta'_1 = \frac{\pi}{4}k'_1, \theta'_2 = \frac{\pi}{4}k'_2,$$
$$k_1, k_2, k'_1, k'_2 \in 2\mathbf{Z}+1,$$
$$k_1 + k_2, k_1 - k_2, k'_1 + k'_2, k'_1 - k'_2 \in 2\mathbf{Z},$$
$$k_1 - k'_1, k_2 - k'_2 \in 4\mathbf{Z},$$
$$k_1 + k_2 - k'_1 - k'_2, k_1 - k_2 - k'_1 + k'_2, \in 8\mathbf{Z},$$

in other words, (A,B) is one of the following elements :

$$\left(\begin{pmatrix} e^{\sqrt{-1}\theta_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} e^{\sqrt{-1}\theta'_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta'_1} \end{pmatrix} \right),$$
$$\text{where } \theta_1 = \frac{\pi}{2}l_1, \theta'_1 = \frac{\pi}{2}l'_1,\ l_1, l'_1 \in \mathbf{Z}, l_1 - l'_1 \in 2\mathbf{Z},$$
$$\left(\begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2} \\ e^{\sqrt{-1}\theta_2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta'_2} \\ e^{\sqrt{-1}\theta'_2} & 0 \end{pmatrix} \right),$$
$$\text{where } \theta_2 = \frac{\pi}{2}l_2, \theta'_2 = \frac{\pi}{2}l'_2,\ l_2, l'_2 \in \mathbf{Z}, l_2 - l'_2 \in 2\mathbf{Z},$$
$$\left(\begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta'_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta'_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta'_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta'_1} \end{pmatrix} \right)$$
$$\text{where } \theta_1 = \frac{\pi}{2}l_1 + \frac{\pi}{4},\ \theta_2 = \frac{\pi}{2}l_2 + \frac{\pi}{4},\ \theta'_1 = \frac{\pi}{2}l'_1 + \frac{\pi}{4},\ \theta'_2 = \frac{\pi}{2}l'_2 + \frac{\pi}{4},$$
$$l_1, l_2, l'_1, l'_2 \in \mathbf{Z},$$
$$l_1 - l'_1, l_2 - l'_2 \in 2\mathbf{Z},$$
$$l_1 + l_2 - l'_1 - l'_2, l_1 - l_2 - l'_1 + l'_2, \in 4\mathbf{Z}. \tag{5.14}$$

The order of $\widetilde{K}_{[\mathfrak{a}]}$ is equal to $8+8+16+16=48=8\times 6=\sharp\widetilde{K}_0\times\sharp\mathbf{Z}_6$.

Lemma 5.2.

$$\widetilde{K}_{[\mathfrak{a}]}/\widetilde{K}_0\cong\mathbf{Z}_6\ . \tag{5.15}$$

Indeed, we compute

$$\begin{aligned}
A &= \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}(\frac{\pi}{2}l_1+\frac{\pi}{4})} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}(\frac{\pi}{2}l_2+\frac{\pi}{4})} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}(\frac{\pi}{2}l_2+\frac{\pi}{4})} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}(\frac{\pi}{2}l_1+\frac{\pi}{4})} \end{pmatrix},\\
A^3 &= \begin{pmatrix} -\sqrt{2}\cos(\frac{\pi}{2}l_1+\frac{\pi}{4}) & 0 \\ 0 & -\sqrt{2}\cos(\frac{\pi}{2}l_1+\frac{\pi}{4}) \end{pmatrix}\\
&= \begin{cases} -\mathrm{I}_2 & \text{if } l_1\equiv 0 \text{ or } 3 \pmod 4 \\ \mathrm{I}_2 & \text{if } l_1\equiv 1 \text{ or } 2 \pmod 4 \end{cases},\\
A^6 &= \mathrm{I}_2\ .
\end{aligned}$$

The generator of $\widetilde{K}_{[\mathfrak{a}]}/\widetilde{K}_0\cong\mathbf{Z}_6$ is represented by the element

$$\left(\begin{pmatrix} \frac{1+\sqrt{-1}}{2} & -\frac{1-\sqrt{-1}}{2} \\ \frac{1+\sqrt{-1}}{2} & \frac{1-\sqrt{-1}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1+\sqrt{-1}}{2} & \frac{1-\sqrt{-1}}{2} \\ -\frac{1+\sqrt{-1}}{2} & -\frac{1-\sqrt{-1}}{2} \end{pmatrix}\right)\ . \tag{5.16}$$

5.1. **Eigenvalue computation.** The Casimir operator $\mathcal{C}_L$ is given as

$$\begin{aligned}
\mathcal{C}_L =&\big(\frac{1}{2}(X_1^2+X_2^2+X_2^3),0\big)+\big(0,\frac{5}{2}(X_1^2+X_2^2+X_2^3)\big)\\
&-\big(X_1,0\big)\big(0,X_1\big)-\big(X_2,0\big)\big(0,X_2\big)-\big(X_3,0\big)\big(0,X_3\big)\ .
\end{aligned} \tag{5.17}$$

Here

$$X_1:=\frac{1}{2}\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix},\ X_2:=\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ X_3:=\frac{1}{2}\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}. \tag{5.18}$$

Lemma 5.3.

$$\begin{aligned}
&[d(\rho_l\boxtimes\rho_m)(\mathcal{C}_L)](v_i^{(l)}\otimes v_a^{(m)})\\
=&-\Big\{\frac{\{i-(l-i)\}^2}{8}+\frac{l+2i(l-i)}{4}\\
&+\frac{5\{a-(m-a)\}^2}{8}+\frac{5\{m+2a(m-a)\}}{4}\\
&-\frac{(i-(l-i))(a-(m-a))}{4}\Big\}(v_i^{(l)}\otimes v_a^{(m)})\\
&+\frac{1}{2}\sqrt{(i+1)(l-i)a(m-a+1)}(v_{i+1}^{(l)}\otimes v_{a-1}^{(m)})\\
&+\frac{1}{2}\sqrt{i(l-i+1)(a+1)(m-a)}(v_{i-1}^{(l)}\otimes v_{a+1}^{(m)})
\end{aligned}$$

(l,m)	$\dim(V_l \otimes V_m)_{\tilde{K}_0}$	eigenvalues of $\mathcal{C}_L$	$-\lambda \leq 6$
$(1,1)$	1	-3	$*$
$(2,0)$	0		
$(0,2)$	0		
$(3,1)$	2	$-3,-3$	$*$
$(1,3)$	2	$-9,-9$	
$(4,0)$	2	$-3,-3$	$*$
$(0,4)$	2	$-15,-15$	$*$
$(2,2)$	2	$-5,\ -5,\ -8$	$*$
$(5,1)$	3	$-5,-5,-8$	$*$
$(6,0)$	1	-6	$*$
$(4,2)$	3	$-6,-9,-9$	$*$
$(3,3)$	4	$-9,-12,-12,-15$	
$(8,0)$	2	$-10,-10$	
$(7,1)$	4	$-12,-12,-8,-8$	
$(6,2)$	5	$-15,-12,-8,-8,-12$	

$$
\begin{aligned}
&\{(l,m) \mid -C_L \leq 6 \text{ and } (V_l \otimes V_m)_{\tilde{K}_0} \neq \{0\}\} \\
=&\{(1,1),(4,0),(2,2),(1,3),(6,0),(5,1),(4,2)\}.
\end{aligned}
$$

It can shown that $(V_l \otimes V_m)_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ for $(l,m) = (1,1),(4,0),(1,3)$ and $(V_l \otimes V_m)_{\tilde{K}_{[\mathfrak{a}]}} \neq \{0\}$ for $(l,m) = (2,2),(6,0),(4,2)$ with multiplicity 1, respectively. But the fixed vector in $(V_2 \otimes V_2)_{\tilde{K}_{[\mathfrak{a}]}} \neq \{0\}$ corresponds to the larger eigenvalue $8 > 6$. Then from the dimension computation

$$
\begin{aligned}
&\dim_{\mathbf{C}} V_6 \boxtimes V_0 + \dim_{\mathbf{C}} V_4 \boxtimes V_2 = 7 \times 1 + 5 \times 3 = 7 + 15 = 22 \\
=&\dim SO(8) - \dim SO(4) = n_{hk}(L),
\end{aligned}
$$

we can conclude

Theorem 5.1. *The Gauss image* $L^6 = \frac{SO(4)}{(\mathbf{Z}_2+\mathbf{Z}_2)\cdot\mathbf{Z}_6} \to Q_6(\mathbf{C})$ *is strictly Hamiltonian stable.*

6. The case $(U,K) = (SO(5) \times SO(5), SO(5))$

(U,K) is of type B_2.

$$
\begin{aligned}
\mathfrak{u} =& \mathfrak{k} + \mathfrak{p}, \quad \mathfrak{a} \subset \mathfrak{p}, \\
\mathfrak{u} =& \mathfrak{o}(5) \oplus \mathfrak{o}(5) \\
\mathfrak{k} =& \{(X, X) | X \in \mathfrak{o}(5)\} \cong \mathfrak{o}(5), \\
\mathfrak{p} =& \{(X, -X) | X \in \mathfrak{o}(5)\}, \\
\mathfrak{a} =& \Big\{ (H, -H) \mid H = H(\xi_1, \xi_2) = \begin{pmatrix} 0 & -\xi_1 & 0 & 0 & 0 \\ \xi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\xi_2 & 0 \\ 0 & 0 & \xi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \xi_1, \xi_2 \in \mathbf{R} \Big\} \\
\cong& \mathfrak{t} = \{H(\xi_1, \xi_2) | \xi_1, \xi_2 \in \mathbf{R}\} \subset \mathfrak{o}(5).
\end{aligned}
$$

6.1. **Description of the subgroups K_0 and $K_{[\mathfrak{a}]}$.**

$$
\begin{aligned}
K =& SO(5) \\
K_0 =& \{A \in K | Ad(A)H = H, \quad \forall H \in \mathfrak{a}\} \\
=& \Big\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix} | A, B \in SO(2) \Big\} = T^2
\end{aligned}
$$

$$
\begin{aligned}
K_{[\mathfrak{a}]} :=& \{A \in K \mid \mathrm{Ad}(A)\mathfrak{a} \subset \mathfrak{t}^2, \text{ preserving orientations of } \mathfrak{a}\} \subset K_{\mathfrak{a}} \\
=& \begin{pmatrix} I_2 & & \\ & I_2 & \\ & & 1 \end{pmatrix} \cdot T^2 \cup \begin{pmatrix} 1 & 0 & & & \\ 0 & -1 & & & \\ & & 1 & 0 & \\ & & 0 & -1 & \\ & & & & 1 \end{pmatrix} \cdot T^2 \\
\cup& \begin{pmatrix} & & 1 & 0 & \\ & & 0 & 1 & \\ 1 & 0 & & & \\ 0 & -1 & & & \\ & & & & -1 \end{pmatrix} \cdot T^2 \cup \begin{pmatrix} & & 1 & 0 & \\ & & 0 & -1 & \\ 1 & 0 & & & \\ 0 & -1 & & & \\ & & & & -1 \end{pmatrix} \cdot T^2.
\end{aligned}
$$

The deck transformation group of the covering map $\mathcal{G} : N^8 \to \mathcal{G}(N^8)$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

6.2. **The groups K, K_1, K_0 and the fibration over K/K_0.**

$$K = SO(5) \supset K_1 = SO(4) \supset K_0 = T^2$$

$$K_1/K_0 = \frac{SO(4)}{T^2} \longrightarrow K/K_0 = \frac{SO(5)}{T^2} \longrightarrow K/K_1 = \frac{SO(5)}{SO(4)} \cong S^4$$

6.3. **Representation of the Casimir operator.**

$$\langle X, Y\rangle_{\mathfrak{so}(5)} := -\mathrm{Tr}(\mathrm{XY})$$

$$\begin{aligned}\mathcal{C}_L &= ((X_{31})^2 + (X_{41})^2 + (X_{32})^2 + (X_{42})^2) \\ &\quad + 2((X_{51})^2 + (X_{52})^2 + (X_{53})^2 + (X_{54})^2) \\ &= \mathcal{C}_{K_1/K_0} + 2\mathcal{C}_{K/K_1} \\ &= 2\mathcal{C}_{K/K_0} - \mathcal{C}_{K_1/K_0}, \text{ w.r.t. } \langle X, Y\rangle = -trXY, \forall X, Y \in \mathfrak{k} \cong o(5)\end{aligned}$$

where $\mathcal{C}_{K/K_0}$ and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operator of K/K_0 and K_1/K_0 relative to $\langle\ ,\ \rangle|_{\mathfrak{k}}$ and $\langle\ ,\ \rangle|_{\mathfrak{k}_1}$, respectively.

6.4. $D(K)$**,** $D(K_1)$**.**

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & -\xi_1 & & & \\ \xi_1 & 0 & & & \\ & & 0 & -\xi_2 & \\ & & \xi_2 & 0 & \\ & & & & 0 \end{pmatrix} \mid \xi_1, \xi_2 \in \mathbf{R} \right\} \subset \mathfrak{k}$$

$$\begin{aligned}\Gamma(K) = \Gamma(K_1) &= \{\xi \in \mathfrak{t} \mid \exp(\xi) = e\} \\ =&\left\{ \xi = \begin{pmatrix} 0 & -\xi_1 & & & \\ \xi_1 & 0 & & & \\ & & 0 & -\xi_2 & \\ & & \xi_2 & 0 & \\ & & & & 0 \end{pmatrix} \mid \xi_1, \xi_2 \in 2\pi\mathbf{Z} \right\}\end{aligned}$$

$$y_i : \mathfrak{t} \ni \xi \mapsto \xi_i \in \mathbf{R}.$$

$$\begin{aligned}&D(K) = D(SO(5)) \\ =&\{\Lambda \in \mathfrak{t}^* \mid \Lambda(\xi) \in 2\pi\mathbf{Z} \text{ for each } \xi \in \Gamma(K), \langle\Lambda, \alpha\rangle \geq 0 \text{ for each } \alpha \in \Pi(K)\} \\ =&\{\Lambda = p_1 y_1 + p_2 y_2 \mid p_1, p_2 \in \mathbf{Z}, p_1 \geq p_2 \geq 0\}\end{aligned}$$

$$\begin{aligned}&D(K_1) = D(SO(4)) \\ =&\{\Lambda = p_1 y_1 + p_2 y_2 \mid p_1, p_2 \in \mathbf{Z}, p_1 \geq |p_2|\}\end{aligned}$$

6.5. **Branching laws of** $(SO(5), SO(4))$**.** Let $\Lambda = k_1y_1 + k_2y_2 \in D(SO(5))$ be the highest weight of an irreducible $SO(5)$-module V_Λ, where $k_1, k_2 \in \mathbf{Z}$ and $k_1 \geq k_2 \geq 0$. Then by the branching laws of $(SO(5), SO(4))$ given in [15], V_Λ contains an irreducible $SO(4)-$module $W_{\Lambda'}$ with the highest weight $\Lambda' = k'_1y_1 + k'_2y_2 \in D(SO(4))$, where $k'_1, k'_2 \in \mathbf{Z}$, $k'_1 \geq |k'_2|$, if and only if

$$k_1 \geq k'_1 \geq k_2 \geq |k'_2|. \tag{6.1}$$

6.6. **Descriptions of** $D(K, K_0)$**,** $D(K_1, K_0)$**.** Let $\{\alpha'_1 = y_1 - y_2, \alpha'_2 = y_1 + y_2\}$ be the fundamental root system of $SO(4)$ and $\{\omega'_1 = \frac{1}{2}(y_1 - y_2), \omega'_2 = \frac{1}{2}(y_1 + y_2)\}$ be the fundamental weight system of $SO(4)$. We use the results of Satoru Yamaguchi [43] as follows:

Each $\Lambda' \in D(SO(4), T^2)$ can be expressed as

$$\Lambda' = k'_1y_1 + k'_2y_2 = m'_1\omega'_1 + m'_2\omega'_2 = p'_1\alpha'_1 + p'_2\alpha'_2,$$

where $m'_i \in \mathbf{Z}$, $m'_i \geq 0$, $p'_i \in \mathbf{Z}$, $p'_i \geq 1$ and

$$p'_1 = (k'_1 - k'_2)/2, \quad p'_2 = (k'_1 + k'_2)/2. \tag{6.2}$$

Let $\{\alpha_1 = y_1 - y_2, \alpha_2 = y_2\}$ be the fundamental root system of $SO(5)$ and $\{\omega_1 = y_1, \omega_2 = \frac{1}{2}(y_1 + y_2)\}$ be the fundamental weight system of $SO(5)$. We use the results of Satoru Yamaguchi [43] as follows:

Each $\Lambda \in D(SO(5), T^2)$ can be expressed as

$$\Lambda = k_1y_1 + k_2y_2 = m_1\Lambda_1 + m_2\Lambda_2 = p_1\alpha_1 + p_2\alpha_2, \tag{6.3}$$

where $m_i \in \mathbf{Z}$, $m_i \geq 0$, $p_i \in \mathbf{Z}$, $p_i \geq 1$ and

$$p_1 = k_1, \quad p_2 = k_1 + k_2.$$

6.7. **Eigenvalue computation.**

$$\begin{aligned} \Lambda &= k_1y_1 + k_2y_2 \in D(K, K_0), \\ &(k_i \in \mathbf{Z},\ k_1 \geq k_2 \geq 0) \\ \Sigma^+(K) &= \{y_1 + y_2, y_1 - y_2, y_1, y_2\} \\ 2\delta_K &= 3y_1 + y_2 \end{aligned}$$

$$\begin{aligned} \Lambda' &= k'_1y_1 + k'_2y_2 \in D(K_1, K_0) \\ &(k'_i \in \mathbf{Z},\ k'_1 \geq |k'_2|) \\ \Sigma^+(K_1) &= \{y_1 - y_2, y_1 + y_2\} \\ 2\delta_{K_2} &= 2y_1 \end{aligned}$$

$$
\begin{aligned}
c_\Lambda =&\langle \Lambda + 2\delta_K, \Lambda\rangle = \frac{1}{2}(k_1^2 + k_2^2 + 3k_1 + k_2),\\
c_{\Lambda'} =&\langle \Lambda' + 2\delta_{K_1}, \Lambda'\rangle = \frac{1}{2}((k_1')^2 + (k_2')^2 + 2k_1'),
\end{aligned}
$$

with respect to the inner product $\langle X, Y\rangle := -\mathrm{Tr}(XY)$ for any $X, Y \in \mathfrak{k} = o(5)$. Hence, we have the following eigenvalue formula.

$$
\begin{aligned}
c_L &= 2c_{K/K_0} - c_{K_1/K_0}\\
&= (k_1^2 + k_2^2 + 3k_1 + k_2) - \frac{1}{2}((k_1')^2 + (k_2')^2 + 2k_1').
\end{aligned}
$$

Since

$$
\mathcal{C}_L = \mathcal{C}_{K/K_0} + \mathcal{C}_{S^4} \geq \mathcal{C}_{K/K_0},
$$

the first eigenvalue of $\mathcal{C}_L$, i.e., $c_L \leq n = 8$ implies $c_\Lambda \leq 8$. Therefore we get

$$
\begin{aligned}
&\{\Lambda \in D(SO(5), T^2) \mid c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{so}_5}) \leq 8\}\\
=&\{y_1, y_1 + y_2, 2y_1, 2y_1 + y_2, 2y_1 + 2y_2\}.
\end{aligned}
$$

Suppose that $(k_1, k_2) = (1, 0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 5$. It follows from the branching laws (6.1) that $(k_1', k_2') = (0, 0) \oplus (1, 0)$. By (6.2), we have $(p_1', p_2') = (0, 0) \oplus (\frac{1}{2}, \frac{1}{2})$. Then $\Lambda'|_{(p_1', p_2') = (\frac{1}{2}, \frac{1}{2})} \notin D(SO(4), T^2)$. When $(p_1', p_2') = (0, 0)$,

$$
c_\Lambda = 2, \quad c_{\Lambda'} = 0, \quad c_L = 2c_\Lambda - c_{\Lambda'} = 4 - 0 = 4 < 8.
$$

On the other hand,

$$
\begin{pmatrix} A & 0 & 0\\ 0 & B & 0\\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 0\\ \vdots\\ 0\\ 1 \end{pmatrix} =
\begin{pmatrix} 0\\ \vdots\\ 0\\ 1 \end{pmatrix},
$$

so

$$
V_{\Lambda_1} \cong (\mathbf{R}^5)^{\mathbf{C}} \supset (\mathbf{R}^5)^{\mathbf{C}}_{K_0} = \mathbf{C}\begin{pmatrix} 0\\ \vdots\\ 0\\ 1 \end{pmatrix}.
$$

But

$$
\begin{pmatrix}
0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0\\
1 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix} 0\\ \vdots\\ 0\\ 1 \end{pmatrix} \neq
\begin{pmatrix} 0\\ \vdots\\ 0\\ 1 \end{pmatrix},
$$

therefore,

$$V_{\Lambda_1} \cong (\mathbf{R}^5)^{\mathbf{C}} \supset (\mathbf{R}^5)^{\mathbf{C}}_{K_0} = \mathbf{C}\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix} \supsetneqq (\mathbf{R}^5)^{\mathbf{C}}_{K_0\cdot \mathbf{Z}_4} = \{0\}.$$

Hence, $\Lambda_1 \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $(k_1, k_2) = (1, 1)$. Then $\dim_{\mathbf{C}} V_\Lambda = 10$. Then $V_\Lambda \cong \mathfrak{o}(5, \mathbf{C})$ and

$$(V_\Lambda)_{K_0} \cong (\mathfrak{t}^2)^{\mathbf{C}} \supset (V_\Lambda)_{K_{\mathfrak{a}}} = \{0\} \Rightarrow (V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}.$$

Hence, $\Lambda|_{(k_1,k_2)=(1,1)} \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $(k_1, k_2) = (2, 0)$. Then $(m_1, m_2) = (2, 0)$ and $\dim_{\mathbf{C}} V_\Lambda = 14$. It follows from the branching laws (6.1) that $(k_1', k_2') = (0, 0) \oplus (1, 0) \oplus (2, 0)$. By (6.2), we have $(p_1', p_2') = (0, 0) \oplus (\frac{1}{2}, \frac{1}{2}) \oplus (1, 1)$. Then $\Lambda'|_{(p_1',p_2')=(\frac{1}{2},\frac{1}{2})} \notin D(SO(4), T^2)$. When $(p_1', p_2') = (0, 0) \Rightarrow (m_1', m_2') = (0, 0)$,

$$c_\Lambda = 5, \quad c_{\Lambda'} = 0, \quad c_L = 2c_\Lambda - c_{\Lambda'} = 10 - 0 = 10 > 8.$$

When $(p_1', p_2') = (1, 1) \Rightarrow (m_1', m_2') = (2, 2)$,

$$c_\Lambda = 5, \quad c_{\Lambda'} = 4, \quad c_L = 2c_\Lambda - c_{\Lambda'} = 10 - 4 = 6 < 8.$$

On the other hand,

$$\begin{aligned} V_{2\Lambda_1} &\cong \mathrm{Sym}_0(\mathbf{C}^5) \\ &= \mathbf{C}\cdot \begin{pmatrix} -\frac{1}{4}I_4 & 0 \\ 0 & 1\end{pmatrix} \oplus \left\{\begin{pmatrix} X & 0 \\ 0 & 0\end{pmatrix} \mid X \in \mathrm{Sym}_0(\mathbf{C}^4)\right\} \\ &\oplus \left\{\begin{pmatrix} 0 & Z \\ {}^tZ & 0\end{pmatrix} \mid Z \in M(4,1;\mathbf{C})\right\} \\ &= W(0,0) \oplus W(2,2) \oplus W(1,1) \end{aligned}$$

$$(V_{2\Lambda_1})_{K_0} = \left\{\begin{pmatrix} c_1 I_2 & & \\ & c_2 I_2 & \\ & & c_3\end{pmatrix} \mid c_1, c_2, c_3 \in \mathbf{C}, 2c_1 + 2c_2 + c_3 = 0\right\}.$$

By direct calculation, we know that

$$\begin{pmatrix}0&0&1&0&0\\0&0&0&-1&0\\1&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&-1\end{pmatrix}\begin{pmatrix}c_1I_2&&\\&c_2I_2&\\&&c_3\end{pmatrix}\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&-1\end{pmatrix}$$
$$=\begin{pmatrix}c_2I_2&&\\&c_1I_2&\\&&c_3\end{pmatrix}.$$

$$(V_{2\Lambda_1})_{K_{[\mathfrak{a}]}}=\Big\{\begin{pmatrix}-\frac{c}{4}I_4&\\&c\end{pmatrix}\mid c\in\mathbf{C}\Big\}=W_{(0,0)}.$$

Thus
$$W_{(2,2)}\cap(V_{2\Lambda_1})_{K_{[\mathfrak{a}]}}=\{0\}.$$

Suppose that $(k_1,k_2)=(2,1)$. Then $(m_1,m_2)=(2,1)$ and $\dim_{\mathbf{C}}V_{2\Lambda_1+\Lambda_2}=35$. It follows from the branching laws (6.1) that $(k_1',k_2')=(1,0)\oplus(1,-1)\oplus(1,1)\oplus(2,0)\oplus(2,-1)\oplus(2,1)$. By (6.2), we have $(m_1',m_2')=(1,1)\oplus(2,0)\oplus(0,2)\oplus(2,2)\oplus(3,1)\oplus(1,3)$, thus

$$V_{\Lambda(2,1)}=W_{\Lambda'(1,1)}\oplus W_{\Lambda'(2,0)}\oplus W_{\Lambda'(0,2)}\oplus W_{\Lambda'(2,2)}\oplus W_{\Lambda'(3,1)}\oplus W_{\Lambda'(1,3)}.$$

It follows (6.2) that $(p_1',p_2')=(\frac{1}{2},\frac{1}{2})\oplus(1,0)\oplus(0,1)\oplus(1,1)\oplus(\frac{3}{2},\frac{1}{2})\oplus(\frac{1}{2},\frac{3}{2})$. Then $\Lambda'|_{(p_1',p_2')=(\frac{1}{2},\frac{1}{2})}$, $\Lambda'|_{(p_1',p_2')=(\frac{3}{2},\frac{1}{2})}$, $\Lambda'|_{(p_1',p_2')=(\frac{1}{2},\frac{3}{2})}\notin D(SO(4),T^2)$.

When $(p_1',p_2')=(1,0)$,
$$c_\Lambda=6,\quad c_{\Lambda'}=2,\quad c_L=2c_\Lambda-c_{\Lambda'}=12-2=10>8.$$
When $(p_1',p_2')=(0,1)$,
$$c_\Lambda=6,\quad c_{\Lambda'}=2,\quad c_L=2c_\Lambda-c_{\Lambda'}=12-2=10>8.$$
When $(p_1',p_2')=(1,1)$, $\Rightarrow(m_1',m_2')=(2,2)$,
$$c_\Lambda=6,\quad c_{\Lambda'}=4,\quad c_L=2c_\Lambda-c_{\Lambda'}=12-4=8.$$
Thus we need to determine the dimension of $(W_{\Lambda'(2,2)})_{K_{[\mathfrak{a}]}}\neq\{0\}$ directly.

$W_{\Lambda'(2,2)}\cong\mathfrak{sl}(2,\mathbb{C})\boxtimes\mathfrak{sl}(2,\mathbb{C})$

$$(W_{\Lambda'(2,2)})_{K_0}\cong(\mathfrak{sl}(2,\mathbb{C})\boxtimes\mathfrak{sl}(2,\mathbb{C}))_{K_0}=\mathbb{C}\boxtimes\mathbb{C}$$
Thus $\dim_{\mathbb{C}}(W_{\Lambda'(2,2)})_{K_0}=1$.

$$\{0\}\neq\mathcal{V}^{\mathbb{C}}\subset V_{2y_1+y_2}$$
By the irreducibility of $V_{2y_1+y_2}$, we have $\mathcal{V}^{\mathbb{C}}=V_{2y_1+y_2}$. Since $\{0\}\neq(\mathcal{V}^{\mathbb{C}})_{K_{[\mathfrak{a}]}}=(W_{\Lambda'(2,2)})_{K_{[\mathfrak{a}]}}\subset(W_{\Lambda'(2,2)})_{K_0}$ and $\dim_{\mathbb{C}}(W_{\Lambda'(2,2)})_{K_0}=1$, we

obtain $\{0\} \neq (\mathcal{V}^{\mathbb{C}})_{K_{[\mathfrak{a}]}} = (W_{\Lambda'(2,2)})_{K_{[\mathfrak{a}]}} = (W_{\Lambda'(2,2)})_{K_0}$ and $\dim_{\mathbb{C}}(W_{\Lambda'(2,2)})_{K_{[\mathfrak{a}]}} = 1$.

Here we note that

$$\begin{aligned}\bigwedge^2 \mathbb{R}^{10} =& \mathfrak{so}(10)\\ =&\mathrm{ad}(\mathfrak{so}(5)) + \mathcal{V},\\ \bigwedge^2 \mathbb{C}^{10} =& \mathfrak{so}(10, \mathbb{C})\\ =&\mathrm{ad}(\mathfrak{so}(5))^{\mathbb{C}} + \mathcal{V}^{\mathbb{C}},\\ \cong& \mathfrak{so}(5, \mathbb{C}) + \mathcal{V}^{\mathbb{C}}.\end{aligned}$$

$n(L^8) = \dim_{\mathbb{C}} V_{2y_1+y_2} = 35$

$n_{hk}(L^8) = \dim(SO(10)) - \dim(SO(5)) = 45 - 10 = 35$

Suppose that $(k_1, k_2) = (2, 2)$. It follows from the branching laws (6.1) that $(k_1', k_2') = (2, 0) \oplus (2, 1) \oplus (2, 2) \oplus (2, -1) \oplus (2, -2)$. By (6.2), we have $(p_1', p_2') = (1, 1) \oplus (\frac{1}{2}, \frac{3}{2}) \oplus (0, 2) \oplus (\frac{3}{2}, \frac{1}{2}) \oplus (2, 0)$, thus $\Lambda'|_{(p_1',p_2')=(\frac{1}{2},\frac{3}{2})}, \Lambda'|_{(p_1',p_2')=(\frac{3}{2},\frac{3}{2})}, \notin D(SO(4), T^2)$. When $(p_1', p_2') = (1, 1)$,

$$c_\Lambda = 8, \quad c_{\Lambda'} = 4, \quad c_L = 2c_\Lambda - c_{\Lambda'} = 16 - 4 = 12 > 8.$$

When $(p_1', p_2') = (0, 2)$,

$$c_\Lambda = 8, \quad c_{\Lambda'} = 6, \quad c_L = 2c_\Lambda - c_{\Lambda'} = 16 - 6 = 10 > 8.$$

When $(p_1', p_2') = (2, 0)$,

$$c_\Lambda = 8, \quad c_{\Lambda'} = 6, \quad c_L = 2c_\Lambda - c_{\Lambda'} = 16 - 6 = 10 > 8.$$

Remark 1. It follows from the branching laws (6.1) that

$$V_{k_1\epsilon_1+k_2\epsilon_2} = \oplus W_{k_1\epsilon_1'+k_2'\epsilon_2},$$

where k_1, k_2, k_1', k_2' are given as follows

(k_1, k_2)	(k_1', k_2')	(p_1', p_2')	$\dim_{\mathbf{C}} V_\Lambda$	c_L
$(1,0)$	$(0,0)\oplus(1,0)$	$(0,0)\oplus(\frac{1}{2},\frac{1}{2})$	5	4
$(1,1)$	$(1,0)\oplus(1,1)\oplus(1,-1)$	$(\frac{1}{2},\frac{1}{2})\oplus(0,1)\oplus(1,0)$	10	$4,4$
$(2,0)$	$(0,0)\oplus(1,0)\oplus(2,0)$	$(0,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(1,1)$	14	$10,6$
$(2,1)$	$(1,0)\oplus(1,-1)\oplus(1,1)$ $\oplus(2,0)\oplus(2,-1)\oplus(2,1)$	$(\frac{1}{2},\frac{1}{2})\oplus(1,0)\oplus(0,1)$ $\oplus(1,1)\oplus(\frac{3}{2},\frac{1}{2})\oplus(\frac{1}{2},\frac{3}{2})$	35	$10,10,8$
$(2,2)$	$(2,0)\oplus(2,1)\oplus(2,2)$ $\oplus(2,-1)\oplus(2,-2)$	$(1,1)\oplus(\frac{1}{2},\frac{3}{2})\oplus(0,2)$ $\oplus(\frac{3}{2},\frac{1}{2})\oplus(2,0)$	35	$12,10,10$

Therefore we conclude that

Theorem 6.1. *The Gauss image $\mathcal{G}(SO(5)/T^2) \subset Q_8(\mathbb{C})$ is strictly Hamiltonian stable.*

7. The case $(U,K) = (SO(10), U(5))$

(U,K) is of BC_2 type.

Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ is the canonical decomposition of $\mathfrak{u}$ as a symmetric Lie algebra of a symmetric pair (U,K) and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$.

$$\sigma : \mathfrak{u} \to \mathfrak{u}, \quad X \mapsto J_5 X J_5^{-1}, \quad J_5 = \begin{pmatrix} 0 & I_5 \\ -I_5 & 0 \end{pmatrix}$$

$$\begin{aligned}
\mathfrak{u} =& so(10)\\
\mathfrak{k} =& \{\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in so(10) | -X^t = X, Y^t = Y\} \cong \mathfrak{u}(5) \ni X + \sqrt{-1}Y,\\
\mathfrak{p} =& \{\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in so(10) | X, Y \in so(5)\},\\
\mathfrak{a} =& \Big\{\begin{pmatrix} H_1 & 0 \\ 0 & -H_1 \end{pmatrix} \mid H_1 = \begin{pmatrix} 0 & -\xi_1 & & & \\ \xi_1 & 0 & & & \\ & & 0 & -\xi_2 & \\ & & \xi_2 & 0 & \\ & & & & 0 \end{pmatrix}, \xi_1, \xi_2 \in \mathbf{R}\Big\}.
\end{aligned}$$

$$n = 18,$$

$$n_{kl}(\mathcal{G}) = \dim SO(n+2) - \dim K = \dim SO(20) - \dim U(5) = 165.$$

Define an $\mathrm{Ad}(U)$-invariant inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$ of $\mathfrak{u}$ by

$$\langle X, Y \rangle_{\mathfrak{u}} := -\mathrm{Tr}(XY)$$

for each $X, Y \in \mathfrak{u}$. The vector space $\mathfrak{p}$ is identified with the Euclidean space $\mathbf{R}^{20}$ with respect to the inner product $\langle \ , \ \rangle_{\mathfrak{u}}$. The 19-dimensional unit sphere $S^{19}(1)$ in $\mathfrak{p}$ is defined as

$$S^{19}(1) := \{X \in \mathfrak{p} \mid \|X\|_{\mathfrak{u}}^2 = \langle X, X\rangle_{\mathfrak{u}} = 1\}.$$

The isotropy linear action $\mathrm{Ad}_{\mathfrak{p}}$ of K on $\mathfrak{p}$ and thus $S^{19}(1)$ induces the group action of K on $\widetilde{\mathrm{Gr}}_2(\mathfrak{p}) \cong Q_{18}(\mathbf{C})$. For each *regular* element H of $\mathfrak{a} \cap S^{19}(1)$, we get a homogeneous isoparametric hypersurface in the unit sphere

$$N^{18} = (\mathrm{Ad}_{\mathfrak{p}} K)H \subset S^{19}(1) \subset \mathbf{R}^{20} \cong \mathfrak{p}.$$

Its Gauss image is

$$\mathcal{G}(N^{18}) = K \cdot [\mathfrak{a}] = [(\mathrm{Ad}_{\mathfrak{p}} K)\mathfrak{a}] \subset \widetilde{\mathrm{Gr}}_2(\mathfrak{p}) \cong Q_{18}(\mathbf{C}).$$

Here N and $\mathcal{G}(N^{18})$ have homogeneous space expressions $N \cong K/K_0$ and $\mathcal{G}(N^{18}) \cong K/K_{[\mathfrak{a}]}$.

7.1. **Description of the subgroups K_0 and $K_{[\mathfrak{a}]}$.**

$$\begin{gathered} K = U(5) \longrightarrow U = SO(10) \\ A + \sqrt{-1}B \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \end{gathered}$$

where $A, B \in gl(5, \mathbf{R})$.

$$K_0 := \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(H) = H \text{ for each } H \in \mathfrak{a}\},$$

$$= \left\{ \begin{pmatrix} a_{11}+\mathbf{i}b_{11} & a_{12}+\mathbf{i}b_{12} & 0 & 0 & 0 \\ -a_{12}+\mathbf{i}b_{12} & a_{11}-\mathbf{i}b_{11} & 0 & 0 & 0 \\ 0 & 0 & a_{22}+\mathbf{i}b_{22} & a_{21}+\mathbf{i}b_{21} & 0 \\ 0 & 0 & -a_{21}+\mathbf{i}b_{21} & a_{22}-\mathbf{i}b_{22} & 0 \\ 0 & 0 & 0 & 0 & a_{33}+\mathbf{i}b_{33} \end{pmatrix} \in U(5) \right\}$$

$$\cong SU(2) \times SU(2) \times U(1)$$

$$K_{\mathfrak{a}} := \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(\mathfrak{a}) = \mathfrak{a}\},$$

$$K_{[\mathfrak{a}]} := \{k \in K_{\mathfrak{a}} \mid \mathrm{Ad}_{\mathfrak{p}}(k) : \mathfrak{a} \longrightarrow \mathfrak{a} \text{ preserves the orientation of } \mathfrak{a}\}$$

$$= \begin{pmatrix} & & 1 & 0 & \\ & & 0 & 1 & \\ 1 & 0 & & & \\ 0 & -1 & & & \\ & & & & 1 \end{pmatrix} \cdot K_0 \cup \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix} \cdot K_0$$

$$\cup \begin{pmatrix} & & 1 & 0 & \\ & & 0 & -1 & \\ 1 & 0 & & & \\ 0 & 1 & & & \\ & & & & 1 \end{pmatrix} \cdot K_0 \cup K_0,$$

The deck transformation group of the covering map $\mathcal{G} : N \to \mathcal{G}(N^{18})$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

7.2. **The groups K, K_2, K_1, K_0 and the corresponding Lie algebras.**

$$\begin{aligned} K = U(5) \supset K_2 = U(4) \times U(1) \supset K_1 = U(2) \times U(2) \times U(1) \\ \supset K_0 = SU(2) \times SU(2) \times U(1) \end{aligned}$$

$$
\begin{aligned}
K &= U(5) \\
K_2 &= U(4) \times U(1) \\
&= \left\{ \begin{pmatrix} A & 0 \\ 0 & b \end{pmatrix} \mid A \in U(4), b \in U(1) \right\} \\
K_1 &= U(2) \times U(2) \times U(1) \\
&= \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & b \end{pmatrix} \mid A, B \in U(2), b \in U(1) \right\} \\
K_0 &= SU(2) \times SU(2) \times U(1) \\
&= \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & b \end{pmatrix} \mid A, B \in SU(2), b \in U(1) \right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{k} &= u(5) = \mathfrak{k}_0 \oplus \mathfrak{k}_{2\xi_1} \oplus \mathfrak{k}_{2\xi_2} \oplus \mathfrak{k}_{\xi_1+\xi_2} \oplus \mathfrak{k}_{\xi_1-\xi_2} \oplus \mathfrak{k}_{\xi_1} \oplus \mathfrak{k}_{\xi_2} \\
\mathfrak{k}_2 &= \left\{ \begin{pmatrix} X & \\ & \sqrt{-1}\theta \end{pmatrix} \mid X \in u(4), \theta \in \mathbf{R} \right\} \\
&= u(4) \oplus u(1) \\
&= \mathfrak{k}_0 \oplus \mathfrak{k}_{2\xi_1} \oplus \mathfrak{k}_{2\xi_2} \oplus \mathfrak{k}_{\xi_1+\xi_2} \oplus \mathfrak{k}_{\xi_1-\xi_2} \\
\mathfrak{k}_1 &= \left\{ \begin{pmatrix} X & & \\ & Y & \\ & & \sqrt{-1}\theta \end{pmatrix} \mid X, Y \in u(2), \theta \in \mathbf{R} \right\} \\
&= u(2) \oplus u(2) \oplus u(1) \\
&= \mathfrak{k}_0 \oplus \mathfrak{k}_{2\xi_1} \oplus \mathfrak{k}_{2\xi_2} \\
\mathfrak{k}_0 &= \left\{ \begin{pmatrix} X & & \\ & Y & \\ & & \sqrt{-1}\theta \end{pmatrix} \mid X, Y \in su(2), \theta \in \mathbf{R} \right\} \\
&= su(2) \oplus su(2) \oplus u(1)
\end{aligned}
$$

7.3. **Two fibrations over K/K_0.**

$$\begin{aligned} K_2/K_0 =& \frac{U(4)\times U(1)}{SU(2)\times SU(2)\times U(1)} \\ \longrightarrow K/K_0 =& \frac{U(5)}{SU(2)\times SU(2)\times U(1)} \\ \longrightarrow K/K_2 =& \frac{U(5)}{U(4)\times U(1)} \cong \mathbf{C}P^4 \end{aligned}$$

$$\begin{aligned} K_1/K_0 =& \frac{U(2)\times U(2)\times U(1)}{SU(2)\times SU(2)\times U(1)} \\ \longrightarrow K/K_0 =& \frac{U(5)}{SU(2)\times SU(2)\times U(1)} \\ \longrightarrow K/K_1 =& \frac{U(5)}{U(2)\times U(2)\times U(1)} \end{aligned}$$

7.4. **$\Gamma(K), D(K)$, etc.**

$$\begin{aligned} \mathfrak{t} =& \Bigg\{ \begin{pmatrix} \sqrt{-1}y_1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{-1}y_2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1}y_3 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{-1}y_4 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{-1}y_5 \end{pmatrix} \\ & \mid y_1, y_2, y_3, y_4, y_5 \in \mathbf{R} \Bigg\} \subset \mathfrak{k} \end{aligned}$$

$$\begin{aligned} &\Gamma(K) = \Gamma(K_2) = \Gamma(K_1) = \Gamma(K_0) \\ =&\{\xi \in \mathfrak{t} \mid \exp(\xi) = e\} \\ =&\Bigg\{ \xi = \begin{pmatrix} \sqrt{-1}\xi_1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{-1}\xi_2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1}\xi_3 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{-1}\xi_4 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{-1}\xi_5 \end{pmatrix} \mid \xi_1, \cdots, \xi_5 \in 2\pi\mathbf{Z} \Bigg\} \end{aligned}$$

$$\Gamma(C(K)) =\{\xi \in \mathfrak{c}(\mathfrak{k}) \mid \exp(\xi) = e\} = \Big\{\xi \in 2\pi\mathbf{Z}\mathrm{I}_5\Big\}$$

$$D(K) = D(U(5))$$
$$=\{\Lambda \in \mathfrak{t}^* \mid \Lambda(\xi) \in 2\pi\mathbf{Z} \text{ for each } \xi \in \Gamma(K), \langle \Lambda, \alpha \rangle \geq 0 \text{ for each } \alpha \in \Pi(K)\}$$
$$=\{\Lambda = p_1y_1 + \cdots + p_5y_5 \mid p_1, \cdots, p_5 \in \mathbf{Z}, p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5\}$$

$$D(K_2) = D(U(4) \times U(1))$$
$$=\{\Lambda = p_1y_1 + \cdots + p_5y_5 \mid p_1, \cdots, p_5 \in \mathbf{Z}, p_1 \geq p_2 \geq p_3 \geq p_4\}$$

$$D(K_1) = D(U(2) \times U(2) \times U(1))$$
$$=\{\Lambda = p_1y_1 + \cdots + p_5y_5 \mid p_1, \cdots, p_5 \in \mathbf{Z}, p_1 \geq p_2, p_3 \geq p_4\}$$

7.5. **Branching laws of** $(U(5), U(4) \times U(1))$**.**

Let $\Lambda = p_1y_1 + \cdots + p_5y_5 \in D(U(5))$ be the highest weight of an irreducible $U(5)$-module V_Λ, where $p_i \in \mathbf{Z}$ $(i = 1, \cdots, 5)$ and $p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5$. Then the irreducible decomposition of V_Λ as a $U(4) \times U(1)$-module contains an irreducible $U(4) \times U(1)$-module $V_{\Lambda'}$ with the highest weight $V_{\Lambda'} = q_1y_1 + \cdots + q_5y_5 \in D(U(4) \times U(1))$, where $q_i \in \mathbf{Z}$ and $q_1 \geq q_2 \geq q_3 \geq q_4$, if and only if
$$p_1 \geq q_1 \geq p_2 \geq q_2 \geq p_3 \geq q_3 \geq p_4 \geq q_4 \geq p_5,$$
$$\sum_{i=1}^{5} p_i = \sum_{i=1}^{5} q_i.$$
Moreover, the multiplicity of $V_{\Lambda'}$ is 1.

7.6. **Branching laws of** $(U(4), U(2) \times U(2))$**.**

Let $\Lambda = p_1y_1 + \cdots + p_4y_4 \in D(U(4))$ be the highest weight of an irreducible $U(4)$-module V_Λ, where $p_i \in \mathbf{Z}$ $(i = 1, \cdots, 4)$ and $p_1 \geq p_2 \geq p_3 \geq p_4$. Then the irreducible decomposition of V_Λ as a $U(2) \times U(2)$-module contains an irreducible $U(2) \times U(2)$-module $V_{\Lambda'}$ with the highest weight $V_{\Lambda'} = q_1y_1 + \cdots + q_4y_4 \in D(U(2) \times U(2))$, where $q_i \in \mathbf{Z}$ and $q_1 \geq q_2, q_3 \geq q_4$, if and only if

(i) $\sum_{i=1}^4 p_i = \sum_{i=1}^4 q_i$;

(ii) $p_1 \geq q_1 \geq p_3$, $p_2 \geq q_2 \geq p_4$;

(iii) in the finite power series expansion in X of $\frac{\prod_{i=1}^3 (X^{r_i+1} - X^{-(r_i+1)})}{(X - X^{-1})^2}$, where $r_i (i = 1, 2, 3)$ are defined as follows
$$r_1 := p_1 - \max(q_1, p_2)$$
$$r_2 := \min(q_1, p_2) - \max(q_2, p_3)$$
$$r_3 := \min(q_2, p_3) - p_4$$

the coefficient of $X^{q_3-q_4+1}$ does not vanish. Moreover, the value of this coefficient is the multiplicity of the $U(2)\times U(2)$-module.

7.7. $D(K, K_0)$.

Each $\Lambda \in D(U(5))$ is expressed as

$$\Lambda = p_1y_1 + \cdots p_5y_5,$$

where $p_i \in \mathbf{Z}$, $p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5$. Then by the branching law of $(U(5), U(4)\times U(1))$,

$$V_\Lambda = \bigoplus_{i=1}^{s} V'_{\Lambda'_i} = \bigoplus_{i=1}^{s} W'_{\Lambda'_{1i}} \boxtimes U_{q_5y_5},$$

where $\Lambda'_i = q_1y_1+q_2y_2+q_3y_3+q_4y_4+q_5y_5 \in D(K_2)$, $\Lambda'_{1i} = q_1y_1+q_2y_2+q_3y_3+q_4y_4 \in D(U(4))$, $q_5y_5 \in D(U(1))$ and

$$p_1 \geq q_1 \geq p_2 \geq q_2 \geq p_3 \geq q_3 \geq p_4 \geq q_4 \geq p_5,$$
$$\sum_{i=1}^{5} p_i = \sum_{j=1}^{5} q_j.$$

By the branching law of $(U(4), U(2)\times U(2))$,

$$W'_{\Lambda'_{1i}} = \bigoplus W''_{\Lambda''} = \bigoplus W''_{\tilde\Lambda_\sigma} \boxtimes W''_{\tilde\Lambda_\rho},$$

where $\Lambda'' = k_1y_1 + k_2y_2 + k_3y_3 + k_4y_4 \in D(U(2)\times U(2))$, $\tilde\Lambda_\sigma = k_1y_1 + k_2y_2 \in D(U(2))$, $\tilde\Lambda_\rho = k_3y_3 + k_4y_4 \in D(U(2))$ and

(i) $\sum_{i=1}^4 k_i = \sum_{i=1}^4 q_i$;

(ii) $q_1 \geq k_1 \geq q_3$, $q_2 \geq k_2 \geq q_4$;

(iii) in the finite power series expansion in X of $\frac{\prod_{i=1}^3 (X^{r_i+1}-X^{-(r_i+1)})}{(X-X^{-1})^2}$, where $r_i (i = 1, 2, 3)$ are defined as follows

$$r_1 := q_1 - \max(k_1, q_2)$$
$$r_2 := \min(k_1, q_2) - \max(k_2, q_3)$$
$$r_3 := \min(k_2, q_3) - q_4$$

the coefficient of $X^{k_3-k_4+1}$ does not vanish. Moreover, the value of this coefficient is the multiplicity of the $U(2)\times U(2)$-module.

By the branching law of $(U(2), SU(2))$,

$$W''_{\tilde\Lambda_\sigma} = W''_{\Lambda_\sigma}, \quad W''_{\tilde\Lambda_\rho} = W''_{\Lambda_\rho},$$

where $\Lambda_\sigma = \frac{k_1-k_2}{2}(y_1 - y_2) \in D(SU(2))$, $\Lambda_\rho = \frac{k_3-k_4}{2}(y_3 - y_4) \in D(SU(2))$.

Hence, one can decompose a K-module V_Λ into the following irreducible K_0-modules.

$$V_\Lambda = \bigoplus \bigoplus W''_{\Lambda_\sigma} \boxtimes W''_{\Lambda_\rho} \boxtimes U_{q_5 y_5}.$$

Now assume that $\Lambda \in D(K, K_0)$. Then there exists at least one nonzero trivial irreducible K_0-module in the above decomposition for some σ and ρ. So in this case, we have

$$\begin{aligned} &k_1 - k_2 = 0, \\ &k_3 - k_4 = 0, \\ &q_5 = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &2k_1 + 2k_3 = \sum_{i=1}^{4} q_i = \sum_{j=1}^{5} p_j \\ &q_2 \geq k_1 = k_2 \geq q_3 \\ &r_1 = q_1 - q_2, \\ &r_2 = k_1 - k_2 = 0, \\ &r_3 = q_3 - q_4, \end{aligned}$$

and in the finite power series expansion in X of

$$\frac{(X^{q_1-q_2+1} - X^{-(q_1-q_2+1)})(X^{q_3-q_4+1} - X^{-(q_3-q_4+1)})}{X - X^{-1}},$$

the coefficient of X does not vanish. Moreover, the value of this coefficient is the multiplicity of the $U(2) \times U(2)$-module.

7.8. **Descriptions of** $D(K, K_0)$**,** $D(K_2, K_0)$**,** $D(K_1, K_0)$**.** Each $\Lambda \in D(K, K_0)$ is

$$\Lambda = p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + p_5 y_5$$

where

$$p_1, \cdots, p_5 \in \mathbf{Z}, p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5.$$

Each $\Lambda' \in D(K_2, K_0)$ is

$$\Lambda' = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4$$

where

$$\begin{aligned} &q_1, \cdots, q_4 \in \mathbf{Z}, \\ &q_1 \geq q_2 \geq q_3 \geq q_4, \\ &\sum_{i=1}^{5} p_i = \sum_{j=1}^{4} q_j. \end{aligned}$$

Each $\Lambda'' \in D(K_1, K_0)$ is

$$\Lambda'' = k_1 y_1 + k_2 y_2 + k_3 y_3 + k_4 y_4$$

where

$$\begin{aligned} & k_1, \cdots, k_4 \in \mathbf{Z}, \\ & k_1 = k_2, k_3 = k_4, \\ & 2k_1 + 2k_3 = \sum_{j=1}^{4} q_j. \end{aligned}$$

7.9. **Representation of the Casimir operator.**

$$\begin{aligned} \mathcal{C}_L &= 2((X_{31})^2 + (X_{41})^2 + (X_{32})^2 + (X_{42})^2 + (Y_{31})^2 + (Y_{41})^2 + (Y_{32})^2 + (Y_{42})^2) \\ &\quad + 4((X_{51})^2 + (X_{52})^2 + (X_{53})^2 + (X_{54})^2 + (Y_{51})^2 + (Y_{52})^2 + (Y_{53})^2 + (Y_{54})^2) \\ &\quad + ((Y_{11})^2 + (Y_{33})^2) \\ &= 2\mathcal{C}_{K_2/K_1} + 4\mathcal{C}_{K/K_2} + \mathcal{C}_{K_1/K_0} \\ &= 4\mathcal{C}_{K/K_0} - 2\mathcal{C}_{K_2/K_0} - \mathcal{C}_{K_1/K_0}, \text{ w.r.t. } \langle X, Y \rangle = -trXY, \forall X, Y \in \mathfrak{u} = so(10) \\ &= 2\mathcal{C}_{K/K_0} - \mathcal{C}_{K_2/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0}, \text{ w.r.t. } \langle X, Y \rangle := -\mathrm{t}r(\mathrm{r}e(XY)) \forall X, Y \in \mathfrak{k} = u(5), \end{aligned}$$

where $\mathcal{C}_{K/K_0}$, $\mathcal{C}_{K_2/K_0}$ and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operator of K/K_0, K_2/K_0 and K_1/K_0 relative to $\langle\ ,\ \rangle|_{\mathfrak{k}}$, $\langle\ ,\ \rangle|_{\mathfrak{k}_2}$ and $\langle\ ,\ \rangle|_{\mathfrak{k}_1}$, respectively.

7.10. **Eigenvalue computation.**

$$\begin{aligned} \Lambda &= p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + p_5 y_5 \in D(K, K_0), \\ &\quad (p_i \in \mathbf{Z},\ p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5) \\ \Sigma^+(K) &= \{y_i - y_j, 1 \leq i < j \leq 5\} \\ 2\delta_K &= \sum_{i=1}^{5} (6 - 2i) y_i = 4y_1 + 2y_2 - 2y_4 - 4y_5 \end{aligned}$$

$$\begin{aligned} \Lambda' &= q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 \in D(K_2, K_0) \\ &\quad (q_i \in \mathbf{Z},\ q_1 \geq q_2 \geq q_3 \geq q_4) \\ \Sigma^+(K_2) &= \{y_i - y_j, 1 \leq i < j \leq 4\} \\ 2\delta_{K_2} &= \sum_{i=1}^{4} (5 - 2i) y_i = 3y_1 + y_2 - y_3 - 3y_4 \end{aligned}$$

$$
\begin{aligned}
\Lambda'' &= k_1y_1 + k_2y_2 + k_3y_3 + k_4y_4 \in D(K_1, K_0)\\
\Sigma^+(K_1) &= \{y_1 - y_2, y_3 - y_4\}\\
2\delta_{K_1} &= y_1 - y_2 + y_3 - y_4
\end{aligned}
$$

$$
\begin{aligned}
c_\Lambda =&\langle \Lambda + 2\delta_K, \Lambda\rangle\\
=&p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + 4p_1 + 2p_2 - 2p_4 - 4p_5,\\
c_{\Lambda'} =&\langle \Lambda' + 2\delta_{K_2}, \Lambda'\rangle\\
=&q_1^2 + q_2^2 + q_3^2 + q_4^2 + 3q_1 + q_2 - q_3 - 3q_4,\\
c_{\Lambda''} =&\langle \Lambda'' + 2\delta_{K_1}, \Lambda''\rangle\\
=&k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_1 - k_2 + k_3 - k_4\\
=&k_1^2 + k_2^2 + k_3^2 + k_4^2 \quad (\because k_1 = k_2, k_3 = k_4),
\end{aligned}
$$

with respect to the inner product $\langle X, Y\rangle := -\mathrm{Tr}(\mathrm{Re}XY)$ for any $X, Y \in \mathfrak{k} = u(5)$. Hence, we have the following eigenvalue formula.

$$
\begin{aligned}
c =& \ 4c_{K/K_0} - 2c_{K_2/K_0} - c_{K_1/K_0}\\
=& \ 2c_\Lambda - c_{\Lambda'} - \frac{1}{2}c_{\Lambda''}\\
=& \ 2(p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + 4p_1 + 2p_2 - 2p_4 - 4p_5)\\
&- (q_1^2 + q_2^2 + q_3^2 + q_4^2 + 3q_1 + q_2 - q_3 - 3q_4)\\
&- \frac{1}{2}(k_1^2 + k_2^2 + k_3^2 + k_4^2)
\end{aligned}
$$

$$
\begin{aligned}
&\{\ \Lambda \in D(K, K_0) \mid -c \leq 18\ \}\\
=&\{\ 0,\ (p_1, p_2, p_3, p_4, p_5) = (0, -1, -1, -1, -1),\\
&(p_1, p_2, p_3, p_4, p_5) = (1, 1, 1, 1, 0),\\
&(p_1, p_2, p_3, p_4, p_5) = (1, 1, 0, 0, 0),\\
&(p_1, p_2, p_3, p_4, p_5) = (0, 0, 0, -1, -1),\\
&(p_1, p_2, p_3, p_4, p_5) = (1, 0, 0, 0, -1),\\
&(p_1, p_2, p_3, p_4, p_5) = (2, 1, 1, 0, 0),\\
&(p_1, p_2, p_3, p_4, p_5) = (0, 0, -1, -1, -2),\\
&(p_1, p_2, p_3, p_4, p_5) = (1, 1, 0, -1, -1)\ \}.
\end{aligned}
$$

Denote the fundamental weight system of $SU(5)$ by $\omega_1, \omega_2, \omega_3, \omega_4$.

Suppose that $\Lambda = (1, 1, 1, 1, 0)$. Then $\dim V_\Lambda = 5$. By the branching law of $(U(5), U(4)\times U(1))$ that $\Lambda' = (0, -1, -1, -1, -1) \oplus (-1, -1, -1, -1, 0)$, where $\Lambda' = (-1,-1,-1,-1,0) \in D(K_2, K_0)$. By the branching law of $(U(4), U(2)\times U(2))$ that $\Lambda'' = (-1,-1,-1,-1) \in D(K_1,K_0)$. Thus $c_\Lambda = 8$, $c_{\Lambda'} = 4$, $c_{\Lambda''} = 4$ and $c_L = 2c_\Lambda - c_{\Lambda'} - \frac{1}{2}c_{\Lambda''} = 10 < 18$.

On the other hand, $\Lambda = \Lambda_0 + \omega_4$, where $\Lambda_0 = \frac{4}{5}\sum_{i=1}^5 y_i$. The group $K = U(5) = C(U(5))\cdot SU(5)$ acts on $\dim V_\Lambda = 5$ and $V_\Lambda \cong \mathbf{C}\otimes\bar{\mathbf{C}}^5$ by $\rho_{\Lambda_0}\boxtimes\bar{\mu}_5$, where $\bar{\mu}_5$ denotes the conjugate representation of the standard representation of $SU(5)$ on $\mathbf{C}^5$. For any element $g_0 = \begin{pmatrix} A & & \\ & B & \\ & & e^{\sqrt{-1}\theta}\end{pmatrix} \in K_0$ any element $u\otimes\mathbf{w}\in\mathbf{C}\otimes\bar{\mathbf{C}}^5$, where $A, B\in SU(2)$ and $\theta\in\mathbf{R}$,

$$\begin{aligned}\rho_\Lambda(g_0)(u\otimes\mathbf{w}) =&\rho_{\Lambda_0}(e^{\frac{\sqrt{-1}}{5}\theta}I_5)(u)\otimes\rho_{\omega_4}(e^{-\frac{\sqrt{-1}}{5}\theta}g_0)\mathbf{w}\\ =& e^{\frac{4\sqrt{-1}}{5}\theta}u\otimes\begin{pmatrix} e^{\frac{\sqrt{-1}}{5}\theta}\bar{A}\begin{pmatrix} w_1\\ w_2\end{pmatrix}\\ e^{\frac{\sqrt{-1}}{5}\theta}\bar{B}\begin{pmatrix} w_3\\ w_4\end{pmatrix}\\ e^{-\frac{4\sqrt{-1}}{5}\theta}w_5\end{pmatrix}.\end{aligned}$$

Hence $(V_\Lambda)_{K_0} = \mathrm{span}_{\mathbf{C}}\{1\otimes\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\}$.

For a generator $g = \begin{pmatrix} & & 1 & 0 & \\ & & 0 & 1 & \\ 1 & 0 & & & \\ 0 & -1 & & & \\ & & & & 1\end{pmatrix} \in K_{[\mathfrak{a}]}\subset K_2$ of $\mathbf{Z}_4$,

$$\begin{aligned}\rho_\Lambda(g)(u\otimes\mathbf{e}_5) &= \rho_{\Lambda_0}(e^{\sqrt{-1}\frac{\pi}{5}}I_5)(u)\otimes\rho_{\omega_4}(e^{-\sqrt{-1}\frac{\pi}{5}}g)(\mathbf{e}_5)\\ &= e^{\sqrt{-1}\frac{4\pi}{5}}u\otimes e^{\sqrt{-1}\frac{\pi}{5}}\mathbf{e}_5 = -u\otimes\mathbf{e}_5.\end{aligned}$$

So $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$, i.e., $\Lambda = (1,1,1,1,0)\notin D(K, K_{[\mathfrak{a}]})$. Similarly, $\Lambda = (0,-1,-1,-1,-1)\notin D(K,K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (1,1,0,0,0)$. Then $\dim V_\Lambda = 10$. By the branching law of $(U(5), U(4)\times U(1))$ that $\Lambda' = (1,1,0,0,0)\oplus(1,0,0,0,1)$, where $\Lambda' = (1,1,0,0,0)\in D(K_2,K_0)$. By the branching law of $(U(4), U(2)\times U(2))$ that $\Lambda'' = (1,1,0,0)\oplus(0,0,1,1)\oplus(1,0,1,0)$, where $\Lambda'' = (1,1,0,0)\oplus(0,0,1,1)\in D(K_1,K_0)$. Thus $c_\Lambda = 8$, $c_{\Lambda'} = 6$, $c_{\Lambda''} = 2$ and $c_L = 2c_\Lambda - c_{\Lambda'} - \frac{1}{2}c_{\Lambda''} = 9 < 18$.

On the other hand, $\Lambda = \Lambda_0 + \omega_2$, where $\Lambda_0 = \frac{2}{5}\sum_{i=1}^5 y_i$. $V_\Lambda \cong \mathbf{C} \oplus \wedge^2 \mathbf{C}^5$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ be the standard basis of $\mathbf{C}^5$. For any element $g_0 \in K_0$ given above and any element $u \otimes \mathbf{e}_i \wedge \mathbf{e}_j \in V_\Lambda$, $(1 \leq i < j \leq 5)$,

$$\begin{aligned}\rho_\Lambda(g_0)(u \otimes \mathbf{e}_i \wedge \mathbf{e}_j) &= \rho_{\Lambda_0}(e^{\frac{\sqrt{-1}}{5}\theta} I_5)(u) \otimes \rho_{\omega_2}(\mathbf{e}_i \wedge \mathbf{e}_j) \\ &= e^{\sqrt{-1}\frac{2}{5}\theta} u \otimes (e^{-\frac{\sqrt{-1}}{5}\theta} g_0 \mathbf{e}_i \wedge e^{-\frac{\sqrt{-1}}{5}\theta} g_0 \mathbf{e}_j).\end{aligned}$$

It is easy to see that $(V_\Lambda)_{K_0} = \mathrm{span}_{\mathbf{C}}\{1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2), 1 \otimes (\mathbf{e}_3 \wedge \mathbf{e}_4)\}$. For the generator $g \in K_{[\mathfrak{a}]}$ of $\mathbf{Z}_4$ given above, we have

$$\begin{aligned}\rho_\Lambda(g)(1 \otimes \mathbf{e}_1 \wedge \mathbf{e}_2) &= -1 \otimes \mathbf{e}_3 \wedge \mathbf{e}_4, \\ \rho_\Lambda(g)(1 \otimes \mathbf{e}_3 \wedge \mathbf{e}_4) &= 1 \otimes \mathbf{e}_1 \wedge \mathbf{e}_2.\end{aligned}$$

Hence $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$, i.e., $\Lambda = (1,1,0,0,0) \notin D(K, K_{[\mathfrak{a}]})$. Similarly, $\Lambda = (0,0,0,-1,-1) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (1,0,0,0,-1)$. Then $\dim V_\Lambda = 24$. By the branching law of $(U(5), U(4) \times U(1))$ that $\Lambda' = (1,0,0,0,-1) \oplus (1,0,0,-1,1) \oplus (0,0,0,0,0) \oplus (0,0,0,-1,1)$, where $\Lambda'_1 = (1,0,0,-1,0)$, $\Lambda'_2 = (0,0,0,0,0) \in D(K_2, K_0)$. By the branching law of $(U(4), U(2) \times U(2))$ that $\Lambda''_1 = (1,0,0,-1) \oplus (1,-1,0,0) \oplus (0,0,0,0) \oplus (0,0,1,-1) \oplus (0,-1,1,0)$, where $\Lambda''_1 = (0,0,0,0) \in D(K_1, K_0)$. Also, $\Lambda''_2 = (0,0,0,0) \in D(K_1, K_0)$. Thus $c_\Lambda = 10$, $c_{\Lambda'_1} = 8$, $c_{\Lambda''_1} = 0$, $c_L = 2c_\Lambda - c_{\Lambda'} - \frac{1}{2}c_{\Lambda''} = 12 < 18$ and $c_{\Lambda'_2} = 0$, $c_{\Lambda''_2} = 0$, $c_L = 20 > 18$.

On the other hand, $\Lambda = \omega_1 + \omega_4$ corresponds to the adjoint representation of $SU(5)$.

$$\begin{aligned}V_\Lambda = \mathbf{C} \otimes (\mathbf{C} \cdot &\begin{pmatrix} -\frac{1}{4}I_4 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbf{C} \cdot \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \\ &\oplus \mathbf{C} \cdot \begin{pmatrix} & & * \\ & & 0 \\ * & 0 & 0 \end{pmatrix} \oplus \mathbf{C} \cdot \begin{pmatrix} & & 0 \\ & & * \\ 0 & * & 0 \end{pmatrix}) \\ &= V'_{(0,0,0,0,0)} \oplus V'_{(1,0,0,-1,0)} \oplus V'_{(1,0,0,0,-1)} \oplus V'_{(0,0,0,-1,1)}.\end{aligned}$$

$$\begin{aligned}(V_\Lambda)_{K_0} = &\{ \begin{pmatrix} c_1 I_2 & & \\ & c_2 I_2 & \\ & & c_3 \end{pmatrix} \mid c_1, c_2, c_3 \in \mathbf{C}, 2c_1 + 2c_2 + c_3 = 0 \} \\ \subset &V'_{(0,0,0,0,0)} \oplus V'_{(1,0,0,-1,0)}.\end{aligned}$$

By direct calculation, we know that for $g \in D(K, K_{[\mathfrak{a}]})$ given above,

$$Ad(g)\begin{pmatrix} c_1 I_2 & & \\ & c_2 I_2 & \\ & & c_3 \end{pmatrix} = \begin{pmatrix} c_2 I_2 & & \\ & c_1 I_2 & \\ & & c_3 \end{pmatrix}.$$

Hence,

$$(V_\Lambda)_{K_{[\mathfrak{a}]}} = \left\{ \begin{pmatrix} -\frac{c}{4} I_4 & \\ & c \end{pmatrix} \mid c \in \mathbf{C} \right\} = V'_{(0,0,0,0,0)}.$$

But this 1-dimensional fixed vector space corresponds to the larger eigenvalue 20.

Suppose that $\Lambda = (2,1,1,0,0)$. Then $\dim V_\Lambda = 45$. By the branching law of $(U(5), U(4) \times U(1))$ that V_Λ can be decomposed the following irreducible $K_1 = U(4) \times U(1)$-submodules:

$$V_\Lambda = V'_{(2,1,1,0,0)} \oplus V'_{(1,1,1,0,1)} \oplus V'_{(2,1,0,0,1)} \oplus V'_{(1,1,0,0,2)},$$

where $\Lambda' = (2,1,1,0,0) \in D(K_2, K_0)$. By the branching law of $(U(4), U(2) \times U(2))$ that $\Lambda'' = (2,1,1,0) \oplus (2,0,1,1) \oplus (1,1,2,0) \oplus (1,1,1,1) \oplus (1,0,2,1)$, where $\Lambda'' = (1,1,1,1) \in D(K_1, K_0)$. Thus $c_\Lambda = 16$, $c_{\Lambda'} = 12$, $c_{\Lambda''} = 4$, $c_L = 2c_\Lambda - c_{\Lambda'} - \frac{1}{2} c_{\Lambda''} = 18$.

On the other hand, from the irreducible $U(4) \times U(1)$-decomposition of V_Λ, we know that $(V_\Lambda)_{K_0} \subset V'_{(2,1,1,0,0)}$. Notice that $\Lambda' = 2y_1 + y_2 + y_3 = \sum_{i=1}^4 y_i + y_1 - y_4 \in D(K_2, K_0)$ corresponds to the tensor product of $C(U(4))$ representation with the highest weight $\sum_{i=1}^4 y_i$, the adjoint representation of $SU(4)$ with the highest weight $y_1 - y_4$ and the trivial representation of $U(1)$. Then for any element $g_0 \in K_0$ and any element $u \otimes X \otimes v \in \mathbf{C} \otimes su(4) \otimes \mathbf{C} \cong V_{\Lambda'}$,

$$\rho_{\Lambda'}(g_0)(u \otimes X \otimes v) = u \otimes Ad \begin{pmatrix} A & \\ & B \end{pmatrix} (X) \otimes v.$$

Thus $(V_\Lambda)_{K_0} = \mathrm{span}\{1 \otimes \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix} \otimes 1\}$. For the element $g \in K_{[\mathfrak{a}]} \subset K_2$,

$$\rho_{\Lambda'}(g)(u \otimes \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix} \otimes v) = e^{\sqrt{-1}\pi} u \otimes \begin{pmatrix} -I_2 & \\ & I_2 \end{pmatrix} \otimes w,$$

it follows that $(V_{[\mathfrak{a}]}) = (V_\Lambda)_{K_0}$, i.e., $\Lambda = (2,1,1,0,0) \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1. Similarly, $\Lambda = (0,0,-1,-1,-2) \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1 and it also gives eigenvalue 18.

Suppose that $\Lambda = (1,1,0,-1,-1)$. Then $\dim V_\Lambda = 75$. By the branching law of $(U(4), U(2) \times U(2))$, V_Λ can be decomposed the following irreducible $K_1 = U(4) \times U(1)$-submodules:

$$V_\Lambda = V'_{(1,1,0,-1,-1)} \oplus V'_{(1,1,-1,-1,0)} \oplus V'_{(1,0,0,-1,0)} \oplus V'_{(1,0,-1,-1,1)},$$

where $\Lambda'_1 = (1,1,-1,-1,0)$ and $\Lambda'_2 = (1,0,0,-1,0) \in D(K_2, K_0)$. For Λ'_2, by the branching law of $(U(4), U(2) \times U(2))$, $\Lambda''_2 = (1,0,0,-1) \oplus (1,-1,0,0) \oplus (0,0,-1,-1) \oplus (0,0,0,0) \oplus (0,-1,1,0)$, where $\Lambda''_2 = (0,0,0,0) \in D(K_1, K_0)$. Therefore, $c_\Lambda = 16$, $c_{\Lambda'_2} = 8$, $c_{\Lambda''_2} = 0$, and $c_L = 2c_\Lambda - c_{\Lambda'_2} - \frac{1}{2}c_{\Lambda''_2} = 24 > 18$. For Λ'_1, by the branching law of $(U(4), U(2) \times U(2))$ that $\Lambda'' = (1,1,-1,-1) \oplus (1,0,0,-1) \oplus (1,-1,1,-1) \oplus (0,0,0,0) \oplus (0,-1,1,0) \oplus (-1,-1,1,1)$, where $\Lambda''_{11} = (1,1,-1,-1)$, $\Lambda''_{12} = (-1,-1,1,1)$, $\Lambda''_{13} = (0,0,0,0) \in D(K_1, K_0)$. Thus $c_\Lambda = 16$, $c_{\Lambda'} = 12$, $c_{\Lambda''_{11}} = c_{\Lambda''_{12}} = 4$, $c_{\Lambda''_{13}} = 0$, $c_L = 2c_\Lambda - c_{\Lambda'} - \frac{1}{2}c_{\Lambda''} = 18, 18$ or 20. Moreover, from the above irreducible K_2-decomposition of V_Λ and eigenvalue calculations, we only need to determine $\dim (V_\Lambda)_{K_{[\mathfrak{a}]}} \cap (V''_{11} \oplus V''_{12})$ since the fixed vectors in this subspace by $K_{[\mathfrak{a}]}$ give eigenvalue 18.

Recall that the irreducible representation of $SU(4)$ with the highest weight $\Lambda'_1 = y_1 + y_2 - y_3 - y_4 = 2\omega_2$ can be described as follows ([11]):

$$\mathrm{Sym}^2(\wedge^2 \mathbf{C}^4) = I(Gr_2(\mathbf{C}^4))_2 \oplus V'_{\Lambda'_1},$$

where $I(Gr_2(\mathbf{C}^4))_2$, the ideal of the Grassmannian $Gr_2(\mathbf{C}^4)$, denotes the space of all homogeneous polynomials of degree 2 on $\mathbb{P}(\wedge^2 \mathbf{C}^{4*})$ that vanish on $Gr_2(\mathbf{C}^4)$. Here, $I(Gr_2(\mathbf{C}^4))_2 \cong \wedge^4 \mathbf{C}^4 \cong \mathbf{C}$ can be written down explicitly in terms of a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ of $\mathbf{C}^4$:

$$\begin{aligned} I(Gr_2(\mathbf{C}^4))_2 = \mathrm{span}\{&(\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) + (\mathbf{e}_1 \wedge \mathbf{e}_4) \cdot (\mathbf{e}_2 \wedge \mathbf{e}_3) \\ &- (\mathbf{e}_1 \wedge \mathbf{e}_3) \cdot (\mathbf{e}_2 \wedge \mathbf{e}_4)\}. \end{aligned}$$

Thus, a basis for $V'_{\Lambda'_1}$ can be given explicitly. For any element $g_0 \in K_0$, denote $g'_0 = \begin{pmatrix} A & \\ & B \end{pmatrix} \in SU(2) \times SU(2) \subset U(4)$. The representation of K_0 on any element $u \otimes X \otimes w \in \mathbf{C} \otimes V'_{\Lambda'} \otimes \mathbf{C}$ is

$$\rho_\Lambda(g)(u \otimes X \otimes w) = \rho_0(1)(u) \otimes \rho_{\Lambda'_1}(g'_0)(X) \otimes \rho_0(e^{\sqrt{-1}\theta})(w).$$

By direct computation,

$$\begin{aligned} (V_\Lambda)_{K_0} \cap V'_{\Lambda'_1} = \mathrm{span}_{\mathbf{C}}\{\, &1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) \otimes 1, \\ &1 \otimes (\mathbf{e}_3 \wedge \mathbf{e}_4) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1, \\ &1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1\}, \end{aligned}$$

where $(\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) \in V''_{11}$, $(\mathbf{e}_3 \wedge \mathbf{e}_4) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \in V''_{12}$ and $(\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \in V''_{13}$. For the generator $g \in K_{[\mathfrak{a}]} \subset K_2$, denote $g' = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}$. The representation of g on $u \otimes X \otimes w$ is

$$\rho_\Lambda(g)(u \otimes X \otimes w) = \rho_0(e^{\frac{\sqrt{-1}}{4}\pi} I_4)(u) \otimes \rho_{\Lambda'_1}(e^{-\frac{\sqrt{-1}}{4}\pi} g')(X) \otimes \rho_0(1)(w).$$

It follows that

$$\begin{aligned}(V_\Lambda)_{K_{[\mathfrak{a}]}} \cap V'_{\Lambda'_1} = \operatorname{span}_{\mathbf{C}}\{&1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1, \\ &1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) \otimes 1, -1 \otimes (\mathbf{e}_3 \wedge \mathbf{e}_4) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1\}.\end{aligned}$$

In particular, $\Lambda = (1, 1, 0, -1, -1) \in D(K, K_{[\mathfrak{a}]})$ and

$$\begin{aligned}&(V_\Lambda)_{K_{[\mathfrak{a}]}} \cap (V''_{11} \cup V''_{12}) \\ &= \operatorname{span}_{\mathbf{C}}\{1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) \otimes 1 - 1 \otimes (\mathbf{e}_3 \wedge \mathbf{e}_4) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1\}\end{aligned}$$

with dimension 1, which corresponds to eigenvalue 18.

Therefore, The Gauss image L is Hamiltonian stable. Moreover,

$$\begin{aligned}&\dim V_{(0,0,-1,-1,-2)} + \dim V_{(2,1,1,0,0)} + \dim V_{(1,1,0,-1,-1)} \\ =&45 + 45 + 75 = 165 \\ =&\dim SO(20) - \dim U(5) = n_{hl}(L).\end{aligned}$$

Hence we can obtain the following

Theorem 7.1. *The Gauss image* $L^{18} = \frac{U(5)}{(SU(2)\times SU(2)\times U(1))\cdot \mathbf{Z}_4} \to Q_{18}(\mathbf{C})$ *is strictly Hamiltonian stable.*

8. The case $(U, K) = (SO(m+2), SO(2) \times SO(m))$ $(m \geq 3)$

(U, K) is of type B_2.

$$\mathfrak{u} = \mathfrak{o}(m+2), \sigma : \mathfrak{u} \to \mathfrak{u}, X \mapsto JXJ^{-1}, J = \begin{pmatrix} I_2 & 0 \\ 0 & -I_m \end{pmatrix},$$

$$\mathfrak{k} = \left\{\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \mid T_1 \in \mathfrak{o}(2), T_2 \in \mathfrak{o}(m)\right\} = \mathfrak{o}(2) + \mathfrak{o}(m),$$

$$\mathfrak{p} = \left\{\begin{pmatrix} 0 & -{}^tX \\ X & 0 \end{pmatrix} \mid X \in M(m, 2; \mathbf{R})\right\},$$

$$\mathfrak{a} = \left\{H = H(\xi_1, \xi_2) = \begin{pmatrix} 0 & -{}^t\xi & 0 \\ \xi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \xi = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \xi_1, \xi_2 \in \mathbf{R}\right\}.$$

8.1. **Description of the subgroups K_0 and $K_{[\mathfrak{a}]}$.**

$$\begin{aligned} K_0 =& \{k \in K | \mathrm{Ad}(k)H = H, \text{ for each } H \in \mathfrak{a}\} \\ =& \Big\{ \begin{pmatrix} I_2 & & \\ & I_2 & \\ & & T \end{pmatrix} | \, T \in SO(m-2) \Big\} \\ & \bigcup \Big\{ \begin{pmatrix} -I_2 & & \\ & -I_2 & \\ & & T \end{pmatrix} | \, T \in SO(m-2) \Big\} \\ \cong & \mathbf{Z}_2 \times SO(m-2). \end{aligned}$$

$$\begin{aligned} K_{[\mathfrak{a}]} &= \{k \in K | \mathrm{Ad}(k)\mathfrak{a} \subset \mathfrak{a} \text{ \& preserving the orientation of } \mathfrak{a}\} \\ &\cong (\mathbf{Z}_2 \times SO(m-2)) \cdot \mathbf{Z}_4 \end{aligned}$$

consists of all elements

$$a = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B' \end{pmatrix} \in K = SO(2) \times SO(m),$$

where

$$\begin{aligned} (A, B) = & \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \\ & \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right), \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right), \\ & \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \\ & \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right). \end{aligned}$$

The deck transformation group of the covering map $\mathcal{G} : N^{2m-2} \to \mathcal{G}(N^{2m-2})$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

8.2. **The groups K, K_1, K_0 and the fibration over K/K_0.**

$$\begin{aligned} T^2 &\cong \frac{SO(2) \times SO(2)}{\mathbf{Z}_2} \\ &\cong \frac{SO(2) \times SO(2) \times SO(m-2)}{\mathbf{Z}_2 \times SO(m-2)} = K_1/K_0 \\ \longrightarrow & K/K_0 = \frac{SO(2) \times SO(m)}{\mathbf{Z}_2 \times SO(m-2)} \\ \longrightarrow & K/K_1 = \frac{SO(2) \times SO(m)}{SO(2) \times SO(2) \times SO(m-2)} \cong \frac{SO(m)}{SO(2) \times SO(m-2)} \cong Q_{m-2}(\mathbf{C}) \end{aligned}$$

8.3. **Representation of the Casimir operator.**

Denote $\langle X, Y\rangle := -\frac{1}{2}\mathrm{tr}XY$ for any $X, Y \in \mathfrak{u}$.

The Casimir operator of L with respect to the induced metric from $Q_{2m-2}(\mathbf{C})$ is given as follows :

$$\mathcal{C}_L = \mathcal{C}_{K/K_0} - \frac{1}{2}\,\mathcal{C}_{K_1/K_0},$$

where $\mathcal{C}_{K/K_0}$ and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operator of K/K_0 and K_1/K_0 relative to $\langle\ ,\ \rangle|_{\mathfrak{k}}$ and $\langle\ ,\ \rangle|_{\mathfrak{k}_1}$, respectively.

8.4. **Branching laws.**

8.4.1. *Branching laws for* $(SO(2p+2), SO(2)\times SO(2p))$ $(p \geq 1)$.

Let $\Lambda = h_0\varepsilon_0 + h_1\varepsilon_1 + \cdots + h_{p-1}\varepsilon_{p-1} + \epsilon h_p\varepsilon_p \in D(SO(2p+2))$, where $\epsilon = 1$ or -1 and

$$h_0, h_1, \cdots, h_p \in \mathbf{Z}, h_0 \geq h_1 \geq \cdots \geq h_p \geq 0,$$

and $\Lambda' = k_0\varepsilon_0 + k_1\varepsilon_1 + \cdots + k_{p-1}\varepsilon_{p-1} + \epsilon' k_p\varepsilon_p \in D(SO(2)\times SO(2p))$ where $\epsilon' = 1$ or -1 and

$$k_0, k_1, \cdots, k_p \in \mathbf{Z}, k_1 \geq \cdots \geq k_p \geq 0.$$

The irreducible decomposition of V_Λ as a $SO(2)\times SO(2p)$-module contains an irreducible $SO(2)\times SO(2p)$-module $V'_{\Lambda'}$ if and only if

$$\begin{aligned} &h_{i-1} \geq k_i \geq h_{i+1}, \quad (1 \leq i \leq p-1)\\ &h_{p-1} \geq k_p \geq 0. \end{aligned}$$

and the coefficient of X^{k_0} in the finite power series

$$X^{\epsilon\epsilon' l_p}\prod_{i=0}^{p-1}\frac{X^{l_i+1} - X^{-l_i-1}}{X - X^{-1}},$$

does not vanish, where

$$\begin{aligned} &l_0 := h_0 - \max\{h_1, k_1\},\\ &l_i := \min\{h_i, k_i\} - \max\{h_{i+1}, k_{i+1}\}, \quad (1 \leq i \leq p-1)\\ &l_p := \min\{h_p, k_p\}. \end{aligned}$$

Moreover, the coefficient of X^{k_0} is the multiplicity of $V'_{\Lambda'}$ appearing in the irreducible decomposition.

8.4.2. *Branching laws for* $(SO(2p+3), SO(2)\times SO(2p+1))$ $(p\geq 1)$.

Let $\Lambda = h_0\varepsilon_0 + h_1\varepsilon_1 + \cdots + h_{p-1}\varepsilon_{p-1} + h_p\varepsilon_p \in D(SO(2p+3))$, where

$$h_0, h_1, \cdots, h_p \in \mathbf{Z}, h_0 \geq h_1 \geq \cdots \geq h_p \geq 0,$$

and $\Lambda' = k_0\varepsilon_0 + k_1\varepsilon_1 + \cdots + k_{p-1}\varepsilon_{p-1} + k_p\varepsilon_p \in D(SO(2)\times SO(2p+1))$ where

$$k_0, k_1, \cdots, k_p \in \mathbf{Z}, k_1 \geq k_2 \geq \cdots \geq k_p \geq 0.$$

The irreducible decomposition of V_Λ as a $SO(2)\times SO(2p+1)$-module contains an irreducible $SO(2)\times SO(2p+1)$-module $V'_{\Lambda'}$ if and only if

$$\begin{aligned} &h_{i-1} \geq k_i \geq h_{i+1}, \quad (1\leq i\leq p-1)\\ &h_{p-1} \geq k_p \geq 0. \end{aligned}$$

and the coefficient of X^{k_0} in the finite power series

$$\Big(\prod_{i=0}^{p-1} \frac{X^{l_i+1} - X^{-l_i-1}}{X - X^{-1}}\Big) \frac{X^{l_p+\frac{1}{2}} - X^{-l_p-\frac{1}{2}}}{X^{\frac{1}{2}} - X^{-\frac{1}{2}}}$$

does not vanish, where

$$\begin{aligned} &l_0 := h_0 - \max\{h_1, k_1\},\\ &l_i := \min\{h_i, k_i\} - \max\{h_{i+1}, k_{i+1}\}, \quad (1\leq i\leq p-1)\\ &l_p := \min\{h_p, k_p\}. \end{aligned}$$

Moreover, the coefficient of X^{k_0} is the multiplicity of $V'_{\Lambda'}$ appearing in the irreducible decomposition.

8.5. $D(K, K_0)$ and eigenvalue computation.

For each $\tilde{\Lambda} \in D(K) = D(SO(2)\times SO(m))$, where $m\geq 3$, $m = 2p$ $(p\geq 2)$ or $m = 2p+1$ $(p\geq 1)$,

$$\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + \cdots + k_p\varepsilon_p,$$

where $k_0\varepsilon_0 \in D(O(2))$, $k_1\varepsilon_1+\cdots+k_p\varepsilon_p \in D(O(m))$, and $k_0, k_1, \cdots, k_p \in \mathbf{Z}$ satisfying

$$\begin{aligned} &k_1 \geq k_2 \geq \cdots \geq k_{p-1} \geq |k_p| \quad \text{if } m = 2p,\\ &k_1 \geq k_2 \geq \cdots \geq k_{p-1} \geq k_p \geq 0 \quad \text{if } m = 2p+1. \end{aligned}$$

Set

$$\Lambda = k_1\varepsilon_1 + \cdots + k_p\varepsilon_p \in D(SO(m)).$$

Then we have

$$\tilde{V}_{\tilde{\Lambda}} = U_{k_0\varepsilon_0} \otimes V_\Lambda.$$

8.5.1. *The case* $m = 2p$ $(p \geq 2)$. Suppose that $m = 2p$ $(p \geq 2)$. Notice that

$$\begin{aligned} D(K, K_0) &= D(SO(2) \times SO(m), \mathbf{Z}_2 \times SO(m-2)) \\ &\subset D(SO(2) \times SO(m), SO(m-2)), \\ D(K_1, K_0) &= D(SO(2) \times SO(2) \times SO(m-2), \mathbf{Z}_2 \times SO(m-2)) \\ &\subset D(SO(2) \times SO(2) \times SO(m-2), SO(m-2)). \end{aligned}$$

Let $\Lambda \in D(SO(2p))$ be the highest weight of an irreducible $SO(2p)$-module V_Λ. It follows from the branching laws for $(SO(2p), SO(2) \times SO(2p-2))$ that V_Λ contains an irreducible $SO(2) \times SO(2p-2)$-module $V_{\Lambda'}$ with the highest weight $\Lambda' \in D(SO(2) \times SO(2p-2))$ and $(V'_{\Lambda'})_{SO(2p-2)} \neq \{0\}$ if and only if

$$\begin{aligned} \Lambda &= k_1\varepsilon_1 + k_2\varepsilon_2 \in D(SO(2p), SO(2p-2)), \\ \Lambda' &= k'_1\varepsilon_1 \in D(SO(2) \times SO(2p-2), SO(2p-2)), \end{aligned}$$

where $k_1, k_2, k'_1 \in \mathbf{Z}$, $k_1 \geq k_2 \geq 0$ and the coefficient of $X^{k'_1}$ in the finite power series expression in X of $\frac{X^{k_1-k_2+1} - X^{-(k_1-k_2+1)}}{X - X^{-1}}$ does not vanish. In particular,

$$-(k_1 - k_2) \leq k'_1 \leq k_1 - k_2.$$

For each $\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(SO(2) \times SO(2p), SO(2p-2))$, $\tilde{\Lambda}' = k_0\varepsilon_0 + k'_1\varepsilon_1 \in D(SO(2) \times SO(2) \times SO(2p-2), SO(2p-2))$. Hence,

$$\begin{aligned} \mathcal{C}_L =& \ \mathcal{C}_{K/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0} \\ &= \langle \tilde{\Lambda} + 2\delta_K, \tilde{\Lambda} \rangle - \frac{1}{2}\langle \tilde{\Lambda}' + 2\delta_{K_1}, \tilde{\Lambda}' \rangle \\ &= k_0^2 + k_1^2 + k_2^2 + 2(p-1)k_1 + 2(p-2)k_2 - \frac{1}{2}(k_0^2 + k'^2_1) \end{aligned}$$

(i) The case $\mathcal{G}(N^6) \cong \frac{SO(2) \times SO(4)}{(\mathbf{Z}_2 \times SO(2)) \cdot \mathbf{Z}_4} \to Q_6(\mathbf{C})$ with $p = 2$

Denote $\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(SO(2) \times SO(4), \mathbf{Z}_2 \times SO(2))$ by $\tilde{\Lambda} = (k_0, k_1, k_2)$. Let $\tilde{\Lambda}' = k_0\varepsilon_0 + k'_1\varepsilon_1 \in D(SO(2) \times SO(2) \times SO(2), \mathbf{Z}_2 \times SO(2))$. Thus $k_0 + k'_1$ is even. Then

$$\begin{aligned} &\{ \ \Lambda \in D(K, K_0) \mid -c_L \leq 6 \ \} \\ &= \{ \ 0, (\pm 1, 1, 0), (0, 1, 1), (\pm 2, 1, 1), (0, 2, 0), \\ &(0, 1, -1), (\pm 2, 1, -1) \ \}. \end{aligned}$$

Suppose that $\tilde{\Lambda} = (k_0, 1, 0)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 4$ and $\tilde{V}_{\tilde{\Lambda}} \cong U_{k_0\varepsilon_0} \otimes \mathbf{C}^4$, where $\Lambda = \varepsilon_1 \in D(K)$ corresponds to the matrix multiplication of $SO(4)$ on $\mathbf{C}^4$. It follows from the branching law of $(SO(4), SO(2) \times SO(2))$ that $k'_1 = \pm 1$. Hence $c_L =$

$\frac{1}{2}k_0^2 + \frac{5}{2}$. Notice that $U_{k_0\varepsilon_0} \otimes \mathbf{C}^4$ can be decomposed into the following $SO(2) \times SO(2) \times SO(2)$-modules:

$$U_{k_0\varepsilon_0} \otimes \mathbf{C}^4 = (U_{k_0\varepsilon_0} \otimes (\mathbf{C}^2 \oplus \{0\})) \oplus (U_{k_0\varepsilon_0} \otimes (\{0\} \oplus \mathbf{C}^2)).$$

$U_{k_0\varepsilon_0} \otimes (\{0\} \oplus \mathbf{C}^2)$ has no nonzero fixed vector by $\mathbf{Z}_2 \times SO(2)$. Since

$$\begin{aligned}
&\rho_{k_0\varepsilon_0+\varepsilon_1} \begin{pmatrix} -I_2 & & \\ & -I_2 & \\ & & T \end{pmatrix} (v \otimes \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \end{pmatrix}) \\
=&e^{\sqrt{-1}\pi k_0} v \otimes \begin{pmatrix} -w_1 \\ -w_2 \\ 0 \\ 0 \end{pmatrix} = e^{\sqrt{-1}\pi(k_0+1)} v \otimes \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \end{pmatrix},
\end{aligned}$$

$(\tilde{V}_{\tilde{\Lambda}})_{\mathbf{Z}_2 \otimes SO(2)} = U_{k_0\varepsilon_0} \otimes (\mathbf{C}^2 \oplus \{0\})$ if k_0 is odd. But since

$$\begin{aligned}
&\rho_{k_0\varepsilon_0+\varepsilon_1} \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & T' \end{pmatrix} (v \otimes \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \end{pmatrix}) \\
=&e^{\sqrt{-1}\frac{\pi}{2}k_0} v \otimes \begin{pmatrix} w_2 \\ w_1 \\ 0 \\ 0 \end{pmatrix},
\end{aligned}$$

$U_{k_0\varepsilon_0} \otimes (\mathbf{C}^2 \oplus \{0\})$ has no nonzero fixed vector by $(\mathbf{Z}_2 \times SO(2)) \cdot \mathbf{Z}_4$, i.e., $(\pm 1, 1, 0) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda}_1 = (k_0, 1, 1)$ and $\tilde{\Lambda}_2 = (k_0, 1, -1)$. Then $\dim \tilde{V}_{\tilde{\Lambda}_1} = \dim \tilde{V}_{\tilde{\Lambda}_2} = 3$ and $\tilde{V}_{\tilde{\Lambda}_1} \oplus \tilde{V}_{\tilde{\Lambda}_2} \cong \mathbf{C} \otimes \wedge^2\mathbf{C}^4$. It follows from the branching law of $(SO(4), SO(2) \times SO(2))$ that

$$\tilde{V}_{\tilde{\Lambda}_1} = \tilde{V}'_{(k_0,1,1)} \oplus \tilde{V}'_{(k_0,-1,-1)} \oplus \tilde{V}'_{(k_0,0,0)}.$$

Hence $(k_0, 0, 0) \in D(K_1, K_0)$. Thus $c_L = \frac{1}{2}k_0^2 + 4$, which equals to 4 if $k_0 = 0$ and 6 if $k_0 = \pm 2$.

Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbf{C}^4$. Then

$$\tilde{V}_{\tilde{\Lambda}_1} = \mathrm{span}\{e_1 \wedge e_2, e_1 \wedge e_3 - e_2 \wedge e_4, e_1 \wedge e_4 + e_2 \wedge e_3\},$$
$$\tilde{V}_{\tilde{\Lambda}_2} = \mathrm{span}\{e_3 \wedge e_4, e_1 \wedge e_3 + e_2 \wedge e_4, e_1 \wedge e_4 - e_2 \wedge e_3\}.$$

Since $e_1 \wedge e_2 \in \wedge^2 \mathbf{C}^4$ is fixed by the representation of $SO(2) \times SO(2)$ with respect to the highest weight $\tilde{\Lambda}_1$,

$$(\tilde{V}_{\tilde{\Lambda}_1})_{K_0} = \mathrm{span}\{1 \otimes (e_1 \wedge e_2)\}.$$

Moreover,

$$\begin{aligned}
&\rho_{\tilde{\Lambda}_1}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & T' \end{pmatrix})(v \otimes (e_1 \wedge e_2)) \\
=&e^{\sqrt{-1}\frac{\pi}{2}k_0} v \otimes (e_2 \wedge e_1).
\end{aligned}$$

Hence, $\tilde{\Lambda}_1 = (0, 1, 1) \notin D(K, K_{[\mathfrak{a}]})$ but $\tilde{\Lambda}_1 = (\pm 2, 1, 1) \in D(K, K_{[\mathfrak{a}]})$ and $(\tilde{V}_{\tilde{\Lambda}_1})_{K_{[\mathfrak{a}]}} \cong \mathbf{C} \otimes \mathbf{C}\{e_1 \wedge e_2\}$ for $k_0 = 2$ or -2, both of which give eigenvalue 6. Similarly, $\tilde{\Lambda}_2 = (0, 1, -1) \notin D(K, K_{[\mathfrak{a}]})$ but $\tilde{\Lambda}_2 = (\pm 2, 1, -1) \in D(K, K_{[\mathfrak{a}]})$ and $(\tilde{V}_{\tilde{\Lambda}_2})_{K_{[\mathfrak{a}]}} \cong \mathbf{C} \otimes \mathbf{C}\{e_3 \wedge e_4\}$ for $k_0 = 2$ or -2, both of which give eigenvalue 6.

Suppose that $\tilde{\Lambda} = (0, 2, 0)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 9$ and $\tilde{V}_{\tilde{\Lambda}} \cong \mathbf{C} \otimes \mathrm{S}^2_0(\mathbf{C}^4)$, where the corresponding representation of $SO(4)$ is just the adjoint representation on $\mathrm{S}^2_0(\mathbf{C}^4)$. It follows from the branching law of $(SO(4), SO(2) \times SO(2))$ that $k_1' = 0, \pm 2$. Thus $c_L = 8 - \frac{1}{2}{k_1'}^2$. When $k_1' = \pm 2$, $c_L = 6$, otherwise $c_L = 8 > 6$. On the other hand, $\mathrm{S}^2_0(\mathbf{C}^4)$ can be decomposed into the following $SO(2) \times SO(2)$-modules:

$$\begin{aligned}
V_{2\varepsilon_1} &\cong \mathrm{S}^2_0(\mathbf{C}^4) \\
&= \mathrm{S}^2_0(\mathbf{C}^2) \oplus \mathrm{S}^2_0(\mathbf{C}^2) \oplus M(2, 2; \mathbf{C}) \oplus \mathbf{C} \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}.
\end{aligned}$$

Thus, $\mathrm{S}_0^2(\mathbf{C}^2) \oplus \mathbf{C}\begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}$ is fixed by $\{-I_2\} \times SO(2)$ and $\dim(\tilde{V}_{\tilde{\Lambda}})_{K_0} = 3$. Moreover,

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & T' \end{pmatrix})(v \otimes \begin{pmatrix} a & b & \\ b & -a & \\ & & 0 \end{pmatrix}) \\
&= v \otimes \begin{pmatrix} -a & b & \\ b & a & \\ & & 0 \end{pmatrix}, \\
&\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & T' \end{pmatrix})(v \otimes \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}) \\
&= v \otimes \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}.
\end{aligned}$$

Hence,

$$(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}} = \mathbf{C} \otimes \mathbf{C}\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \oplus \mathbf{C} \otimes \mathbf{C}\begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}.$$

Notice that the first summand lies in the $SO(2) \times SO(2) \times SO(2)$-module $V'_{2\varepsilon_1} \oplus V'_{-2\varepsilon_1}$, which gives eigenvalue 6 and the second summand lies in the $SO(2) \times SO(2) \times SO(2)$-module with respect to weight $(0, 0, 0) \in D(K_1, K_0)$, which gives eigenvalue $8 > 6$. Therefore, $\tilde{\Lambda} = (0, 2, 0) \in D(K, K_{[\mathfrak{a}]})$ and the multiplicity corresponding to eigenvalue 6 is 1.

Since $\tilde{\Lambda} = (2, 1, 1), (-2, 1, 1), (2, 1, -1), (-2, 1, -1), (0, 2, 0) \in D(K, K_{[\mathfrak{a}]})$ give the smallest eigenvalue 6 with multiplicity 1 and

$$\begin{aligned}
&\dim \tilde{V}_{(2,1,1)} + \dim \tilde{V}_{(-2,1,1)} + \dim \tilde{V}_{(2,1,-1)} \\
&\quad + \dim \tilde{V}_{(-2,1,-1)} + \dim \tilde{V}_{(0,2,0)} \\
&= 3 + 3 + 3 + 3 + 9 = 21 \\
&= \dim SO(8) - \dim SO(2) \times SO(4),
\end{aligned}$$

hence, $\mathcal{G}(N^6) \subset Q_6(\mathbf{C})$ is strictly Hamiltonian stable.

(ii) The case $\mathcal{G}(N^{4p-2}) \cong \frac{SO(2)\times SO(2p)}{(\mathbf{Z}_2\times SO(2p-2))\cdot\mathbf{Z}_4} \to Q_{4p-2}(\mathbf{C})$ with $p \geq 3$.

Suppose that $\tilde{\Lambda} = (k_0, k_1, k_2) = (\pm 4, 0, 0) \in D(K, K_{[\mathfrak{a}]})$. Then $k_1' = 0$ and $c_L = 8 < 4p-2$ for $p \geq 3$. Therefore, $\mathcal{G}(N^{4p-2}) \cong \frac{SO(2)\times SO(2p)}{(\mathbf{Z}_2\times SO(2p-2))\cdot\mathbf{Z}_4} \to Q_{4p-2}(\mathbf{C})$ is not Hamiltonian stable if $p \geq 3$.

Theorem 8.1.

$$(SO(2)\times SO(2p))/(\mathbf{Z}_2 \times SO(2p-2))\mathbf{Z}_4 \quad (p \geq 2)$$

is not Hamiltonian stable if and only if $(m-2)-1 = 2p-3 \geq 3$. *If* $p = 2$, *then it is strictly Hamiltonian stable. (And they all are rigid.)*

8.5.2. *The case* $m = 2p+1$ $(p \geq 1)$. Suppose that $m = 2p+1$ $(p \geq 1)$. Let $\Lambda \in D(SO(2p+1))$ be the highest weight of an irreducible $SO(2p+1)$-module V_Λ. It follows from the branching laws for $(SO(2p+1), SO(2)\times SO(2p-1))$ that V_Λ contains an irreducible $SO(2)\times SO(2p-1)$-module $V_{\Lambda'}$ with the highest weight $\Lambda' \in D(SO(2)\times SO(2p-1))$ and $(V'_{\Lambda'})_{SO(2p-1)} \neq \{0\}$ if and only if

$$\Lambda = k_1\varepsilon_1 + k_2\varepsilon_2 \in D(SO(2p+1), SO(2p-1)),$$
$$\Lambda' = k_1'\varepsilon_1 \in D(SO(2)\times SO(2p-1), SO(2p-1)),$$

where $k_1, k_2, k_1' \in \mathbf{Z}$, $k_1 \geq k_2 \geq 0$ and the coefficient of $X^{k_1'}$ in the finite power series expression in X of $\frac{X^{k_1-k_2+1}-X^{-(k_1-k_2+1)}}{X-X^{-1}}$ does not vanish. In particular,

$$-(k_1-k_2) \leq k_1' \leq k_1 - k_2.$$

For each $\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(SO(2)\times SO(2p+1), SO(2p-1))$ for $p \geq 2$, or $\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 \in D(SO(2)\times SO(3))$ with $p = 1$, $\tilde{\Lambda}' = k_0\varepsilon_0 + k_1'\varepsilon_1 \in D(SO(2)\times SO(2)\times SO(2p-1), SO(2p-1))$, where $k_0, k_1, k_1' \in \mathbf{Z}$, $k_1 \geq 0$, $|k_1'| \leq k_1$, $k_0 + k_1'$ is even for $\tilde{\Lambda}' \in D(K_1, K_0)$. Hence,

$$\begin{aligned}
\mathcal{C}_L =\ & \mathcal{C}_{K/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0} \\
=\ & \langle \tilde{\Lambda} + 2\delta_K, \tilde{\Lambda}\rangle - \frac{1}{2}\langle \tilde{\Lambda}' + 2\delta_{K_1}, \tilde{\Lambda}'\rangle \\
=\ & \begin{cases} k_0^2 + k_1^2 + k_2^2 + (2p-1)k_1 + (2p-3)k_2 - \frac{1}{2}(k_0^2 + {k'}_1^2), & p \geq 2; \\ \frac{1}{2}k_0^2 + k_1^2 + k_1 - \frac{1}{2}{k'}_1^2, & p = 1. \end{cases}
\end{aligned}$$

(i) The case $\mathcal{G}(N^4) \cong \frac{SO(2)\times SO(3)}{\mathbf{Z}_2\cdot\mathbf{Z}_4} \to Q_4(\mathbf{C})$ with $p = 1$

Denote $\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 \in D(SO(2)\times SO(3), \mathbf{Z}_2)$ by $\tilde{\Lambda} = (k_0, k_1)$. Let $\tilde{\Lambda}' = k_0\varepsilon_0 + k_1'\varepsilon_1 \in D(SO(2)\times SO(2), \mathbf{Z}_2)$. Thus $k_0 + k_1'$ is even. Then

$$\{\ \Lambda \in D(K,K_0) \mid -c_L \leq 4\ \} = \{\ (\pm 2,0),(\pm 2,1),(\pm 1,1), \\ (0,1),(0,2)\ \}.$$

Suppose that $\tilde{\Lambda} = (\pm 2, 0)$. Notice that for any $v \otimes w \in \tilde{V}_{k_0\varepsilon_0} \cong \mathbf{C} \otimes \mathbf{C}$,

$$\rho_{k_0\varepsilon_0}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & -1 \end{pmatrix})(v \otimes w) = e^{\sqrt{-1}k_0\frac{\pi}{2}} v \otimes w,$$

$\tilde{\Lambda} = k_0\varepsilon_0 \in D(K, K_{[\mathfrak{a}]})$ if and only if $k_0 \in 4\mathbf{Z}$. Hence here $\tilde{\Lambda} = (\pm 2, 0) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (k_0, 1)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 3$. The complex representation of $K = SO(2) \times SO(3)$ with the highest weight $\tilde{\Lambda}$ corresponds to

$$\begin{aligned} \tilde{V}_{\tilde{\Lambda}} &= U_{k_0\varepsilon_0} \otimes V_{\varepsilon_1} \cong U_{k_0\varepsilon_0} \otimes \mathbf{C}^3 \\ &= (U_{k_0\varepsilon_0} \otimes \mathbf{C}^2) \oplus (U_{k_0\varepsilon_0} \otimes \mathbf{C}^1). \end{aligned}$$

For each $v \otimes w \in U_{k_0\varepsilon_0} \otimes \mathbf{C}^3$ and $\mathrm{diag}(-I_2, -I_2, 1) \in K_0$, where $w = (w_1, w_2, w_2)^t \in \mathbf{C}^3$, the representation of K_0 is given by

$$\begin{aligned} &\rho_{\tilde{\Lambda}}(\begin{pmatrix} -I_2 & & \\ & -I_2 & \\ & & 1 \end{pmatrix})(v \otimes w) \\ =&e^{\sqrt{-1}k_0\pi} v \otimes (-w_1, -w_2, w_3)^t \end{aligned}$$

Then $(V_{\tilde{\Lambda}})_{K_0} = \mathbf{C} \otimes \mathbf{C}(0,0,w_3)^t \cong \mathbf{C} \otimes \mathbf{C}$ if k_0 is even and $(V_{\tilde{\Lambda}})_{K_0} = \mathbf{C} \otimes \mathbf{C}(w_1, w_2, 0)^t \cong \mathbf{C} \otimes \mathbf{C}^2$ if k_0 is odd. Moreover,

$$\begin{aligned} &\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & -1 \end{pmatrix})(v \otimes w) \\ =&e^{\sqrt{-1}k_0\frac{\pi}{2}} v \otimes \begin{pmatrix} -w_2 \\ -w_1 \\ -w_3 \end{pmatrix}. \end{aligned}$$

Thus $\tilde{\Lambda} \in D(K, K_{[\mathfrak{a}]})$ if and only if $k_0 \equiv 2 \bmod 4$ and its multiplicity is 1. In particular, $\tilde{\Lambda} = (0,1)$ or $(\pm 1, 1) \notin D(K, K_{[\mathfrak{a}]})$

and $\tilde{\Lambda} = (\pm 2, 1) \in D(K, K_{[\mathfrak{a}]})$. For $\tilde{\Lambda} = (\pm 2, 1)$, it follows from the branching laws of $(SO(3), SO(2))$ that $|k_1'| \leq k_1$ thus $k_1' = 0$ such that $k_0 + k_1'$ is even. Hence, $c_L = 4$.

Suppose that $\tilde{\Lambda} = (0, 2)$. Then $\dim_{\mathbf{C}} \tilde{V}_{\tilde{\Lambda}} = 5$. It follows from the branching law of $(SO(3), SO(2))$ that $k_1' = 0, \pm 2$. When $k_1' = \pm 2$, $c_L = 4$, otherwise $c_L > 4$. On the other hand, $\Lambda = 2\varepsilon_1 \in D(SO(3))$ corresponds to $V_\Lambda \cong \mathrm{S}_0^2(\mathbf{C}^3)$ and the representation of $SO(3)$ on $\mathrm{S}_0^2(\mathbf{C}^3)$ is just the adjoint representation. Thus, $\mathrm{S}_0^2(\mathbf{C}^3)$ can be decomposed into the following $SO(2)$-modules:

$$
\begin{aligned}
&V_{2\varepsilon_1} \cong \mathrm{S}_0^2(\mathbf{C}^3) \\
&= \mathrm{S}_0^2(\mathbf{C}^2) \oplus \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{pmatrix} \mid a, b \in \mathbf{C} \right\} \\
&\oplus \mathbf{C} \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix} \\
&= \mathbf{C} \cdot \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{pmatrix} \oplus \mathbf{C} \cdot \begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & -1 \end{pmatrix} \\
&\oplus \mathbf{C} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \sqrt{-1} \\ 1 & \sqrt{-1} & 0 \end{pmatrix} \oplus \mathbf{C} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\sqrt{-1} \\ 1 & -\sqrt{-1} & 0 \end{pmatrix} \\
&\oplus \mathbf{C} \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix} \\
&= V'_{2\varepsilon_1} \oplus V'_{-2\varepsilon_1} \oplus V'_{\varepsilon_1} \oplus V'_{-\varepsilon_1} \oplus V'_0.
\end{aligned}
$$

It is easy to see that $(\tilde{V}_{\tilde{\Lambda}})_{K_0} \cong (\mathbf{C} \otimes \mathrm{S}_0^2(\mathbf{C}^2)) \oplus (\mathbf{C} \otimes \mathbf{C} \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix})$ and $(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}} \cong \mathbf{C} \otimes \mathbf{C} \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \oplus (\mathbf{C} \otimes \mathbf{C} \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix})$. Hence, $\tilde{\Lambda} = (0, 2) \in D(K, K_{[\mathfrak{a}]})$. Notice that the first summand lies in $\mathbf{C} \otimes (V'_{2\varepsilon_1} \oplus V'_{-2\varepsilon_1})$, which gives eigenvalue 4 and the second summand lies in $\mathbf{C} \otimes V'_0$, which gives eigenvalue $6 > 4$.

Therefore, $\mathcal{G}(N^4) \subset Q_4(\mathbf{C})$ is Hamiltonian stable. Moreover,

$$
\begin{aligned}
&\dim \tilde{V}_{(2,1)} + \dim \tilde{V}_{(-2,1)} + \dim \tilde{V}_{(0,2)} = 3 + 3 + 5 \\
&= 11 = \dim SO(6) - \dim SO(2) \times SO(3),
\end{aligned}
$$

hence, $\mathcal{G}(N^4) \subset Q_4(\mathbf{C})$ is strictly Hamiltonian stable.

(ii) The case $\mathcal{G}(N^8) \cong \frac{SO(2) \times SO(5)}{(\mathbf{Z}_2 \times SO(3)) \cdot \mathbf{Z}_4} \to Q_8(\mathbf{C})$ with $p = 2$

Denote $\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(SO(2) \times SO(5), \mathbf{Z}_2 \times SO(3))$ by $\tilde{\Lambda} = (k_0, k_1, k_2)$. Let $\tilde{\Lambda}' = k_0\varepsilon_0 + k_1'\varepsilon_1 \in D(SO(2) \times SO(2) \times SO(3), \mathbf{Z}_2 \times SO(3))$. Then

$$\{\ \Lambda \in D(K, K_0) \mid -c_L \leq 8\ \} = \{\ (\pm 4, 0, 0), (\pm 1, 1, 0), (\pm 3, 1, 0), (0, 1, 1), (\pm 2, 1, 1), (0, 2, 0)\ \}.$$

Suppose that $\tilde{\Lambda} = (\pm 4, 0, 0)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 1$. It follows from the branching law of $(SO(5), SO(2) \times SO(3))$ that $k_1' = 0$. Thus $c_L = 8$. On the other hand, for $v \otimes w \in \tilde{V}_{k_0\varepsilon_0} \cong \mathbf{C} \otimes \mathbf{C}$ and the generator $\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & T' \end{pmatrix} \in K_{[\mathfrak{a}]}$,

$$\rho_{k_0\varepsilon_0} \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & T' \end{pmatrix} (v \otimes w) = e^{\sqrt{-1}\frac{\pi}{2}k_0} v \otimes w = v \otimes w,$$

if $k_0 \in 4\mathbf{Z}\backslash\{0\}$. So $\tilde{\Lambda} = (\pm 4, 0, 0) \in D(K, K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (k_0, 1, 0)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 5$ and $\tilde{V}_{\tilde{\Lambda}} \cong U_{k_0\varepsilon_0} \otimes \mathbf{C}^5$, where $\Lambda = \varepsilon_1 \in D(K)$ corresponds to the matrix multiplication of $SO(5)$ on $\mathbf{C}^5$. It follows from the branching law of $(SO(5), SO(2) \times SO(3))$ that $k_1' = \pm 1$. Hence $c_L = \frac{1}{2}k_0^2 + \frac{7}{2}$. Notice that $U_{k_0\varepsilon_0} \otimes \mathbf{C}^5$ can be decomposed into the following $SO(2) \times SO(3)$-modules:

$$U_{k_0\varepsilon_0} \otimes \mathbf{C}^5 = (U_{k_0\varepsilon_0} \otimes (\mathbf{C}^2 \oplus \{0\})) \oplus (U_{k_0\varepsilon_0} \otimes (\{0\} \oplus \mathbf{C}^3)).$$

$U_{k_0}\varepsilon_0 \otimes (\{0\} \oplus \mathbf{C}^3)$ has no nonzero fixed vector by $\mathbf{Z}_2 \times SO(3)$. If k_0 is odd, then

$$\begin{aligned}
&\rho_{k_0\varepsilon_0+\varepsilon_1}\begin{pmatrix} -I_2 & & \\ & -I_2 & \\ & & T \end{pmatrix}(v \otimes \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}) \\
=&e^{\sqrt{-1}\pi k_0} v \otimes \begin{pmatrix} -w_1 \\ -w_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = v \otimes \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\end{aligned}$$

i.e., $(\tilde{V}_{\tilde{\Lambda}})_{\mathbf{Z}_2 \otimes SO(3)} = U_{k_0\varepsilon_0} \otimes (\mathbf{C}^2 \oplus \{0\})$ if k_0 is odd. But since

$$\begin{aligned}
&\rho_{k_0\varepsilon_0+\varepsilon_1}\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & T' \end{pmatrix}(v \otimes \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}) \\
=&e^{\sqrt{-1}\frac{\pi}{2}k_0} v \otimes \begin{pmatrix} w_2 \\ w_1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\end{aligned}$$

$U_{k_0\varepsilon_0} \otimes (\mathbf{C}^2 \oplus \{0\})$ has no nonzero fixed vector by $(\mathbf{Z}_2 \times SO(3)) \cdot \mathbf{Z}_4$, i.e., neither $(\pm 1, 1, 0)$ and $(\pm 3, 1, 0)$ is in $D(K, K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (k_0, 1, 1)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 10$ and $\tilde{V}_{\tilde{\Lambda}} \cong \mathbf{C} \otimes \wedge^2 \mathbf{C}^5$. It follows from the branching law of $(SO(5), SO(2) \times SO(3))$ that $k_1' = 0$. Thus $c_L = \frac{1}{2}k_0^2 + 6$. On the other hand, since $e_1 \wedge e_2 \in \wedge^2 \mathbf{C}^5$ is fixed by $SO(2) \times SO(3)$, $v \otimes (e_1 \wedge e_2) \in \mathbf{C} \otimes \wedge^2 \mathbf{C}^5$ is fixed by $\mathbf{Z}_2 \times SO(3) \subset SO(2) \times SO(2) \times SO(3)$. Moreover,

$$\begin{aligned}
&\rho_{k_0\varepsilon_0+\varepsilon_1+\varepsilon_2}\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & T' \end{pmatrix}(v \otimes (e_1 \wedge e_2)) \\
=&e^{\sqrt{-1}\frac{\pi}{2}k_0} v \otimes (e_2 \wedge e_1).
\end{aligned}$$

Hence, $\tilde{\Lambda} = (0,1,1) \notin D(K, K_{[\mathfrak{a}]})$ but $\tilde{\Lambda} = (\pm 2, 1, 1) \in D(K, K_{[\mathfrak{a}]})$ and $(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}} \cong \mathbf{C} \otimes \mathbf{C}\{e_1 \wedge e_2\}$ for $k_0 = 2$ or -2, both of which give eigenvalue 8.

Suppose that $\tilde{\Lambda} = (0,2,0)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 14$ and $\tilde{V}_{\tilde{\Lambda}} \cong \mathbf{C} \otimes \mathrm{S}^2_0(\mathbf{C}^5)$, where the representation of $SO(5)$ with highest weight $2\varepsilon_1$ is just the adjoint representation on $\mathrm{S}^2_0(\mathbf{C}^5)$. It follows from the branching law of $(SO(5), SO(2) \times SO(3))$ that $k'_1 = 0, \pm 2$. Thus $c_L = 10 - \frac{1}{2}{k'_1}^2$. When $k'_1 = \pm 2$, $c_L = 8$, otherwise $c_L = 10 > 8$. On the other hand, $\mathrm{S}^2_0(\mathbf{C}^5)$ can be decomposed into the following $SO(2) \times SO(3)$-modules:

$$\begin{aligned} V_{2\varepsilon_1} &\cong \mathrm{S}^2_0(\mathbf{C}^5) \\ &= \mathrm{S}^2_0(\mathbf{C}^2) \oplus \mathrm{S}^2_0(\mathbf{C}^3) \oplus M(2,3;\mathbf{C}) \\ &\quad \oplus \left\{ \begin{pmatrix} zI_2 & \\ 0 & wI_3 \end{pmatrix} \mid z, w \in \mathbf{C}, 2z + 3w = 0 \right\}. \end{aligned}$$

Thus, $\mathrm{S}^2_0(\mathbf{C}^2)$ is fixed by $\{-I_2\} \times SO(3)$ and

$$(\tilde{V}_{\tilde{\Lambda}})_{K_0} \cong \mathbf{C} \otimes \mathrm{S}^2_0(\mathbf{C}^2) \oplus \mathbf{C} \otimes \mathbf{C} \begin{pmatrix} 3I_2 & \\ & -2I_3 \end{pmatrix}.$$

Moreover,

$$\begin{aligned} &\rho_{2\varepsilon_1} \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & T' \end{pmatrix} (v \otimes \begin{pmatrix} a & b & \\ b & -a & \\ & & 0 \end{pmatrix}) \\ &= v \otimes \begin{pmatrix} -a & b & \\ b & a & \\ & & 0 \end{pmatrix}. \end{aligned}$$

Hence, $(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}} = \mathbf{C} \otimes \mathbf{C} \cdot \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \oplus \mathbf{C} \otimes \mathbf{C} \begin{pmatrix} 3I_2 & \\ & -2I_3 \end{pmatrix}$.

Therefore, $\tilde{\Lambda} = (0,2,0) \in D(K, K_{[\mathfrak{a}]})$. Notice the first summand lies in $\tilde{V}'_{(0,2,0)} \oplus \tilde{V}'_{(0,-2,0)}$ which gives eigenvalue 8 and the second summand lies in $\tilde{V}'_{(0,0,0)}$ which gives eigenvalue 10. Hence the multiplicity corresponding to eigenvalue 8 is 1.

Since $\tilde{\Lambda} = (4,0,0)$, $(-4,0,0)$, $(2,1,1)$, $(-2,1,1)$, $(0,2,0) \in D(K, K_{[\mathfrak{a}]})$ give the smallest eigenvalue 8 with multiplicity 1 and

$$\begin{aligned}
&\dim \tilde{V}_{(4,0,0)} + \dim \tilde{V}_{(-4,0,0)} + \dim \tilde{V}_{(2,1,1)} + \dim \tilde{V}_{(-2,1,1)} \\
&+ \dim \tilde{V}_{(0,2,0)} \\
&= 1 + 1 + 10 + 10 + 14 = 36 \\
&> 34 = \dim SO(10) - \dim SO(2) \times SO(5),
\end{aligned}$$

hence, $\mathcal{G}(N^8) \subset Q_8(\mathbf{C})$ is Hamiltonian stable but not strictly Hamiltonian stable.

(iii) The case $\mathcal{G}(N^{4p}) \cong \frac{SO(2)\times SO(2p+1)}{(\mathbf{Z}_2\times SO(2p-1))\cdot \mathbf{Z}_4} \to Q_{4p}(\mathbf{C})$ with $p \geq 3$.

Suppose that $\tilde{\Lambda} = (k_0, k_1, k_2) = (\pm 4, 0, 0) \in D(K, K_{[\mathfrak{a}]})$. Then $k_1' = 0$ and $c_L = 8 < 4p$ for $p \geq 3$. Therefore, $\mathcal{G}(N^{4p}) \cong \frac{SO(2)\times SO(2p+1)}{(\mathbf{Z}_2\times SO(2p-1))\cdot \mathbf{Z}_4} \to Q_{4p}(\mathbf{C})$ is not Hamiltonian stable if $p \geq 3$.

Theorem 8.2.

$$(SO(2) \times SO(2p+1))/(\mathbf{Z}_2 \times SO(2p-1))\mathbf{Z}_4 \quad (p \geq 1)$$

is not Hamiltonian stable if and only if $(m-2) - 1 = 2p - 2 \geq 3$. *If* $p = 1$ *then it is strictly Hamiltonian stable and if* $p = 2$, *then it is Hamiltonian stable but not strictly Hamiltonian stable.*

9. The case $(U, K) = (SU(m+2), S(U(2) \times U(m)))$ $(m \geq 2)$

9.1. Description of the subgroups K_0 and $K_{[\mathfrak{a}]}$.

$$\begin{aligned}
U =& SU(m+2), \quad (m \geq 2) \\
K =& S(U(2) \times U(m)) = \Big\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \in SU(m+2) | A \in U(2), B \in U(m) \Big\} \\
\mathfrak{a} =& \Big\{ H(\xi_1, \xi_2) = \begin{pmatrix} 0 & H_{12} \\ -\bar{H}_{12}{}^t & 0 \end{pmatrix} \mid H_{12} = \begin{pmatrix} \xi_1 & 0 & 0 & \cdots & 0 \\ 0 & \xi_2 & 0 & \cdots & 0 \end{pmatrix}, \\
& \xi_1, \xi_2 \in \mathbf{R} \Big\}
\end{aligned}$$

$$\begin{aligned}
K_0 =& \Big\{ \begin{pmatrix} e^{i\theta_1} & & & & \\ & e^{i\theta_2} & & & \\ & & e^{i\theta_1} & & \\ & & & e^{i\theta_2} & \\ & & & & T_{22} \end{pmatrix} \mid T_{22} \in U(m-2) \Big\} \\
\cong& S(U(1) \times U(1) \times U(m-2))
\end{aligned}$$

$$K_{[\mathfrak{a}]} = K_0 \cup (K_0 \cdot Q) \cup (K_0 \cdot Q^2) \cup (K_0 \cdot Q^3),$$

where

$$Q = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & I_{m-2} \end{pmatrix} \in U(m+2).$$

The deck transformation group of the covering map $\mathcal{G} : N^{8m-2} \longrightarrow \mathcal{G}(N^{4m-2})$ $(m \geq 2)$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

When $m = 2$, (U, K) is of B_2 type.

$$\begin{aligned} K = S(U(2) \times U(2)) &\supset K_1 = S(U(1) \times U(1) \times U(1) \times U(1)) \\ &\supset K_0 = S(U(1) \times U(1)) \end{aligned}$$

When $m \geq 3$, (U, K) is of BC_2 type.

$$\begin{aligned} K = S(U(2) \times U(m)) &\supset K_2 = S(U(2) \times U(2) \times U(m-2)) \\ &\supset K_1 = S(U(1) \times U(1) \times U(1) \times U(1) \times U(m-2)) \\ &\supset K_0 = S(U(1) \times U(1) \times U(m-2)) \end{aligned}$$

9.2. **Two fibrations over K/K_0.**

$$\begin{aligned} K_2/K_0 &= \frac{S(U(2) \times U(2) \times U(m-2))}{S(U(1) \times U(1) \times U(m-2))} \cong \frac{U(2) \times U(2) \times U(m-2)}{U(1) \times U(1) \times U(m-2)} \\ &\cong \frac{U(2) \times U(2)}{U(1) \times U(1)} \cong ?? \\ \longrightarrow K/K_0 &= \frac{S(U(2) \times U(m))}{S(U(1) \times U(1) \times U(m-2))} \\ &\cong \frac{U(2) \times U(m)}{U(1) \times U(1) \times U(m-2)} \\ \longrightarrow K/K_2 &= \frac{S(U(2) \times U(m))}{S(U(2) \times U(2) \times U(m-2))} = \frac{U(m)}{U(2) \times U(m-2)} \cong \mathrm{Gr}_2(\mathbf{C}^m) \ , \end{aligned}$$

$$
\begin{aligned}
K_1/K_0 &= \frac{S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))}{S(U(1)\times U(1)\times U(m-2))}\\
&\cong \frac{U(1)\times U(1)\times U(1)\times U(1)\times U(m-2)}{U(1)\times U(1)\times U(m-2)}\\
&\cong \frac{U(1)\times U(1)\times U(1)\times U(1)}{U(1)\times U(1)} \cong S^1\times S^1\\
\longrightarrow K/K_0 &= \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))}\\
&\cong \frac{U(2)\times U(m)}{U(1)\times U(1)\times U(m-2)}
\end{aligned}
$$

$$
\begin{aligned}
\longrightarrow K/K_1 &= \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))}\\
&\cong \frac{U(2)\times U(m)}{U(1)\times U(1)\times U(1)\times U(1)\times U(m-2)}\\
&\cong \frac{U(2)}{U(1)\times U(1)}\times\frac{U(m)}{U(1)\times U(1)\times U(m-2)}\\
&\cong \mathbf{C}P^1\times F_{1,1,m-2}(\mathbf{C}^m)\ .
\end{aligned}
$$

9.3. $D(SU(m))$ **and** $D(S(U(2)\times U(m)))$**.**

Let $\tilde{U} := U(m+2)$, $\tilde{K} := U(2)\times U(m)$, $\tilde{K}_2 := U(2)\times U(2)\times U(m-2)$, $\tilde{K}_1 := U(1)\times U(1)\times U(1)\times U(1)\times U(m-2)$ and $\tilde{K}_0 := U(1)\times U(1)\times U(m-2)$. Then $\tilde{U} = C(\tilde{U})\cdot U$, $\tilde{K} = C(\tilde{U})\cdot K$, $\tilde{K}_2 = C(\tilde{U})\cdot K_2$, $\tilde{K}_1 = C(\tilde{U})\cdot K_1$, $\tilde{K}_0 = C(\tilde{U})\cdot K_0$ and $\tilde{\mathfrak{u}} = \mathfrak{u}(m+2) = \mathfrak{c}(\tilde{\mathfrak{u}})\oplus\mathfrak{su}(m+2)$, $\tilde{\mathfrak{k}} = \mathfrak{c}(\tilde{\mathfrak{u}})\oplus\mathfrak{k}$, $\tilde{\mathfrak{k}}_2 = \mathfrak{c}(\tilde{\mathfrak{u}})\oplus\mathfrak{k}_2$, $\tilde{\mathfrak{k}}_1 = \mathfrak{c}(\tilde{\mathfrak{u}})\oplus\mathfrak{k}_1$, $\tilde{\mathfrak{k}}_0 = \mathfrak{c}(\tilde{\mathfrak{u}})\oplus\mathfrak{k}_0$.

For each $\tilde{\Lambda}\in D(\tilde{U})$,

$$\tilde{\Lambda} = \tilde{\Lambda}^0 + \Lambda$$

where $\tilde{\Lambda}^0\in\mathfrak{c}(\tilde{\mathfrak{u}})$ and $\Lambda\in D(U)$.

$$
\begin{aligned}
&D(\tilde{U}) = D(U(m+2))\\
=&\{\tilde{\Lambda} = \tilde{p}_1 y_1+\cdots+\tilde{p}_{m+2}y_{m+2}\mid \tilde{p}_1,\cdots,\tilde{p}_{m+2}\in\mathbf{Z}, \tilde{p}_i-\tilde{p}_{i+1}\geq 0\ (i=1,\cdots,m+1)\}
\end{aligned}
$$

$$
\begin{aligned}
&D(C(\tilde{U})) = D(C(U(m+2)))\\
=&\{\Lambda = p_0(y_1+\cdots+y_{m+2})\mid p_0\in\frac{1}{m+2}\mathbf{Z},\}
\end{aligned}
$$

$$
\begin{aligned}
&D(U) = D(SU(m+2))\\
&=\{\Lambda = p_1 y_1 + \cdots + p_{m+2} y_{m+2} \mid \\
&\qquad \sum_{i=1}^{m+2} p_i = 0, p_i - p_{m+2} \in \mathbf{Z}, p_i - p_{i+1} \geq 0 \ (i = 1, \cdots, m+1)\}
\end{aligned}
$$

If

$$\tilde{\Lambda} = \tilde{p}_1 y_1 + \cdots + \tilde{p}_{m+2} y_{m+2} \in D(U(m+2)),$$

then

$$\tilde{\Lambda} = \Lambda^0 + \Lambda,$$

where

$$\Lambda^0 = (\frac{1}{m+2} \sum_{i=1}^{m+2} \tilde{p}_i)(\sum_{i=1}^{m+2} y_i) \in D(C(U(m+2)))$$

and

$$\Lambda = (\tilde{p}_1 - \frac{1}{m+2} \sum_{i=1}^{m+2} \tilde{p}_i) y_1 + \cdots + (\tilde{p}_{m+2} - \frac{1}{m+2} \sum_{i=1}^{m+2} \tilde{p}_i) y_{m+2} \in D(SU(m+2)).$$

Assume that

$$\Lambda = p_1 y_1 + \cdots + p_{m+2} y_{m+2} \in D(SU(m+2))$$

with $\sum_{i=1}^{m+2} p_i = 0$, $p_i - p_{m+2} \in \mathbf{Z}$, $p_i - p_{i+1} \geq 0$ $(i = 1, \cdots, m+1)$.

$p_{m+2} \in \frac{1}{m+2}\mathbf{Z}$. thus $p_i \in \frac{1}{m+2}\mathbf{Z}$. Since $p_i - p_j \in \mathbf{Z}$, then there is $r \in \mathbf{Z}$ with $0 \leq r < m+2$ such that $p_i = \frac{1}{m+2}((m+2)k_i + r)$ for some $k_i \in \mathbf{Z}$.

If we set $\tilde{p}_i := p_i - \frac{1}{m+2}((m+2)k + r)$ $(i = 1, \cdots, m+2)$ for arbitrary $k \in \mathbf{Z}$, then

$$
\begin{aligned}
\tilde{\Lambda} :=& \tilde{p}_1 y_1 + \cdots + \tilde{p}_{m+2} y_{m+2}\\
=& -(\frac{1}{m+2}((m+2)k + r))(y_1 + \cdots + y_{m+2}) + p_1 y_1 + \cdots + p_{m+2} y_{m+2}\\
=& -(k + \frac{r}{m+2})(y_1 + \cdots + y_{m+2}) + p_1 y_1 + \cdots + p_{m+2} y_{m+2}\\
=& -(k + \frac{r}{m+2})(y_1 + \cdots + y_{m+2}) + \Lambda \in D(U(m+2))
\end{aligned}
$$

$$-(k + \frac{r}{m+2})(y_1 + \cdots + y_{m+2}) \in D(C(U(m+2))).$$

$$
\begin{aligned}
D(\tilde{K}) &= D(U(2)\times U(m)) \\
=&\{\tilde{\Lambda} = \tilde{q}_1 y_1 + \tilde{q}_2 y_2 + \tilde{q}_3 y_3 + \cdots + \tilde{q}_{m+2} y_{m+2} \mid \\
&\qquad \tilde{q}_i \in \mathbf{Z}\ (i = 1, \cdots, m+2), \\
&\qquad \tilde{q}_1 - \tilde{q}_2 \geq 0, \tilde{q}_i - \tilde{q}_{i+1} \geq 0\ (i = 3, \cdots, m+1)\ \}
\end{aligned}
$$

$$
\begin{aligned}
D(K) &= D(S(U(2)\times U(m))) \\
=&\{\Lambda = q_1 y_1 + q_2 y_2 + q_3 y_3 + \cdots + q_{m+2} y_{m+2} \mid \\
&\qquad \sum_{i=1}^{m+2} q_i = 0, \\
&\qquad q_i - q_j \in \mathbf{Z}\ (i, j = 1, 2, \cdots, m+2), \\
&\qquad q_1 - q_2 \geq 0, q_i - q_{i+1} \geq 0\ (i = 3, 4, \cdots, m+1)\ \}
\end{aligned}
$$

$$D(\tilde{K}) \longrightarrow D(K)$$

is surjective.

Assume that

$$\tilde{\Lambda} = \tilde{q}_1 y_1 + \tilde{q}_2 y_2 + \tilde{q}_3 y_3 + \cdots + \tilde{q}_{m+2} y_{m+2} \in D(\tilde{K}).$$

Then

$$\tilde{\Lambda} = \ (\frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)(\sum_{i=1}^{m+2} y_i) + \sum_{i=1}^{m+2}(\tilde{q}_i - \sum_{j=1}^{m+2}\tilde{q}_j) y_i$$

$$\Lambda^0 = (\frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)(\sum_{i=1}^{m+2} y_i) \in D(C(U(m+2)))$$

$$\Lambda = \sum_{i=1}^{m+2}(\tilde{q}_i - \sum_{j=1}^{m+2}\tilde{q}_j) y_i \in D(K)$$

Assume that

$$\Lambda = q_1 y_1 + q_2 y_2 + q_3 y_3 + \cdots + q_{m+2} y_{m+2} \in D(K).$$

Then

$$
\begin{aligned}
\tilde{\Lambda} =& \tilde{q}_1 + \cdots + \tilde{q}_{m+2} \\
=& -(k + \frac{r}{m+2})(y_1 + \cdots + y_{m+2}) + q_1 y_1 + \cdots + q_{m+2} y_{m+2} \in D(\tilde{K}),
\end{aligned}
$$

where k, r are defined as above.

$$
\begin{aligned}
&D(\tilde{K_2}) = D(U(2)\times U(2)\times U(m-2))\\
=&\{\tilde{\Lambda} = \tilde{q}_1y_1 + \tilde{q}_2y_2 + \tilde{q}_3y_3 + \tilde{q}_4y_4 + \tilde{q}_5y_5 + \cdots + \tilde{q}_{m+2}y_{m+2} \mid\\
&\qquad \tilde{q}_i \in \mathbf{Z}\ (i = 1,\cdots,m+2),\\
&\qquad \tilde{q}_1 - \tilde{q}_2, \tilde{q}_3 - \tilde{q}_4, \tilde{q}_i - \tilde{q}_{i+1} \geq 0\ (i = 5,\cdots,m+1)\ \}
\end{aligned}
$$

$$
\begin{aligned}
&D(K_2) = D(S(U(2)\times U(2)\times U(m-2)))\\
=&\{\Lambda = q_1y_1 + q_2y_2 + q_3y_3 + q_4y_4 + q_5y_5 + \cdots + q_{m+2}y_{m+2} \mid\\
&\qquad \sum_{i=1}^{m+2} q_i = 0,\\
&\qquad q_i - q_j \in \mathbf{Z}\ (i,j = 1,2,\cdots,m+2),\\
&\qquad q_1 - q_2, q_3 - q_4, q_i - q_{i+1} \geq 0\ (i = 5,6,\cdots,m+1)\ \}
\end{aligned}
$$

$$D(\tilde{K_2}) \longrightarrow D(K_2)$$

is surjective.

Assume that

$$\tilde{\Lambda} = \tilde{q}_1y_1 + \tilde{q}_2y_2 + \tilde{q}_3y_3 + \tilde{q}_4y_4 + \tilde{q}_5y_5 + \cdots + \tilde{q}_{m+2}y_{m+2} \in D(\tilde{K_2}).$$

Then

$$
\begin{aligned}
\tilde{\Lambda} =& \tilde{q}_1y_1 + \tilde{q}_2y_2 + \tilde{q}_3y_3 + \tilde{q}_4y_4 + \tilde{q}_5y_5 + \cdots + \tilde{q}_{m+2}y_{m+2}\\
=& (\frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)(\sum_{i=1}^{m+2}y_i)\\
&+ (\tilde{q}_1 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_1 + (\tilde{q}_2 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_2\\
&+ (\tilde{q}_3 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_3 + (\tilde{q}_4 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_4\\
&+ (\tilde{q}_5 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_5 + \cdots + (\tilde{q}_{m+2} - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_{m+2}
\end{aligned}
$$

$$\Lambda^0 = (\frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)(\sum_{i=1}^{m+2}y_i) \in D(C(U(m+2)))$$

$$\begin{aligned}\Lambda =&(\tilde{q}_1 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_1 + (\tilde{q}_2 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_2 \\ &+ (\tilde{q}_3 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_3 + (\tilde{q}_4 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_4 \\ &+ (\tilde{q}_5 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_5 + \cdots + (\tilde{q}_{m+2} - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_{m+2} \\ \in& D(K_2)\end{aligned}$$

Assume that

$$\Lambda = q_1y_1 + q_2y_2 + q_3y_3 + q_4y_4 + q_5y_5 + \cdots + q_{m+2}y_{m+2} \in D(K_2).$$

Then

$$\begin{aligned}\tilde{\Lambda} =&\tilde{q}_1 + \cdots + \tilde{q}_{m+2} \\ =&-(\frac{1}{m+2}((m+2)k+r))(y_1 + \cdots + y_{m+2}) + q_1y_1 + \cdots + q_{m+2}y_{m+2}.\end{aligned}$$

$$\begin{aligned}D(\tilde{K_1}) &= D(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2)) \\ =&\{\tilde{\Lambda} = \tilde{q}_1y_1 + \tilde{q}_2y_2 + \tilde{q}_3y_3 + \tilde{q}_4y_4 + \tilde{q}_5y_5 + \cdots + \tilde{q}_{m+2}y_{m+2} \mid \\ &\qquad \tilde{q}_i \in \mathbf{Z}\ (i = 1, \cdots, m+2), \\ &\qquad \tilde{q}_i - \tilde{q}_{i+1} \geq 0\ (i = 5, \cdots, m+1)\ \}\end{aligned}$$

$$\begin{aligned}D(K_1) &= D(S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))) \\ =&\{\Lambda = q_1y_1 + q_2y_2 + q_3y_3 + q_4y_4 + q_5y_5 + \cdots + q_{m+2}y_{m+2} \mid \\ &\qquad \sum_{i=1}^{m+2} q_i = 0, \\ &\qquad q_i - q_j \in \mathbf{Z}\ (i,j = 1, \cdots, m+2), \\ &\qquad q_i - q_{i+1} \geq 0\ (i = 5, \cdots, m+1)\ \}\end{aligned}$$

$$D(\tilde{K_1}) \longrightarrow D(K_1)$$

is surjective.

Assume that

$$\tilde{\Lambda} = \tilde{q}_1y_1 + \tilde{q}_2y_2 + \tilde{q}_3y_3 + \tilde{q}_4y_4 + \tilde{q}_5y_5 + \cdots + \tilde{q}_{m+2}y_{m+2} \in D(\tilde{K_1}).$$

Then

$$\begin{aligned}\tilde{\Lambda} =&\tilde{q}_1 y_1 + \tilde{q}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4 + \tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2} \\ =& (\frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)(\sum_{i=1}^{m+2} y_i) \\ &+ (\tilde{q}_1 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_1 + (\tilde{q}_2 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_2 \\ &+ (\tilde{q}_3 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_3 + (\tilde{q}_4 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_4 \\ &+ (\tilde{q}_5 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_5 + \cdots + (\tilde{q}_{m+2} - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_{m+2} \\ &= \Lambda^0 + \Lambda,\end{aligned}$$

where

$$\Lambda^0 = (\frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)(\sum_{i=1}^{m+2} y_i) \in D(C(U(m+2)))$$

$$\begin{aligned}\Lambda =&(\tilde{q}_1 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_1 + (\tilde{q}_2 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_2 \\ &+ (\tilde{q}_3 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_3 + (\tilde{q}_4 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_4 \\ &+ (\tilde{q}_5 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_5 + \cdots + (\tilde{q}_{m+2} - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_{m+2} \\ \in& D(K_1)\end{aligned}$$

Assume that

$$\Lambda = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 + q_5 y_5 + \cdots + q_{m+2} y_{m+2} \in D(K_1).$$

Then

$$\begin{aligned}\tilde{\Lambda} =&\tilde{q}_1 + \cdots + \tilde{q}_{m+2} \\ =& -(\frac{1}{m+2}((m+2)k+r))(y_1 + \cdots + y_{m+2}) + q_1 y_1 + \cdots + q_{m+2} y_{m+2}.\end{aligned}$$

$$
\begin{aligned}
&D(\tilde{K_0}) = D(U(1) \times U(1) \times U(m-2)) \\
=&\{\tilde{\Lambda} = \tilde{q}_1 y_1 + \tilde{q}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4 + \tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2} \mid \\
&\qquad \tilde{q}_3 = \tilde{q}_1 \in \frac{1}{2}\mathbf{Z}, \tilde{q}_4 = \tilde{q}_2 \in \frac{1}{2}\mathbf{Z}, \\
&\qquad \tilde{q}_i \in \mathbf{Z} \ (i = 5, \cdots, m+2), \\
&\qquad \tilde{q}_i - \tilde{q}_{i+1} \geq 0 \ (i = 5, 6, \cdots, m+1) \ \}
\end{aligned}
$$

$$
\begin{aligned}
&D(K_0) = D(S(U(1) \times U(1) \times U(m-2))) \\
=&\{\Lambda = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 + q_5 y_5 + \cdots + q_{m+2} y_{m+2} \mid \\
&\qquad \sum_{i=1}^{m+2} q_i = 2q_1 + 2q_2 + \sum_{i=5}^{m+2} q_i = 0, q_3 = q_1, q_4 = q_2, \\
&\qquad q_i - q_j \in \mathbf{Z} \ (i, j = 1, 2, 5, \cdots, m+2), \\
&\qquad q_i - q_{i+1} \geq 0 \ (i = 5, 6, \cdots, m+1) \ \}
\end{aligned}
$$

$$
D(\tilde{K_0}) \longrightarrow D(K_0)
$$

is surjective.

Assume that

$$
\tilde{\Lambda} = \tilde{q}_1 y_1 + \tilde{q}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4 + \tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2} \in D(\tilde{K_0}).
$$

Then

$$
\begin{aligned}
\tilde{\Lambda} =& \tilde{q}_1 y_1 + \tilde{q}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4 + \tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2} \\
=& \ (\frac{1}{m+2} \sum_{i=1}^{m+2} \tilde{q}_i)(\sum_{i=1}^{m+2} y_i) \\
&+ (\tilde{q}_1 - \frac{1}{m+2} \sum_{i=1}^{m+2} \tilde{q}_i) y_1 + (\tilde{q}_2 - \frac{1}{m+2} \sum_{i=1}^{m+2} \tilde{q}_i) y_2 \\
&+ (\tilde{q}_3 - \frac{1}{m+2} \sum_{i=1}^{m+2} \tilde{q}_i) y_3 + (\tilde{q}_4 - \frac{1}{m+2} \sum_{i=1}^{m+2} \tilde{q}_i) y_4 \\
&+ (\tilde{q}_5 - \frac{1}{m+2} \sum_{i=1}^{m+2} \tilde{q}_i) y_5 + \cdots + (\tilde{q}_{m+2} - \frac{1}{m+2} \sum_{i=1}^{m+2} \tilde{q}_i) y_{m+2}
\end{aligned}
$$

$$
\Lambda^0 = (\frac{1}{m+2} \sum_{i=1}^{m+2} \tilde{q}_i)(\sum_{i=1}^{m+2} y_i) \in D(C(U(m+2)))
$$

$$
\begin{aligned}
\Lambda =&(\tilde{q}_1 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_1 + (\tilde{q}_2 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_2 \\
&+ (\tilde{q}_3 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_3 + (\tilde{q}_4 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_4 \\
&+ (\tilde{q}_5 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_5 + \cdots + (\tilde{q}_{m+2} - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{q}_i)y_{m+2} \\
&\in D(K_0)
\end{aligned}
$$

Assume that

$$\Lambda = q_1y_1 + q_2y_2 + q_3y_3 + q_4y_4 + q_5y_5 + \cdots + q_{m+2}y_{m+2} \in D(K_0).$$

Then

$$
\begin{aligned}
\tilde{\Lambda} =&\tilde{q}_1y_1 + \cdots + \tilde{q}_{m+2}y_{m+2} \\
=& -(\frac{1}{m+2}((m+2)k+r))(y_1 + \cdots + y_{m+2}) + q_1y_1 + \cdots + q_{m+2}y_{m+2} \\
\in& D(\tilde{K}_0).
\end{aligned}
$$

9.4. **Branching laws of** $(U(m), U(2)\times U(m-2))$**.** By the branching laws of $(SU(m), S(U(m)\times U(2)))$ given in [24], we have the following branching laws of $(U(m), U(2)\times U(m-2))$.

Let $\tilde{\Lambda} = \tilde{p}_1y_1 + \cdots + \tilde{p}_my_m \in D(U(m))$ be the highest weight of an irreducible $U(m)$-module $V_{\tilde{\Lambda}}$, where $\tilde{p}_i \in \mathbf{Z}$ $(i = 1, \cdots, m)$, $\tilde{p}_i - \tilde{p}_{i+1} \geq 0$ $(i = 1, \cdots, m-1)$. Then the irreducible decomposition

$$V_{\tilde{\Lambda}} = \bigoplus_{\tilde{\Lambda}' \in D(U(2)\times U(m-2))} V'_{\tilde{\Lambda}'},$$

of $V_{\tilde{\Lambda}}$ as a $U(2)\times U(m-2)$-module contains an irreducible $U(2)\times U(m-2)$-module $V'_{\tilde{\Lambda}'}$ with the highest weight $\tilde{\Lambda}' = \tilde{q}_1y_1 + \cdots + \tilde{q}_my_m \in D(U(2)\times U(m-2))$, where $\tilde{q}_i \in \mathbf{Z}$ $(i = 1, \cdots, m)$, $\tilde{q}_1 - \tilde{q}_2 \geq 0$, $\tilde{q}_i - \tilde{q}_{i+1} \geq 0$ $(i = 3, \cdots, m-1)$, if and only if the following conditions are satisfied:

(i) $\tilde{q}_i \in \mathbf{Z}$ for $i = 1, \cdots, m$;
(ii) $\tilde{p}_{i-2} \geq \tilde{q}_i \geq \tilde{p}_i$ $(i = 3, \cdots, m)$;

(iii) In the finite power series expansion in X of $\frac{\prod_{i=2}^{m}(X^{r_i+1}-X^{-(r_i+1)})}{(X-X^{-1})^{m-2}}$, where

$$\begin{aligned} r_2 &:= \tilde{p}_1 - \max(\tilde{q}_3, \tilde{p}_2) \\ r_i &:= \min(\tilde{q}_i, \tilde{p}_{i-1}) - \max(\tilde{q}_{i+1}, \tilde{p}_i), \quad (3 \leq i \leq m-1) \\ r_m &:= \min(\tilde{q}_m, \tilde{p}_{m-1}) - \tilde{p}_m \end{aligned}$$

the coefficient of $X^{\tilde{q}_1-\tilde{q}_2+1}$ does not vanish. Moreover, the value of this coefficient is the multiplicity of the $U(2)\times U(m-2)$-module $V'_{\tilde{\Lambda}'}$.

9.5. **Branching laws of** $(U(3), U(2)\times U(1))$**.** Let $\Lambda = p_1y_1 + p_2y_2 + p_3y_3 \in D(SU(3))$ be the highest weight of an irreducible $SU(3)$-module V_Λ, where $p_i - p_j \in \mathbf{Z}$ $(i,j = 1,2,3)$, $p_1 \geq p_2 \geq p_3$, $\sum_{i=1}^{3} p_i = 0$. Then the irreducible decomposition $V_\Lambda = \oplus_{\Lambda'} V'_{\Lambda'}$ of V_Λ as an $S(U(2)\times U(1))$-module contains an irreducible $S(U(2)\times U(1))$-module $V'_{\Lambda'}$ with the highest weight $\Lambda' = q_1y_1 + q_2y_2 + q_3y_3 \in D(S(U(2)\times U(1)))$, with $q_i - q_j \in \mathbf{Z}$ $(i,j = 1,2,3)$, $q_1 \geq q_2$, $\sum_{i=1}^{3} q_i = 0$ if and only if $p_1 \geq q_1 \geq p_2 \geq q_2 \geq p_3$. Equivalently,

$$V_{p_1y_1+p_2y_2+p_3y_3} = \bigoplus_{\alpha=0}^{p_1-p_2} \bigoplus_{\beta=0}^{p_2-p_3} V'_{(p_1-\alpha)y_1+(p_2-\beta)y_2+(p_3+\alpha+\beta)y_3},$$

where $\Lambda = p_1y_1 + p_2y_2 + p_3y_3 \in D(SU(3))$.

Let $\tilde{\Lambda} = \tilde{p}_1y_1 + \tilde{p}_2y_2 + \tilde{p}_3y_3 \in D(U(3))$ be the highest weight of an irreducible $U(3)$-module $\tilde{V}_{\tilde{\Lambda}}$, where $\tilde{p}_i \in \mathbf{Z}$ for $i = 1,2,3$, $\tilde{p}_1 \geq \tilde{p}_2 \geq \tilde{p}_3$. Then the irreducible decomposition

$$\tilde{V}_{\tilde{p}_1y_1+\tilde{p}_2y_2+\tilde{p}_3y_3} = \bigoplus_{\alpha=0}^{\tilde{p}_1-\tilde{p}_2} \bigoplus_{\beta=0}^{\tilde{p}_2-\tilde{p}_3} \tilde{V}'_{(\tilde{p}_1-\alpha)y_1+(\tilde{p}_2-\beta)y_2+(\tilde{p}_3+\alpha+\beta)y_3}$$

of $\tilde{V}_{\tilde{\Lambda}}$ as a $U(2)\times U(1)$-module holds.

9.6. **Descriptions of** $D(\tilde{K}, \tilde{K}_0)$**,** $D(\tilde{K}_2, \tilde{K}_0)$**,** $D(\tilde{K}_1, \tilde{K}_0)$**.**

Let

$$\tilde{\Lambda} = \tilde{p}_1y_1 + \tilde{p}_2y_2 + \tilde{p}_3y_3 + \cdots + \tilde{p}_{m+2}y_{m+2} \in D(\tilde{K}) = D(U(2)\times U(m)),$$

where $\tilde{p}_1, \cdots, \tilde{p}_{m+2} \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{p}_3 \geq \cdots \geq \tilde{p}_{m+2}$. Thus $\Lambda_\sigma = \tilde{p}_1y_1 + \tilde{p}_2y_2 \in D(U(2))$, $\Lambda_\tau = \tilde{p}_3y_3 + \cdots + \tilde{p}_{m+2}y_{m+2} \in D(U(m))$ and

$$\tilde{\rho}_{\tilde{\Lambda}} = \sigma \boxtimes \tau \in \mathcal{D}(\tilde{K}) = \mathcal{D}(U(2)\times U(m)),$$

where $\sigma \in \mathcal{D}(U(2))$, $\tau \in \mathcal{D}(U(m))$.

By the branching law of $(U(m), U(2)\times U(m-2))$, we have the decomposition into irreducible $U(2)\times U(m-2)$-modules:

$$V_\tau = \bigoplus V'_{\tilde{\Lambda}'},$$

where $\tilde{\Lambda}' = \sum_{i=3}^{m+2} \tilde{q}_i y_i \in D(U(2)\times U(m-2))$ with $\tilde{q}_3, \cdots, \tilde{q}_{m+2} \in \mathbf{Z}$, $\tilde{q}_i - \tilde{q}_{i+1} \geq 0$ $(i = 3, 5, \cdots, m+1)$. Notice that $\Lambda_\varsigma := \tilde{q}_3 y_3 + \tilde{q}_4 y_4 \in D(U(2))$ and $\Lambda_\gamma := \tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2} \in D(U(m-2))$,

$$V_{\tilde{\Lambda}} = \bigoplus (V_\sigma \boxtimes V_\varsigma \boxtimes V_\gamma).$$

By the branching law of $(U(2), U(1)\times U(1))$,

$$\begin{aligned} V_\sigma = V_{\tilde{p}_1 y_1 + \tilde{p}_2 y_2} &= \bigoplus_{\alpha=0}^{\tilde{p}_1 - \tilde{p}_2} V'_{(\tilde{p}_1-\alpha)y_1 + (\tilde{p}_2+\alpha)y_2}, \\ V_\varsigma = V_{\tilde{q}_3 y_3 + \tilde{q}_4 y_4} &= \bigoplus_{\beta=0}^{\tilde{q}_3 - \tilde{q}_4} V'_{(\tilde{q}_3-\beta)y_3 + (\tilde{q}_4+\beta)y_4}. \end{aligned}$$

Thus we have the decomposition into irreducible $U(1)\times U(1)\times U(1)\times U(1)\times U(m-2)$-modules :

$$\begin{aligned} &V_{\tilde{\Lambda}} \\ =&\bigoplus \bigoplus_{\alpha=0}^{\tilde{p}_1-\tilde{p}_2} \bigoplus_{\beta=0}^{\tilde{q}_3-\tilde{q}_4} (V'_{(\tilde{p}_1-\alpha)y_1+(\tilde{p}_2+\alpha)y_2} \boxtimes V'_{(\tilde{q}_3-\beta)y_3+(\tilde{q}_4+\beta)y_4} \boxtimes V_{\tilde{q}_5 y_5+\cdots+\tilde{q}_{m+2}y_{m+2}}). \end{aligned}$$

As a $U(1)\times U(1)$-module,

$$\begin{aligned} &V'_{(\tilde{p}_1-\alpha)y_1+(\tilde{p}_2+\alpha)y_2} \boxtimes V'_{(\tilde{q}_3-\beta)y_3+(\tilde{q}_4+\beta)y_4} \\ =&V''_{\frac{1}{2}(\tilde{p}_1+\tilde{q}_3-\alpha-\beta)(y_1+y_3)+\frac{1}{2}(\tilde{p}_2+\tilde{q}_4+\alpha+\beta)(y_2+y_4)} \ . \end{aligned}$$

Hence we have the decomposition into irreducible $U(1)\times U(1)\times U(m-2)$-modules :

$$\begin{aligned} &V_{\tilde{\Lambda}} \\ =&\bigoplus (V_{\tilde{p}_1 y_1+\tilde{p}_2 y_2} \boxtimes V_{\tilde{q}_3 y_3+\tilde{q}_4 y_4} \boxtimes V_{\tilde{q}_5 y_5+\cdots+\tilde{q}_{m+2}y_{m+2}}) \\ =&\bigoplus \bigoplus_{\alpha=0}^{\tilde{p}_1-\tilde{p}_2} \bigoplus_{\beta=0}^{\tilde{q}_3-\tilde{q}_4} (V'_{(\tilde{p}_1-\alpha)y_1+(\tilde{p}_2+\alpha)y_2} \boxtimes V'_{(\tilde{q}_3-\beta)y_3+(\tilde{q}_4+\beta)y_4} \boxtimes V_{\tilde{q}_5 y_5+\cdots+\tilde{q}_{m+2}y_{m+2}}) \\ =&\bigoplus \bigoplus_{\alpha=0}^{\tilde{p}_1-\tilde{p}_2} \bigoplus_{\beta=0}^{\tilde{q}_3-\tilde{q}_4} V''_{\frac{1}{2}(\tilde{p}_1+\tilde{q}_3-\alpha-\beta)(y_1+y_3)+\frac{1}{2}(\tilde{p}_2+\tilde{q}_4+\alpha+\beta)(y_2+y_4)} \boxtimes V_{\tilde{q}_5 y_5+\cdots+\tilde{q}_{m+2}y_{m+2}} \ . \end{aligned}$$

Thus $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ if and only if there exist $\alpha, \beta \in \mathbf{Z}$ with $0 \leq \alpha \leq \tilde{p}_1 - \tilde{p}_2$ and $0 \leq \beta \leq \tilde{q}_3 - \tilde{q}_4$,

$$V''_{\frac{1}{2}(\tilde{p}_1+\tilde{q}_3-\alpha-\beta)(y_1+y_3)+\frac{1}{2}(\tilde{p}_2+\tilde{q}_4+\alpha+\beta)(y_2+y_4)} \boxtimes V_{\tilde{q}_5 y_5+\cdots+\tilde{q}_{m+2}y_{m+2}}$$

is a trivial $U(1) \times U(1) \times U(m-2)$-module, that is,

$$\begin{cases} \tilde{p}_1 + \tilde{q}_3 - \alpha - \beta = 0, \\ \tilde{p}_2 + \tilde{q}_4 + \alpha + \beta = 0, \\ \tilde{q}_5 = \cdots = \tilde{q}_{m+2} = 0. \end{cases}$$

Thus

$$\begin{aligned} \tilde{p}_5 = \tilde{p}_6 = \cdots = \tilde{p}_m &= 0, \\ \tilde{p}_3 \geq \tilde{p}_4 &\geq 0, \\ \tilde{p}_{m+2} \leq \tilde{p}_{m+1} &\leq 0, \\ \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_{m+1} + \tilde{p}_{m+2} &= 0. \end{aligned}$$

If $m \geq 4$, then each $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ is

$$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_{m+1} y_{m+1} + \tilde{p}_{m+2} y_{m+2},$$

where $\tilde{p}_i \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{p}_3 \geq \tilde{p}_4 \geq 0 \geq \tilde{p}_{m+1} \geq \tilde{p}_{m+2}$,

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_{m+1} + \tilde{p}_{m+2} = 0.$$

If $m = 3$, then each $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ is

$$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_5 y_5,$$

where $\tilde{p}_i \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{p}_3 \geq \tilde{p}_4 \geq \tilde{p}_5$, $\tilde{p}_3 \geq 0$, $\tilde{p}_5 \leq 0$,

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_5 = 0.$$

If $m = 2$, then each $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ is

$$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4,$$

where $\tilde{p}_i \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{p}_3 \geq \tilde{p}_4$,

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 = 0.$$

Each $\tilde{\Lambda}' \in D(\tilde{K}_2, \tilde{K}_0)$ is

$$\tilde{\Lambda}' = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4,$$

where $\tilde{p}_1, \tilde{p}_2, \tilde{q}_3, \tilde{q}_4 \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{q}_3 \geq \tilde{q}_4$,

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{q}_3 + \tilde{q}_4 = 0,$$

in another words, $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_{m+1} + \tilde{p}_{m+2} = 0$ if $m \geq 4$, $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_5 = 0$ if $m = 3$.

Each $\tilde{\Lambda}'' \in D(\tilde{K}_1, \tilde{K}_0)$ is

$$\tilde{\Lambda}'' = \tilde{q}_1' y_1 + \tilde{q}_2' y_2 + \tilde{q}_3' y_3 + \tilde{q}_4' y_4,$$

where $\tilde{q}_1', \tilde{q}_2', \tilde{q}_3', \tilde{q}_4' \in \mathbf{Z}$, $\tilde{q}_1' + \tilde{q}_3' = 0$, $\tilde{q}_2' + \tilde{q}_4' = 0$, $\tilde{q}_1' = -\alpha + \tilde{p}_1$, $\tilde{q}_2' = \alpha + \tilde{p}_2$ for some $\alpha = 0, \cdots, \tilde{p}_1 - \tilde{p}_2$, and $\tilde{q}_3' = -\beta + \tilde{q}_3$, $\tilde{q}_4' = \beta + \tilde{q}_4$ for some $\beta = 0, \cdots, \tilde{q}_3 - \tilde{q}_4$.

Notice that the coefficient of $X^{\tilde{q}_3 - \tilde{q}_4 + 1}$ in

$$\begin{aligned}
&\frac{1}{X - X^{-1}} (X^{\tilde{p}_3 - \tilde{p}_4 + 1} - X^{-(\tilde{p}_3 - \tilde{p}_4 + 1)})(X^{\tilde{p}_{m+1} - \tilde{p}_{m+2} + 1} - X^{-(\tilde{p}_{m+1} - \tilde{p}_{m+2} + 1)}) \\
= &\sum_{i=0}^{\tilde{p}_3 - \tilde{p}_4} (X^{(\tilde{p}_3 - \tilde{p}_4) + (\tilde{p}_{m+1} - \tilde{p}_{m+2}) - 2i + 1} - X^{(\tilde{p}_3 - \tilde{p}_4) - (\tilde{p}_{m+1} - \tilde{p}_{m+2}) - 2i - 1})
\end{aligned}$$

is equal to the multiplicity of the $U(2) \times U(m-2)$-module with respect to the highest weight $\Lambda = \sum_{i=3}^{m+2} \tilde{q}_i y_i$.

9.7. **Representation of the Casimir operator.**

Denote $\langle X, Y \rangle := -\mathrm{tr} XY$ for any $X, Y \in \mathfrak{u}$.

The Casimir operator of L with respect to the induced metric from $Q_{4m-2}(\mathbf{C})$ is given as follows : if $m = 2$,

$$\begin{aligned}
\mathcal{C}_L = &\frac{1}{2}((E_1)^2 + (E_2)^2) + (X_1)^2 + (Y_1)^2 + (X_2)^2 + (Y_2)^2 \\
= &\mathcal{C}_{K/K_0} - \frac{1}{2}\, \mathcal{C}_{K_1/K_0},
\end{aligned}$$

if $m \geq 3$,

$$\begin{aligned}
\mathcal{C}_L = &\frac{1}{2}((E_1)^2 + (E_2)^2) + (X_1)^2 + (Y_1)^2 + (X_2)^2 + (Y_2)^2 \\
&+ 2 \sum_{i=1}^{m-2} \big((X_{1i})^2 + (Y_{1i})^2 + (X_{2i})^2 + (Y_{2i})^2\big) \\
= &2\, \mathcal{C}_{K/K_0} - \mathcal{C}_{K_2/K_0} - \frac{1}{2}\, \mathcal{C}_{K_1/K_0},
\end{aligned}$$

where $\mathcal{C}_{K/K_0}, \mathcal{C}_{K_2/K_0}$ and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operator of K/K_0, K_2/K_0 and K_1/K_0 relative to $\langle\ ,\ \rangle|_{\mathfrak{k}}$, $\langle\ ,\ \rangle|_{\mathfrak{k}_2}$ and $\langle\ ,\ \rangle|_{\mathfrak{k}_1}$, respectively.

9.8. **Eigenvalue computation when $m = 2$.**

$$\begin{aligned}
c_{\tilde{\Lambda}} = &\langle \tilde{\Lambda} + 2\delta_{\tilde{K}}, \tilde{\Lambda} \rangle \\
= &\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 + \tilde{p}_4^2 + \tilde{p}_1 - \tilde{p}_2 + \tilde{p}_3 - \tilde{p}_4 \\
c_{\tilde{\Lambda}''} = &\langle \tilde{\Lambda}'' + 2\delta_{\tilde{K}_1}, \tilde{\Lambda}'' \rangle \\
= &(\tilde{q}_1')^2 + (\tilde{q}_2')^2 + (\tilde{q}_3')^2 + (\tilde{q}_4')^2.
\end{aligned}$$

Then the eigenvalue formula is

$$
\begin{aligned}
c =& c_{\tilde{\Lambda}} - \frac{1}{2} c_{\tilde{\Lambda}''} \\
=& \tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 + \tilde{p}_4^2 + (\tilde{p}_1 - \tilde{p}_2) + (\tilde{p}_3 - \tilde{p}_4) \\
&- \frac{1}{2}((\tilde{q}_1')^2 + (\tilde{q}_2')^2 + (\tilde{q}_3')^2 + (\tilde{q}_4')^2)
\end{aligned}
$$

Denote $\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 \in D(\tilde{K}, \tilde{K}_0)$ by $\tilde{\Lambda} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4)$. Then

$$
\begin{aligned}
&\{\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0) | -c_L \leq 6\} \\
=&\{0, (1, 1, -1, -1), (1, 0, 0, -1), (1, -1, 0, 0), (1, -1, 1, -1), \\
&(1, 1, 0, -2), (2, 0, -1, -1), (0, -1, 1, 0), (0, 0, 1, -1), \\
&(0, -2, 1, 1), (-1, -1, 2, 0), (-1, -1, 1, 1)\}.
\end{aligned}
$$

Suppose that $\tilde{\Lambda} = (1, 1, -1, -1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 1$. By the branching law of $(U(2), U(1) \times U(1))$, $(\tilde{q}_1', \tilde{q}_2', \tilde{q}_3', \tilde{q}_4') = (1, 1, -1, -1) \in D(\tilde{K}_1, \tilde{K}_0)$. Then

$$
c_{\tilde{\Lambda}} = 4, \quad c_{\tilde{\Lambda}''} = 4, \quad c_L = c_{\tilde{\Lambda}} - \frac{1}{2} c_{\tilde{\Lambda}''} = 2 < 6.
$$

On the other hand, $V_{\tilde{\Lambda}} = \mathbf{C} \boxtimes \mathbf{C}$, which is fixed by the $\rho_{\tilde{\Lambda}}|_{\tilde{K}_0}$-action. But for a generator $g = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$, $\rho_{\tilde{\Lambda}}(g) = -\mathrm{Id}$ on $V_{\tilde{\Lambda}}$. Hence, $\tilde{\Lambda} = (1, 1, -1, -1) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$. Similarly, $\tilde{\Lambda} = (-1, -1, 1, 1) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (1, 0, 0, -1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 4$. It follows from the branching laws of $(U(2), U(1) \times U(1))$ that $(\tilde{q}_1', \tilde{q}_2') = (1, 0) \oplus (0, 1)$ and $(\tilde{q}_3', \tilde{q}_4') = (0, -1) \oplus (-1, 0)$. Then $(\tilde{q}_1', \tilde{q}_2', \tilde{q}_3', \tilde{q}_4') = (1, 0, -1, 0)$ or $(0, 1, 0, -1) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence,

$$
c_{\tilde{\Lambda}} = 4, \quad c_{\tilde{\Lambda}''} = 2, \quad c_L = c_{\tilde{\Lambda}} - \frac{1}{2} c_{\tilde{\Lambda}''} = 3 < 6.
$$

Recall that

$$
\mathcal{D}(SU(2)) = \{(V_\ell, \rho_\ell) | \ell \in \mathbf{Z}, \ell > 0\},
$$

where V_ℓ denotes the complex vector space of complex homogeneous polynomials of degree ℓ with two variables z_0, z_1 for any $\ell \in \mathbf{Z}$ and the representation ρ_ℓ of $SU(2)$ on V_ℓ is defined by

$$(\rho_\ell \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} f)(z_0, z_1) = f((z_0, z_1) \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix})$$

for each $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2)$. Set

$$v_i := \frac{1}{\sqrt{i!(\ell - i)!}} z_0^{\ell-i} z_1^i \in V_\ell \quad (i = 0, 1, \cdots, \ell)$$

and the standard Hermitian inner product of V_ℓ invariant under ρ_ℓ is defined such that $\{v_0, v_1, \cdots, v_\ell\}$ is a unitary basis of V_ℓ.

Here,

$$V_{\tilde{\Lambda}} = (W'_{\frac{1}{2}(y_1+y_2)} \otimes V_1) \boxtimes (W'_{-\frac{1}{2}(y_1+y_2)} \otimes V_1).$$

Here V_ℓ denotes the complex vector space of complex homogeneous polynomials of degree ℓ with two variables z_0, z_1 for any $\ell \in \mathbf{Z}$. Denote the basis of V_1 by v_0, v_1. The representation of $\tilde{K}_0$ on $v_i \otimes v_j \in V_{\tilde{\Lambda}}$ $(i, j = 0, 1)$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}} \begin{pmatrix} e^{\sqrt{-1}s} & & & \\ & e^{\sqrt{-1}t} & & \\ & & e^{\sqrt{-1}s} & \\ & & & e^{\sqrt{-1}t} \end{pmatrix} (v_i \otimes v_j) \\
=&[\rho_1 \begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}} \end{pmatrix}](v_i) \otimes [\rho_1 \begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}} \end{pmatrix}](v_j) \\
=&e^{\sqrt{-1}(s-t)[1-(i+j)]} v_i \otimes v_j.
\end{aligned}$$

Then $(V_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_1 \otimes v_0, v_0 \otimes v_1\}$. But for $\mathrm{diag}(1, 1, -1, -1) \in \tilde{K}_{[\mathfrak{a}]}$ and $i, j = 0, 1$,

$$\rho_{\tilde{\Lambda}}(\mathrm{diag}(1, 1, -1, -1))(v_i \otimes v_j) = -v_i \otimes v_j.$$

So $(V_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ and $\tilde{\Lambda} = (1, 0, 0, -1) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$. Similarly, $\tilde{\Lambda} = (0, -1, 1, 0) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (1, -1, 0, 0)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 3$. It follows from the branching laws of $(U(2), U(1) \times U(1))$ that $(\tilde{q}'_1, \tilde{q}'_2) = (1, -1) \oplus (0, 0) \oplus (-1, 1)$ and $(\tilde{q}'_3, \tilde{q}'_4) = (0, 0)$. Then $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (0, 0, 0, 0) \in D(\tilde{K}, \tilde{K}_0)$. Hence,

$$c_{\tilde{\Lambda}} = 4, \quad c_{\tilde{\Lambda}''} = 0, \quad c_L = c_{\tilde{\Lambda}} - \frac{1}{2} c_{\tilde{\Lambda}''} = 4 < 6.$$

On the other hand,

$$V_{\tilde{\Lambda}} \cong V_2 \boxtimes \mathbf{C}.$$

The representation of $\tilde{K}_0$ on $v_i \otimes w \in V_{\tilde{\Lambda}}$ is given by

$$\begin{aligned}&\rho_{\tilde{\Lambda}} \begin{pmatrix} e^{\sqrt{-1}s} & & & \\ & e^{\sqrt{-1}t} & & \\ & & e^{\sqrt{-1}s} & \\ & & & e^{\sqrt{-1}t} \end{pmatrix} (v_i \otimes w)\\ =&e^{\sqrt{-1}(s-t)(1-i)} v_i \otimes w.\end{aligned}$$

Then $(V_{\tilde{\Lambda}})_{\tilde{K}_0} = \operatorname{span}_{\mathbf{C}}\{v_1 \otimes w\}$. But for $\begin{pmatrix} 0 & e^{\sqrt{-1}s} & & \\ -e^{\sqrt{-1}t} & 0 & & \\ & & 0 & e^{\sqrt{-1}s} \\ & & e^{\sqrt{-1}t} & 0 \end{pmatrix} \in$ $\tilde{K}_{[\mathfrak{a}]}$,

$$\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & e^{\sqrt{-1}s} & & \\ -e^{\sqrt{-1}t} & 0 & & \\ & & 0 & e^{\sqrt{-1}s} \\ & & e^{\sqrt{-1}t} & 0 \end{pmatrix} (v_1 \otimes w) = -v_1 \otimes w.$$

So $(V_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ and $\tilde{\Lambda} = (1,-1,0,0) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$. Similarly, $\tilde{\Lambda} = (0,0,1,-1) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (1,-1,1,-1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 9$. It follows from the branching laws of $(U(2), U(1)\times U(1))$ that $(\tilde{q}'_1, \tilde{q}'_2) = (1,-1) \oplus (0,0)$ and $(\tilde{q}'_3, \tilde{q}'_4) = (1,-1) \oplus (0,0)$. Then $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (1,-1,-1,1)$, $(-1,1,1,-1)$ or $(0,0,0,0) \in D(\tilde{K}, \tilde{K}_0)$. When $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (0,0,0,0)$,

$$c_{\tilde{\Lambda}} = 8, \quad c_{\tilde{\Lambda}''} = 0, \quad c_L = c_{\tilde{\Lambda}} - \frac{1}{2} c_{\tilde{\Lambda}''} = 8 > 6.$$

When $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (1,-1,-1,1)$ or $(-1,1,1,-1)$,

$$c_{\tilde{\Lambda}} = 8, \quad c_{\tilde{\Lambda}''} = 4, \quad c_L = c_{\tilde{\Lambda}} - \frac{1}{2} c_{\tilde{\Lambda}''} = 6.$$

On the other hand,

$$V_{\tilde{\Lambda}} \cong V_2 \boxtimes V_2.$$

The representation of $\tilde{K}_0$ on $v_i \otimes v_j \in V_{\tilde{\Lambda}}$ $(i,j=0,1,2)$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}} \begin{pmatrix} e^{\sqrt{-1}s} & & & \\ & e^{\sqrt{-1}t} & & \\ & & e^{\sqrt{-1}s} & \\ & & & e^{\sqrt{-1}t} \end{pmatrix} (v_i \otimes v_j) \\
=&[\rho_2 \begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}} \end{pmatrix}](v_i) \otimes [\rho_2 \begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}} \end{pmatrix}](v_j) \\
=&e^{\sqrt{-1}(s-t)[2-(i+j)]} v_i \otimes v_j.
\end{aligned}$$

Hence

$$(V_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_0 \otimes v_2, v_1 \otimes v_1, v_2 \otimes v_0\}.$$

Moreover, the action of the generator $\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix}$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $v_i \otimes v_j$ is given by

$$\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix})(v_i \otimes v_j) = (-1)^{3-i} v_{2-i} \otimes v_{2-j}.$$

Therefore,

$$(V_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}\{v_0 \otimes v_2 - v_2 \otimes v_0, v_1 \otimes v_1\}$$

and $\tilde{\Lambda} = (1,-1,1,-1) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$. Notice that the $\tilde{K}_{[\mathfrak{a}]}$-fixed vector $v_1 \otimes v_1 \in V_0'$, which corresponds eigenvalue 8 and the $\tilde{K}_{[\mathfrak{a}]}$-fixed vector $v_0 \otimes v_2 - v_2 \otimes v_0 \in V'_{y_1-y_2-y_3+y_4} \oplus V'_{-y_1+y_2+y_3-y_4}$, which gives eigenvalue 6.

Suppose that $\tilde{\Lambda} = (2,0,-1,-1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 3$. It follows from the branching laws of $(U(2), U(1) \times U(1))$ that $(\tilde{q}_1', \tilde{q}_2') = (2,0) \oplus (1,1) \oplus (0,2)$ and $(\tilde{q}_3', \tilde{q}_4') = (-1,-1)$. Then $(\tilde{q}_1', \tilde{q}_2', \tilde{q}_3', \tilde{q}_4') = (1,1,-1,-1) \in D(\tilde{K}, \tilde{K}_0)$. Hence,

$$c_{\tilde{\Lambda}} = 8, \quad c_{\tilde{\Lambda}''} = 4, \quad c_L = c_{\tilde{\Lambda}} - \frac{1}{2} c_{\tilde{\Lambda}''} = 6.$$

On the other hand,

$$V_{\tilde{\Lambda}} \cong (V_2 \otimes \mathbf{C}) \boxtimes \mathbf{C}.$$

The representation of $\tilde{K}_0$ on $v_i \otimes w \in V_{\tilde{\Lambda}}$ $(i = 0, 1, 2)$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}} \begin{pmatrix} e^{\sqrt{-1}s} & & & \\ & e^{\sqrt{-1}t} & & \\ & & e^{\sqrt{-1}s} & \\ & & & e^{\sqrt{-1}t} \end{pmatrix} (v_i \otimes w) \\
=&e^{\sqrt{-1}(s+t)}[\rho_2 \begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}} \end{pmatrix}](v_i) \otimes e^{-\sqrt{-1}(s+t)} w \\
=&e^{\sqrt{-1}(s-t)(1-i)} v_i \otimes w.
\end{aligned}$$

Hence

$$(V_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_1 \otimes 1\}.$$

Moreover, the action of the generator $\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix}$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $v_i \otimes w$ is given by

$$\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix})(v_i \otimes 1) = (-1)^{1-i} v_{2-i} \otimes 1.$$

Therefore,

$$(V_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}\{v_1 \otimes 1\}$$

and $\tilde{\Lambda} = (2, 0, -1, -1) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 6. Similarly, $\tilde{\Lambda} = (-1, -1, 2, 0) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 6 and with multiplicity 1.

Suppose that $\tilde{\Lambda} = (1, 1, 0, -2)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 3$. It follows from the branching laws of $(U(2), U(1) \times U(1))$ that $(\tilde{q}'_1, \tilde{q}'_2) = (1, 1)$ and $(\tilde{q}'_3, \tilde{q}'_4) = (-2, 0) \oplus (-1, -1) \oplus (0, -2)$. Then $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (1, 1, -1, -1) \in D(\tilde{K}, \tilde{K}_0)$. Hence,

$$c_{\tilde{\Lambda}} = 8, \quad c_{\tilde{\Lambda}''} = 4, \quad c_L = c_{\tilde{\Lambda}} - \frac{1}{2} c_{\tilde{\Lambda}''} = 6.$$

On the other hand,

$$V_{\tilde{\Lambda}} \cong \mathbf{C} \boxtimes (V_2 \otimes \mathbf{C}).$$

The representation of $\tilde{K}_0$ on $w \otimes v_i \in V_{\tilde{\Lambda}}$ $(i = 0, 1, 2)$ is given by

$$
\begin{aligned}
&\rho_{\tilde{\Lambda}} \begin{pmatrix} e^{\sqrt{-1}s} & & & \\ & e^{\sqrt{-1}t} & & \\ & & e^{\sqrt{-1}s} & \\ & & & e^{\sqrt{-1}t} \end{pmatrix} (w \otimes v_i) \\
=& e^{\sqrt{-1}(s+t)} w \otimes e^{-\sqrt{-1}(s+t)} [\rho_2 \begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}} \end{pmatrix}](v_i) \\
=& e^{\sqrt{-1}(s-t)(1-i)} w \otimes v_i.
\end{aligned}
$$

Hence

$$(V_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{1 \otimes v_1\}.$$

Moreover, the action of the generator $\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix}$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $v_i \otimes w$ is given by

$$\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix})(1 \otimes v_i) = 1 \otimes v_{2-i}.$$

Therefore,

$$(V_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}_{\mathbf{C}}\{1 \otimes v_1\}$$

and $\tilde{\Lambda} = (2, 0, -1, -1) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 6. Similarly, $\tilde{\Lambda} = (0, -2, 1, 1) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 6 and with multiplicity 1.

Observe that

$$
\begin{aligned}
&\dim_{\mathbf{C}} V_{(2,0,-1,-1)} + \dim_{\mathbf{C}} V_{(-1,-1,2,0)} + \dim_{\mathbf{C}} V_{(1,1,0,-2)} \\
&+ \dim_{\mathbf{C}} V_{(0,-2,1,1)} + \dim_{\mathbf{C}} V_{(1,-1,1,-1)} = 3 + 3 + 3 + 3 + 9 \\
&= 21 = \dim SO(8) - \dim S(U(2) \times U(2)) = n_{hk}(\mathcal{G}).
\end{aligned}
$$

Therefore, $\mathcal{G}(N^6) \subset Q_6(\mathbf{C})$ is strictly Hamiltonian stable.

9.9. **Eigenvalue computation when $m = 3$.**

$$\begin{aligned}
c_{\tilde{\Lambda}} =&\langle \tilde{\Lambda} + 2\delta_{\tilde{K}}, \tilde{\Lambda}\rangle \\
=&\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 + \tilde{p}_4^2 + \tilde{p}_5^2 + \tilde{p}_1 - \tilde{p}_2 + 2(\tilde{p}_3 - \tilde{p}_5), \\
c_{\tilde{\Lambda}'} =&\langle \tilde{\Lambda}' + 2\delta_{\tilde{K}_2}, \tilde{\Lambda}'\rangle \\
=&\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{q}_3^2 + \tilde{q}_4^2 + \tilde{p}_1 - \tilde{p}_2 + \tilde{q}_3 - \tilde{q}_4, \\
c_{\tilde{\Lambda}''} =&\langle \tilde{\Lambda}'' + 2\delta_{\tilde{K}_1}, \tilde{\Lambda}''\rangle \\
=&(\tilde{q}_1')^2 + (\tilde{q}_2')^2 + (\tilde{q}_3')^2 + (\tilde{q}_4')^2.
\end{aligned}$$

Then the eigenvalue formula is

$$\begin{aligned}
c =&2c_{\tilde{\Lambda}} - c_{\tilde{\Lambda}'} - \frac{1}{2}c_{\tilde{\Lambda}''} \\
=& \tilde{p}_1^2 + \tilde{p}_2^2 + 2(\tilde{p}_3^2 + \tilde{p}_4^2 + \tilde{p}_5^2) + (\tilde{p}_1 - \tilde{p}_2) + 4(\tilde{p}_3 - \tilde{p}_5) \\
&- (\tilde{q}_3^2 + \tilde{q}_4^2) - (\tilde{q}_3 - \tilde{q}_4) - \frac{1}{2}((\tilde{q}_1')^2 + (\tilde{q}_2')^2 + (\tilde{q}_3')^2 + (\tilde{q}_4')^2)
\end{aligned}$$

Denote $\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_5 y_5 \in D(\tilde{K}, \tilde{K}_0)$ by $\tilde{\Lambda} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5)$. Then

$$\begin{aligned}
&\{\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0) | - c_L \leq 10\} \\
=&\{0, (1, -1, 1, 0, -1), (2, 0, 0, -1, -1), (0, -2, 1, 1, 0), \\
&(1, 1, 0, 0, -2), (-1, -1, 2, 0, 0), (1, -1, 0, 0, 0), \\
&(1, 0, 0, 0, -1), (0, -1, 1, 0, 0), (1, 1, 0, -1, -1), \\
&(-1, -1, 1, 1, 0), (0, 0, 1, 0, -1)\}.
\end{aligned}$$

Suppose that $\tilde{\Lambda} = (1, -1, 1, 0, -1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 24$. It follows from the branching laws of $(U(3), U(2) \times U(1))$ that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1, -1, 0) \oplus (0, 0, 0)$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0, 0, 0)$, by the branching laws of $(U(2), U(1) \times U(1))$, $(\tilde{q}_1', \tilde{q}_2', \tilde{q}_3', \tilde{q}_4', \tilde{q}_5') = (0, 0, 0, 0, 0)$. Hence,

$$c_L = 2c_{\tilde{\Lambda}} - c_{\tilde{\Lambda}'} - \frac{1}{2}c_{\tilde{\Lambda}''} = 16 > 10.$$

When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1, -1, 0)$, by the branching laws of $(U(2), U(1) \times U(1))$, $(\tilde{q}_1', \tilde{q}_2', \tilde{q}_3', \tilde{q}_4', \tilde{q}_5') = (1, -1, -1, 1, 0)$, $(0, 0, 0, 0, 0)$ or $(-1, 1, 1, -1, 0) \in D(\tilde{K}, \tilde{K}_0)$, respectively. Hence,

$$c_L = 2c_{\tilde{\Lambda}} - c_{\tilde{\Lambda}'} - \frac{1}{2}c_{\tilde{\Lambda}''} = 10, 12 \text{ or } 10,$$

respectively.

On the other hand, now

$$\tilde{K}_0 = \left\{ \begin{pmatrix} e^{\sqrt{-1}s} & & & & \\ & e^{\sqrt{-1}t} & & & \\ & & e^{\sqrt{-1}s} & & \\ & & & e^{\sqrt{-1}t} & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix} ; s,t,l \in \mathbf{R} \right\} \cong U(1)\times U(1)\times U(1).$$

$$\begin{aligned}(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} &\subset (W_{y_1-y_2} \boxtimes W_{y_3-y_4} \boxtimes W_0) \oplus (W_{y_1-y_2} \boxtimes W_0 \boxtimes W_0) \\ &\cong (V_2 \boxtimes V_2 \boxtimes \mathbf{C}) \oplus (V_2 \boxtimes \mathbf{C} \boxtimes \mathbf{C}),\end{aligned}$$

where the latter is the $\tilde{K}_2 = U(2) \times U(2) \times U(1)$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u_i \otimes v_j \otimes w \in V_2 \boxtimes V_2 \boxtimes \mathbf{C}$ $(i,j = 0,1,2)$ is given by

$$\begin{aligned}&\rho_{\tilde{\Lambda}}(\begin{pmatrix} e^{\sqrt{-1}s} & & & & \\ & e^{\sqrt{-1}t} & & & \\ & & e^{\sqrt{-1}s} & & \\ & & & e^{\sqrt{-1}t} & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u_i \otimes v_j \otimes w) \\ =&\rho_{y_1-y_2}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix}(u_i) \otimes \rho_{y_3-y_4}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix}(v_j) \otimes w \\ =&e^{\sqrt{-1}(s-t)(2-i-j)} u_i \otimes v_j \otimes w.\end{aligned}$$

The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u_i \otimes v \otimes w \in V_2 \boxtimes \mathbf{C} \boxtimes \mathbf{C}$ $(i = 0,1,2)$ is given by

$$\begin{aligned}&\rho_{\tilde{\Lambda}}(\begin{pmatrix} e^{\sqrt{-1}s} & & & & \\ & e^{\sqrt{-1}t} & & & \\ & & e^{\sqrt{-1}s} & & \\ & & & e^{\sqrt{-1}t} & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u_i \otimes v \otimes w) \\ =&\rho_{y_1-y_2}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix}(u_i) \otimes v \otimes w \\ =&e^{\sqrt{-1}(s-t)(1-i)} u_i \otimes v \otimes w\end{aligned}$$

Thus,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{Span}_{\mathbf{C}}\{u_2 \otimes v_0 \otimes w, u_0 \otimes v_2 \otimes w, u_1 \otimes v_1 \otimes w, u_1 \otimes v \otimes w\}.$$

Moreover, the action of the generator $\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix}$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u_i \otimes v_{2-i} \otimes w$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u_i \otimes v_{2-i} \otimes w) \\
=&\rho_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (u_i) \otimes \rho_2 \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} (v_{2-i}) \otimes w \\
=& (-1)^{1-i} u_{2-i} \otimes v_i \otimes w
\end{aligned}$$

and the action on $u_i \otimes v \otimes w$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u_i \otimes v \otimes w) \\
=&\rho_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (u_i) \otimes v \otimes w = (-1)^{2-i} u_{2-i} \otimes v \otimes w
\end{aligned}$$

Therefore,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}_{\mathbf{C}}\{u_2 \otimes v_0 \otimes w - u_0 \otimes v_2 \otimes w, u_1 \otimes v_1 \otimes w\},$$

$\dim_{\mathbf{C}} (\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = 2$ and $\tilde{\Lambda} = (1, -1, 1, 0, -1) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$. Notice that the $\tilde{K}_{[\mathfrak{a}]}$-fixed vector $u_1 \otimes v_1 \otimes w \in V_{\tilde{\Lambda}''}$, which corresponds eigenvalue 12, where $\tilde{\Lambda}'' = 0$. And the $\tilde{K}_{[\mathfrak{a}]}$-fixed vector $u_2 \otimes v_0 \otimes w - u_0 \otimes v_2 \otimes w \in V_{\tilde{\Lambda}''_1} \oplus V_{\tilde{\Lambda}''_2}$, which gives eigenvalue 10, where $\tilde{\Lambda}''_1 = (1, -1, -1, 1, 0)$ and $\tilde{\Lambda}''_2 = (-1, 1, 1, -1, 0)$.

Suppose that $\tilde{\Lambda} = (2, 0, 0, -1, -1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 9$. It follows from the branching laws of $(U(3), U(2) \times U(1))$ that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0, -1, -1) \oplus (-1, -1, 0)$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (-1, -1, 0)$, by the branching laws of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4, \tilde{q}'_5) = (1, 1, -1, -1, 0)$. Hence,

$$c_L = 2c_{\tilde{\Lambda}} - c_{\tilde{\Lambda}'} - \frac{1}{2} c_{\tilde{\Lambda}''} = 10.$$

On the other hand,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset (W_{2y_1} \boxtimes W_{-(y_3+y_4)} \boxtimes W_0) \cong V_2 \boxtimes \mathbf{C} \boxtimes \mathbf{C},$$

and the representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u_i \otimes v \otimes w \in V_2 \boxtimes \mathbf{C} \boxtimes \mathbf{C}$ $(i = 0, 1, 2)$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(\begin{pmatrix} e^{\sqrt{-1}s} & & & & \\ & e^{\sqrt{-1}t} & & & \\ & & e^{\sqrt{-1}s} & & \\ & & & e^{\sqrt{-1}t} & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u_i \otimes v \otimes w) \\
=&\rho_{2y_1}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix}(u_i) \otimes \rho_{-y_3-y_4}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix}(v) \otimes w \\
=&e^{\sqrt{-1}(s-t)(1-i)} u_i \otimes v \otimes w.
\end{aligned}$$

Thus,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{Span}_{\mathbf{C}}\{u_1 \otimes v \otimes w\}.$$

Moreover, the action of the generator $\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix}$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u_i \otimes v \otimes w$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u_i \otimes v \otimes w) \\
=&\rho_{2y_1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(u_i) \otimes \rho_{-(y_3+y_4)}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}(v) \otimes w = (-1)^{1+i} u_{2-i} \otimes v \otimes w
\end{aligned}$$

Therefore,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}_{\mathbf{C}}\{u_1 \otimes v \otimes w\},$$

$\dim_{\mathbf{C}}(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = 1$ and $\tilde{\Lambda} = (2, 0, 0, -1, -1) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 10. Similarly, $\tilde{\Lambda} = (0, -2, 1, 1, 0) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 10 and with multiplicity 1 and dimension 9.

Suppose that $\tilde{\Lambda} = (1, 1, 0, 0, -2)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 6$. It follows from the branching laws of $(U(3), U(2) \times U(1))$ that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0, 0, -2) \oplus$

$(0,-1,-1)\oplus(0,-2,0)$. When $(\tilde{q}_3,\tilde{q}_4,\tilde{q}_5)=(0,-2,0)$, by the branching laws of $(U(2),U(1)\times U(1))$, $(\tilde{q}_1',\tilde{q}_2',\tilde{q}_3',\tilde{q}_4',\tilde{q}_5')=(1,1,-1,-1,0)$. Hence,

$$c_L=2c_{\tilde{\Lambda}}-c_{\tilde{\Lambda}'}-\frac{1}{2}c_{\tilde{\Lambda}''}=10.$$

On the other hand,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0}\subset(W_0\boxtimes W_{-2y_4})\boxtimes W_0)\cong\mathbf{C}\boxtimes V_2\boxtimes\mathbf{C},$$

and the representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u\otimes v_i\otimes w\in\mathbf{C}\boxtimes V_2\boxtimes\mathbf{C}$ $(i=0,1,2)$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(\begin{pmatrix} e^{\sqrt{-1}s} &&&& \\ & e^{\sqrt{-1}t} &&& \\ && e^{\sqrt{-1}s} && \\ &&& e^{\sqrt{-1}t} & \\ &&&& e^{\sqrt{-1}l}\end{pmatrix})(u\otimes v_i\otimes w)\\
=&\rho_{y_1+y_2}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t}\end{pmatrix}(u)\otimes\rho_{-2y_4}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t}\end{pmatrix}(v_i)\otimes w\\
=&e^{\sqrt{-1}(s-t)(1-i)}u\otimes v_i\otimes w.
\end{aligned}$$

Thus,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0}=\mathrm{Span}_{\mathbf{C}}\{u\otimes v_1\otimes w\}.$$

Moreover, the action of the generator $\begin{pmatrix} 0 & 1 &&& \\ -1 & 0 &&& \\ && 0 & -1 & \\ && -1 & 0 & \\ &&&& e^{\sqrt{-1}l}\end{pmatrix}$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u_i\otimes v\otimes w$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 &&& \\ -1 & 0 &&& \\ && 0 & -1 & \\ && -1 & 0 & \\ &&&& e^{\sqrt{-1}l}\end{pmatrix})(u\otimes v_i\otimes w)\\
=&\rho_{y_1+y_2}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}(u)\otimes\rho_{-2y_4}\begin{pmatrix} 0 & -1 \\ -1 & 0\end{pmatrix}(v_i)\otimes w=u\otimes v_{2-i}\otimes w
\end{aligned}$$

Therefore,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}}=\mathrm{span}_{\mathbf{C}}\{u\otimes v_1\otimes w\},$$

$\dim_{\mathbf{C}}(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = 1$ and $\tilde{\Lambda} = (1,1,0,0,-2) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 10. Similarly, $\tilde{\Lambda} = (-1,-1,2,0,0) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 10 and with multiplicity 1 and dimension 6.

Suppose that $\tilde{\Lambda} = (1,-1,0,0,0)$. Then $(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1,-1,0,0,0)$. It follows from the branching laws of $(U(2), U(1)\times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4, \tilde{q}'_5) = (0,0,0,0,0) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence,

$$c_L = 2c_{\tilde{\Lambda}} - c_{\tilde{\Lambda}'} - \frac{1}{2}c_{\tilde{\Lambda}''} = 4 < 10.$$

On the other hand,

$$(\tilde{V}_{\tilde{\Lambda}}) = W_{y_1-y_2} \boxtimes W_0 \cong V_2 \boxtimes \mathbf{C},$$

and the representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u_i \otimes w \in V_2 \boxtimes \mathbf{C}$ $(i = 0,1,2)$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(\begin{pmatrix} e^{\sqrt{-1}s} & & & & \\ & e^{\sqrt{-1}t} & & & \\ & & e^{\sqrt{-1}s} & & \\ & & & e^{\sqrt{-1}t} & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u_i \otimes w) \\
=&\rho_2\begin{pmatrix} e^{\sqrt{-1}\frac{s-t}{2}} & \\ & e^{-\sqrt{-1}\frac{s-t}{2}} \end{pmatrix}(u_i) \otimes w \\
=&e^{\sqrt{-1}(s-t)(1-i)} u_i \otimes w.
\end{aligned}$$

Thus,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{Span}_{\mathbf{C}}\{u_1 \otimes w\}.$$

Moreover, the action of the generator $\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix}$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u_1 \otimes w$ is given by

$$\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u_1 \otimes w) = -u_1 \otimes w.$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ and $\tilde{\Lambda} = (1,-1,0,0,0) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (1,0,0,0,-1)$. It follows from the branching laws of $(U(3), U(2)\times U(1))$ that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,0,-1) \oplus (0,-1,0)$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,-1,0)$, by the branching laws of $(U(2), U(1)\times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4, \tilde{q}'_5) = (1,0,-1,0,0) \oplus (0,1,0,-1,0) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence,

$$c_L = 2c_{\tilde{\Lambda}} - c_{\tilde{\Lambda}'} - \frac{1}{2}c_{\tilde{\Lambda}''} = 5,5 < 10.$$

On the other hand, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset (W_{y_1} \boxtimes W_{-y_4} \boxtimes W_0) \cong (V_1 \boxtimes V_1 \boxtimes \mathbf{C})$, where the latter is the $\tilde{K}_2 = U(2)\times U(2)\times U(1)$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u_i \otimes v_j \otimes w \in V_1 \boxtimes V_1 \boxtimes \mathbf{C}$ $(i,j = 0,1)$ is given by

$$\begin{aligned}&\rho_{\tilde{\Lambda}}(\begin{pmatrix} e^{\sqrt{-1}s} & & & & \\ & e^{\sqrt{-1}t} & & & \\ & & e^{\sqrt{-1}s} & & \\ & & & e^{\sqrt{-1}t} & \\ & & & & e^{\sqrt{-1}l}\end{pmatrix})(u_i \otimes v_j \otimes w)\\ =&e^{\sqrt{-1}(s-t)(1-i-j)} u_i \otimes v_j \otimes w.\end{aligned}$$

Thus,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{Span}_{\mathbf{C}}\{u_1 \otimes v_0 \otimes w, u_0 \otimes v_1 \otimes w\}.$$

Moreover, the action of the generator $\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l}\end{pmatrix}$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u_i \otimes v_{1-i} \otimes w$ $(i = 0,1)$ is given by

$$\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l}\end{pmatrix})(u_i \otimes v_{1-i} \otimes w) = (-1)^{1-i} u_{1-i} \otimes v_i \otimes w.$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ and $\tilde{\Lambda} = (1,0,0,0,-1) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (0,-1,1,0,0)$. It follows from the branching laws of $(U(3), U(2)\times U(1))$ that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,0,1) \oplus (1,0,0)$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1,0,0)$, by the branching laws of $(U(2), U(1)\times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4, \tilde{q}'_5) = (-1,0,1,0,0) \oplus (0,-1,0,1,0) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence,

$$c_L = 2c_{\tilde{\Lambda}} - c_{\tilde{\Lambda}'} - \frac{1}{2}c_{\tilde{\Lambda}''} = 5,5 < 10.$$

On the other hand, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset (W_{-y_2} \boxtimes W_{y_3} \boxtimes W_0) \cong (V_1 \boxtimes V_1 \boxtimes \mathbf{C})$, where the latter is the $\tilde{K}_2 = U(2) \times U(2) \times U(1)$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u_i \otimes v_j \otimes w \in V_1 \boxtimes V_1 \boxtimes \mathbf{C}$ $(i, j = 0, 1)$ is given by

$$\begin{aligned}&\rho_{\tilde{\Lambda}}(\begin{pmatrix} e^{\sqrt{-1}s} & & & & \\ & e^{\sqrt{-1}t} & & & \\ & & e^{\sqrt{-1}s} & & \\ & & & e^{\sqrt{-1}t} & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u_i \otimes v_j \otimes w) \\ =&e^{\sqrt{-1}(s-t)(1-i-j)} u_i \otimes v_j \otimes w.\end{aligned}$$

Thus,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{Span}_{\mathbf{C}}\{u_1 \otimes v_0 \otimes w, u_0 \otimes v_1 \otimes w\}.$$

Moreover, the action of the generator $\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix}$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u_i \otimes v_{1-i} \otimes w$ $(i = 0, 1)$ is given by

$$\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u_i \otimes v_{1-i} \otimes w) = (-1)^i u_{1-i} \otimes v_i \otimes w.$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ and $\tilde{\Lambda} = (1, 0, 0, 0, -1) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (1, 1, 0, -1, -1)$. It follows from the branching laws of $(U(3), U(2) \times U(1))$ that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0, -1, -1) \oplus (-1, -1, 0)$. For the element $(\tilde{p}_1, \tilde{p}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1, 1, -1, -1, 0)$ in $D(\tilde{K}_2, \tilde{K}_0)$, by the branching laws of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (1, 1, -1, -1) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence,

$$c_L = 2c_{\tilde{\Lambda}} - c_{\tilde{\Lambda}'} - \frac{1}{2}c_{\tilde{\Lambda}''} = 6 < 10.$$

On the other hand, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset (W_{y_1+y_2} \boxtimes W_{-y_3-y_4} \boxtimes W_0) \cong (\mathbf{C} \boxtimes \mathbf{C} \boxtimes \mathbf{C})$, where the latter is the $\tilde{K}_2 = U(2) \times U(2) \times U(1)$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u \otimes v \otimes w \in \mathbf{C} \boxtimes \mathbf{C} \boxtimes \mathbf{C}$ is given by

$$\begin{aligned}&\rho_{\tilde{\Lambda}}(\begin{pmatrix} e^{\sqrt{-1}s} & & & & \\ & e^{\sqrt{-1}t} & & & \\ & & e^{\sqrt{-1}s} & & \\ & & & e^{\sqrt{-1}t} & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u \otimes v \otimes w)\\ =&e^{\sqrt{-1}(s+t)} u \otimes e^{-\sqrt{-1}(s+t)} v \otimes w = u \otimes v \otimes w.\end{aligned}$$

It says that $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{Span}_{\mathbf{C}}\{1 \otimes 1 \otimes 1\}$. Moreover, the action of the generator $\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix}$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u \otimes v \otimes w$ is given by

$$\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u \otimes v \otimes w) = -u \otimes v \otimes w.$$

Therefore $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ and $\tilde{\Lambda} = (1, 1, 0, -1, -1) \notin D(\tilde{K}, \tilde{K}_0)$. Similarly, $\tilde{\Lambda} = (-1, -1, 1, 1, 0) \notin D(\tilde{K}, \tilde{K}_0)$.

Suppose that $\tilde{\Lambda} = (0, 0, 1, 0, -1)$. It follows from the branching laws of $(U(3), U(2) \times U(1))$ that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1, 0, -1) \oplus (0, 0, 0) \oplus (1, -1, 0) \oplus (0, -1, 1)$. For the element $(\tilde{p}_1, \tilde{p}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0, 0, 0, 0, 0)$ in $D(\tilde{K}_2, \tilde{K}_0)$, by the branching laws of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (0, 0, 0, 0) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence,

$$c_L = 2c_{\tilde{\Lambda}} - c_{\tilde{\Lambda}'} - \frac{1}{2} c_{\tilde{\Lambda}''} = 12 > 10.$$

For the element $(\tilde{p}_1, \tilde{p}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0, 0, 1, -1, 0)$ in $D(\tilde{K}_2, \tilde{K}_0)$, by the branching laws of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (0, 0, 0, 0) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence,

$$c_L = 2c_{\tilde{\Lambda}} - c_{\tilde{\Lambda}'} - \frac{1}{2} c_{\tilde{\Lambda}''} = 8 < 10.$$

On the other hand, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset \tilde{V}'_{(0,0,0,0,0)} \oplus \tilde{V}'_{(0,0,1,-1,0)}$. We only concern on $\tilde{V}'_{(0,0,1,-1,0)}$ since it corresponds to the smaller eigenvalue 8. Notice that $\tilde{V}'_{(0,0,1,-1,0)} = (W_0 \boxtimes W_{y_3 - y_4} \boxtimes W_0) \cong \mathbf{C} \boxtimes V_2 \boxtimes \mathbf{C}$, which is the

$\tilde{K}_2 = U(2) \times U(2) \times U(1)$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u \otimes v_i \otimes w \in \tilde{V}'_{(0,0,1,-1,0)}$ $(i = 0, 1, 2)$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(\begin{pmatrix} e^{\sqrt{-1}s} & & & & \\ & e^{\sqrt{-1}t} & & & \\ & & e^{\sqrt{-1}s} & & \\ & & & e^{\sqrt{-1}t} & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u \otimes v_i \otimes w) \\
=&e^{\sqrt{-1}(s-t)(1-i)} u \otimes v_i \otimes w.
\end{aligned}$$

Thus $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{Span}_{\mathbf{C}}\{1 \otimes v_1 \otimes 1\} \oplus \tilde{V}'_{(0,0,0,0,0)}$.

Moreover, the action of the generator $\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix}$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u \otimes v_1 \otimes w$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & e^{\sqrt{-1}l} \end{pmatrix})(u \otimes v_1 \otimes w) \\
=&u \otimes \rho_2(\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}) v_1 \otimes w = -u \otimes v_1 \otimes w.
\end{aligned}$$

Therefore, $1 \otimes v_1 \otimes 1 \notin (\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}}$ and $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \tilde{V}'_{(0,0,0,0,0)}$, which gives the larger eigenvalue 10.

Therefore, $\mathcal{G}(N^{10}) \subset Q_{10}(\mathbf{C})$ is Hamiltonian stable. Moreover, since

$$\begin{aligned}
&\dim_{\mathbf{C}} V_{(1,-1,1,0,-1)} + \dim_{\mathbf{C}} V_{(2,0,0,-1,-1)} + \dim_{\mathbf{C}} V_{(0,-2,1,1,0)} \\
&+ \dim_{\mathbf{C}} V_{(1,1,0,0,-2)} + \dim_{\mathbf{C}} V_{(-1,-1,2,0,0)} \\
&= 24 + 9 + 9 + 6 + 6 = 54 \\
&= \dim SO(12) - \dim S(U(2) \times U(3)) = n_{hk}(\mathcal{G}),
\end{aligned}$$

$\mathcal{G}(N^{10}) \subset Q_{10}(\mathbf{C})$ is strictly Hamiltonian stable.

9.10. **Eigenvalue computation when $m \geq 4$.**

$$\begin{aligned}
c_{\tilde{\Lambda}} =& \langle \tilde{\Lambda} + 2\delta_{\tilde{K}}, \tilde{\Lambda} \rangle \\
=& \tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 + \tilde{p}_4^2 + \tilde{p}_{m+1}^2 + \tilde{p}_{m+2}^2 \\
& + \tilde{p}_1 - \tilde{p}_2 + (m-1)\tilde{p}_3 + (m-3)\tilde{p}_4 \\
& - (m-3)\tilde{p}_{m+1} - (m-1)\tilde{p}_{m+2}, \\
c_{\tilde{\Lambda}'} =& \langle \tilde{\Lambda}' + 2\delta_{\tilde{K}_2}, \tilde{\Lambda}' \rangle \\
=& \tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{q}_3^2 + \tilde{q}_4^2 + \tilde{p}_1 - \tilde{p}_2 + \tilde{q}_3 - \tilde{q}_4, \\
c_{\tilde{\Lambda}''} =& \langle \tilde{\Lambda}'' + 2\delta_{\tilde{K}_1}, \tilde{\Lambda}'' \rangle \\
=& (\tilde{q}_1')^2 + (\tilde{q}_2')^2 + (\tilde{q}_3')^2 + (\tilde{q}_4')^2.
\end{aligned}$$

Then the eigenvalue formula is

$$\begin{aligned}
c =& 2c_{\tilde{\Lambda}} - c_{\tilde{\Lambda}'} - \frac{1}{2} c_{\tilde{\Lambda}''} \\
=& \tilde{p}_1^2 + \tilde{p}_2^2 + 2(\tilde{p}_3^2 + \tilde{p}_4^2 + \tilde{p}_{m+1}^2 + \tilde{p}_{m+2}^2) \\
& + (\tilde{p}_1 - \tilde{p}_2) + 2(m-1)(\tilde{p}_3 - \tilde{p}_{m+2}) + 2(m-3)(\tilde{p}_4 - \tilde{p}_{m+1}) \\
& - (\tilde{q}_3^2 + \tilde{q}_4^2) - (\tilde{q}_3 - \tilde{q}_4) - \frac{1}{2}((\tilde{q}_1')^2 + (\tilde{q}_2')^2 + (\tilde{q}_3')^2 + (\tilde{q}_4')^2)
\end{aligned}$$

In case $\tilde{\Lambda} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_{m+1}, \tilde{p}_{m+2}) = (\tilde{p}_1, \tilde{p}_2, 0, 0, 0, 0)$:

Since

$$\tilde{p}_3 = \tilde{p}_4 = \tilde{p}_{m+1} = \tilde{p}_{m+2} = 0,$$

we have

$$\tilde{q}_3 = \tilde{q}_4 = \tilde{q}_5 = \cdots = \tilde{q}_{m+2} = 0$$

and thus

$$\tilde{q}_3' = \tilde{q}_4' = 0.$$

$\tilde{p}_1 + \tilde{p}_2 = 0$, namely, $\tilde{p}_2 = -\tilde{p}_1$.

$\tilde{q}_1' = -\alpha + \tilde{p}_1$, $\tilde{q}_2' = \alpha + \tilde{p}_2 = \alpha - \tilde{p}_1 = -\tilde{q}_1'$ for some $\alpha = 0, 1 \cdots, \tilde{p}_1 - \tilde{p}_2 = 2\tilde{p}_1$. $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ implies that $\tilde{q}_1' = \tilde{q}_2' = 0$ since $\tilde{q}_1' + \tilde{q}_3' = 0$ and $\tilde{q}_2' + \tilde{q}_4' = 0$. Then

$$\begin{aligned}
c =& 2c_{\tilde{\Lambda}} - c_{\tilde{\Lambda}'} - \frac{1}{2} c_{\tilde{\Lambda}''} \\
=& \tilde{p}_1^2 + \tilde{p}_2^2 + (\tilde{p}_1 - \tilde{p}_2) - \frac{1}{2}((\tilde{q}_1')^2 + (\tilde{q}_2')^2) \\
=& 2\tilde{p}_1^2 + 2\tilde{p}_1 - (\tilde{q}_1')^2 \\
=& 2\tilde{p}_1(\tilde{p}_1 + 1)
\end{aligned}$$

Now $\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 = 2\tilde{p}_1 \frac{1}{2}(y_1 - y_2)$. Set $\ell := 2\tilde{p}_1$. Then $\tilde{V}_{\tilde{\Lambda}} \cong V_\ell \boxtimes \mathbf{C}$, where V_ℓ denotes the complex vector space of complex homogeneous polynomials of degree ℓ with two variables z_0, z_1. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $v_i \otimes w \in \tilde{V}_{\tilde{\Lambda}}$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}} \begin{pmatrix} e^{\sqrt{-1}s} & & & & \\ & e^{\sqrt{-1}t} & & & \\ & & e^{\sqrt{-1}s} & & \\ & & & e^{\sqrt{-1}t} & \\ & & & & T \end{pmatrix} (v_i \otimes w) \\
=&[\rho_\ell \begin{pmatrix} e^{\sqrt{-1}(s-t)/2} & 0 \\ 0 & e^{-\sqrt{-1}(s-t)/2} \end{pmatrix}](v_i) \otimes w \\
=&v_i(e^{\frac{\sqrt{-1}(s-t)}{2}} z_0, e^{-\frac{\sqrt{-1}(s-t)}{2}} z_1) \otimes w \\
=&e^{\frac{\sqrt{-1}(s-t)}{2}(\ell - 2i)} v_i \otimes w
\end{aligned}$$

Hence, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_{\tilde{p}_1} \otimes w\}$. On the other hand,

$$\begin{aligned}
&\rho_{\tilde{\Lambda}} \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & T \end{pmatrix} (v_{\tilde{p}_1} \otimes w) \\
=&[\rho_\ell \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}](v_{\tilde{p}_1}) \otimes w \\
=&(-1)^{\tilde{p}_1} v_{\tilde{p}_1} \otimes w.
\end{aligned}$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}_{\mathbf{C}}\{v_{\tilde{p}_1} \otimes w\}$ for $\tilde{p}_1$ is even. In particular, for $m \geq 4$, $\tilde{\Lambda} = 2y_1 - 2y_2 \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$ with $\tilde{p}_1 = 2$ has eigenvalue $c = 2\tilde{p}_1(\tilde{p}_1 + 1) = 12 < 4m - 2$, which means that

$$L = \frac{S(U(2) \times U(m))}{S(U(1) \times U(1) \times U(m-2)) \cdot \mathbf{Z}_4} \longrightarrow Q_{4m-2}(\mathbf{C})$$

is NOT Hamiltonian stable for $m \geq 4$.

Theorem 9.1. $L = \frac{S(U(2) \times U(m))}{S(U(1) \times U(1) \times U(m-2)) \cdot \mathbf{Z}_4} \longrightarrow Q_{4m-2}(\mathbf{C})$ $(m \geq 2)$ *is not Hamiltonian stable if and only if* $m \geq 4$. *If* $m = 2$ *or* 3, *it is strictly Hamiltonian stable.*

10. The case $(U, K) = (Sp(m+2), Sp(2) \times Sp(m))$ $(m \geq 2)$

When $m = 2$, (U, K) is of B_2 type. When $m \geq 3$, (U, K) is of BC_2 type.

$$\mathfrak{u} := \mathfrak{sp}(m+2) \subset \mathfrak{u}(2m+4)$$

$$\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in \mathfrak{sp}(m+2),$$

where $A \in \mathfrak{u}(m+2)$, $B \in M(m+2, \mathbf{C})$ with ${}^tB = B$.

$$\begin{aligned}
&\mathfrak{k} \\
=&\mathfrak{sp}(2) + \mathfrak{sp}(m) \\
=&\left\{ \begin{pmatrix} A_{11} & 0 & B_{11} & 0 \\ 0 & A_{22} & 0 & B_{22} \\ -\bar{B}_{11} & 0 & \bar{A}_{11} & 0 \\ 0 & -\bar{B}_{22} & 0 & \bar{A}_{22} \end{pmatrix} \mid A_{11} \in \mathfrak{u}(2), B_{11} \in M(2, \mathbf{C}), B_{11}^t = B_{11}, \right. \\
&\left. A_{22} \in \mathfrak{u}(m), B_{22} \in M(m, \mathbf{C}), B_{22}^t = B_{22} \right\}
\end{aligned}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & A_{12} & 0 & B_{12} \\ -\bar{A_{12}}^t & 0 & B_{12}^t & 0 \\ 0 & -\bar{B}_{12} & 0 & \bar{A}_{12} \\ -\bar{B}_{11}^t & 0 & -A_{12}^t & 0 \end{pmatrix} \mid A_{12} \in M(2, m; \mathbf{C}), B_{12} \in M(2, m; \mathbf{C}) \right\}$$

$$\begin{aligned}
\mathfrak{a} = \Bigg\{ H(\xi_1, \xi_2) = &\begin{pmatrix} 0 & H_{12} & 0 & 0 \\ -\bar{H_{12}}^t & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{H}_{12} \\ 0 & 0 & -H_{12}^t & 0 \end{pmatrix} \mid \\
&H_{12} = \begin{pmatrix} \xi_1 & 0 & 0 & \cdots & 0 \\ 0 & \xi_2 & 0 & \cdots & 0 \end{pmatrix}, \xi_1, \xi_2 \in \mathbf{R} \Bigg\}
\end{aligned}$$

10.1. **Description of the subgroups K_0 and $K_{[\mathfrak{a}]}$.** $U := Sp(m+2)$

$$\begin{aligned}
K :=&Sp(2) \times Sp(m) \\
=&\left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in Sp(2), B \in Sp(m) \right\}
\end{aligned}$$

$$\begin{aligned}
K_2 :=&Sp(2) \times Sp(2) \times Sp(m-2) \\
=&\left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & B_2 \end{pmatrix} \mid A, B_1 \in Sp(2), B_2 \in Sp(m-2) \right\}
\end{aligned}$$

$$
\begin{aligned}
K_1 :=&Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)\\
=&\Big\{\begin{pmatrix} a_1 & 0 & 0 & 0 & 0\\ 0 & \bar{a}_1 & 0 & 0 & 0\\ 0 & 0 & a_2 & 0 & 0\\ 0 & 0 & 0 & \bar{a}_2 & 0\\ 0 & 0 & 0 & 0 & C \end{pmatrix} \mid a_1, a_2 \in U(2), C \in Sp(m-2)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
K_0 =&Sp(1)\times Sp(1)\times Sp(m-2)\\
=&\Big\{\begin{pmatrix} a_1 & 0 & 0 & 0 & 0\\ 0 & a_2 & 0 & 0 & 0\\ 0 & 0 & a_1 & 0 & 0\\ 0 & 0 & 0 & a_2 & 0\\ 0 & 0 & 0 & 0 & C \end{pmatrix} \mid a_1, a_2 \in Sp(1) = SU(2), C \in Sp(m-2)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&K_{\mathfrak{a}} := \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(\mathfrak{a}) = \mathfrak{a}\},\\
&K_{[\mathfrak{a}]} := \{k \in K_{\mathfrak{a}} \mid \mathrm{Ad}_{\mathfrak{p}}(k) : \mathfrak{a} \longrightarrow \mathfrak{a} \text{ preserves the orientation of } \mathfrak{a}\}\\
&= (Q\cdot K_0)\cup(Q^2\cdot K_0)\cup(Q^3\cdot K_0)\cup K_0,
\end{aligned}
$$

where

$$
D = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & I_{m-2} \end{pmatrix} \text{ and } Q := \begin{pmatrix} D & 0\\ 0 & D \end{pmatrix}.
$$

The deck transformation group of the covering map $\mathcal{G} : N^{8m-2} \to \mathcal{G}(N^{8m-2})$ $(m \geq 2)$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

10.2. **The groups K, K_2, K_1, K_0.** When $m = 2$,

$$
\begin{aligned}
K = Sp(2)\times Sp(2)) &\supset K_1 = Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\\
&\supset K_0 = Sp(1)\times Sp(1)
\end{aligned}
$$

When $m \geq 3$,

$$
\begin{aligned}
K = Sp(2)\times Sp(m) &\supset K_2 = Sp(2)\times Sp(2)\times Sp(m-2)\\
&\supset K_1 = Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)\\
&\supset K_0 = Sp(1)\times Sp(1)\times Sp(m-2)
\end{aligned}
$$

10.3. **Two fibrations over K/K_0.**

$$\begin{aligned}
K_2/K_0 &= \frac{Sp(2)\times Sp(2)\times Sp(m-2)}{Sp(1)\times Sp(1)\times Sp(m-2)}\\
&\cong \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)}\\
\longrightarrow K/K_0 &= \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)}\\
\longrightarrow K/K_2 &= \frac{Sp(2)\times Sp(m)}{Sp(2)\times Sp(2)\times Sp(m-2)} = \frac{Sp(m)}{Sp(2)\times Sp(m-2)} \cong \mathrm{Gr}_2(\mathbf{H}^m)\ .
\end{aligned}$$

$$\begin{aligned}
K_1/K_0 &= \frac{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)}{Sp(1)\times Sp(1)\times Sp(m-2))}\\
&\cong \frac{Sp(1)\times Sp(1)}{Sp(1)}\times\frac{Sp(1)\times Sp(1)}{Sp(1)} \cong S^3\times S^3\\
\longrightarrow K/K_0 &= \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2))}
\end{aligned}$$

$$\begin{aligned}
\longrightarrow K/K_1 &= \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)}\\
&\cong \mathbf{H}P^1\times\frac{Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)}
\end{aligned}$$

10.4. $D(Sp(m))$ **and** $D(Sp(2)\times Sp(m))$**.**

Let $G = Sp(m)$ and $K = Sp(2)\times Sp(m-2)$ in this subsection. Their Lie algebras are $\mathfrak{g}$ and $\mathfrak{k}$, respectively. A maximal abelian subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ is given as

$$\mathfrak{t} = \Bigg\{\xi = \begin{pmatrix} \sqrt{-1}\xi_1 & & & & & \\ & \ddots & & & & \\ & & \sqrt{-1}\xi_m & & & \\ & & & -\sqrt{-1}\xi_1 & & \\ & & & & \ddots & \\ & & & & & -\sqrt{-1}\xi_m \end{pmatrix}$$
$$\mid \xi_1,\cdots,\xi_m\in\mathbf{R}\Bigg\}\subset\mathfrak{k}\subset\mathfrak{g}$$

$$
\begin{aligned}
&\Gamma(G) = \Gamma(K) = \{\xi \in \mathfrak{t} \mid \exp(\xi) = e\} \\
&=\left\{\xi = 2\pi \begin{pmatrix} \sqrt{-1}k_1 & & & & & \\ & \ddots & & & & \\ & & \sqrt{-1}k_m & & & \\ & & & -\sqrt{-1}k_1 & & \\ & & & & \ddots & \\ & & & & & -\sqrt{-1}k_m \end{pmatrix} \right. \\
&\qquad \left. \mid k_1, \cdots, k_m \in \mathbf{Z} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&D(G) = D(Sp(m)) \\
=&\{\Lambda \in \mathfrak{t}^* \mid \Lambda(\xi) \in 2\pi\mathbf{Z} \text{ for each } \xi \in \Gamma(K), \langle \Lambda, \alpha \rangle \geq 0 \text{ for each } \alpha \in \Pi(G)\} \\
=&\{\Lambda = p_1 y_1 + \cdots + p_m y_m \mid p_1, \cdots, p_m \in \mathbf{Z}, p_1 \geq p_2 \geq \cdots \geq p_m \geq 0\}
\end{aligned}
$$

$$
\begin{aligned}
&D(K) = D(Sp(2) \times Sp(m-2))) \\
=&\{\Lambda \in \mathfrak{t}^* \mid \Lambda(\xi) \in 2\pi\mathbf{Z} \text{ for each } \xi \in \Gamma(K), \langle \Lambda, \alpha \rangle \geq 0 \text{ for each } \alpha \in \Pi(K)\} \\
=&\{\Lambda = p_1 y_1 + \cdots + p_m y_m \mid p_1, \cdots, p_m \in \mathbf{Z}, p_1 \geq p_2 \geq 0, p_3 \geq \cdots \geq p_m \geq 0\}
\end{aligned}
$$

10.5. **Representation of the Casimir operator.**

Denote $\langle X, Y \rangle := -\frac{1}{2}\mathrm{tr}XY$ for any $X, Y \in sp(m+2) \subset u(2m+4)$.

When $m = 2$,

$$
\begin{aligned}
\mathcal{C}_L &= \frac{1}{2}\mathcal{C}_{K_1/K_0} + \mathcal{C}_{k/K_1} \\
&= \mathcal{C}_{K/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0},
\end{aligned}
$$

When $m \geq 3$,

$$
\begin{aligned}
\mathcal{C}_L &= \mathcal{C}_{K_2/K_1} + 2\mathcal{C}_{K/K_2} + \frac{1}{2}\mathcal{C}_{K_1/K_0} \\
&= 2\mathcal{C}_{K/K_0} - \mathcal{C}_{K_2/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0},
\end{aligned}
$$

where $\mathcal{C}_{K/K_0}$, $\mathcal{C}_{K_2/K_0}$ and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operator of K/K_0, K_2/K_0 and K_1/K_0 relative to $\langle\ ,\ \rangle|_{\mathfrak{k}}$, $\langle\ ,\ \rangle|_{\mathfrak{k}_2}$ and $\langle\ ,\ \rangle|_{\mathfrak{k}_1}$, respectively.

10.6. **Branching laws of** $(Sp(2), Sp(1) \times Sp(1))$ **(**[19]**,** [40]**.)**

Let $\Lambda = p_1 y_1 + p_2 y_2 \in D(Sp(2))$ be the highest weight of an irreducible $Sp(2)$-module V_Λ, where $p_1, p_2 \in \mathbf{Z}$ and $p_1 \geq p_2 \geq 0$. Then the irreducible decomposition of V_Λ as a $Sp(1) \times Sp(1)$-module contains an irreducible $Sp(1) \times Sp(1)$-module $V_{\Lambda'}$ with the highest weight $\Lambda' =$

$q_1y_1 + q_2y_2 \in D(Sp(1) \times Sp(1))$, where $q_1, q_2 \in \mathbf{Z}$ and $q_1 \geq 0, q_2 \geq 0$, if and only if

(i) $p_1 \geq q_2 \geq 0$;

(ii) in the finite power series expansion in X of $\frac{\prod_{i=0}^{1}(X^{r_i+1} - X^{-(r_i+1)})}{X - X^{-1}}$, where $r_i (i = 0, 1)$ are defined as follows

$$\begin{aligned} r_0 :=& p_1 - \max(p_2, q_2), \\ r_1 :=& \min(p_2, q_2), \end{aligned}$$

the coefficient of X^{q_1+1} does not vanish. Moreover, the value of this coefficient is the multiplicity of the $Sp(1) \times Sp(1)$-module.

10.7. **Descriptions of $D(K, K_0)$, $D(K_1, K_0)$ when $m = 2$.**

Each $\Lambda \in D(K, K_0)$ is

$$\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4,$$

where $p_1, \cdots, p_4 \in \mathbf{Z}$, $p_1 \geq p_2 \geq 0$, $p_3 \geq p_4 \geq 0$.

Each $\Lambda' \in D(K_1, K_0)$ is

$$\Lambda = q_1y_1 + q_2y_2 + q_3y_3 + q_4y_4,$$

where

$$\begin{aligned} &q_i \in \mathbf{Z}, q_i \geq 0 \quad (i = 1, \cdots, 4), \\ &q_1 = q_3, \quad q_2 = q_4. \end{aligned}$$

10.8. **Eigenvalue computation when $m = 2$.**

$$\begin{aligned} \Lambda &= p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 \in D(K, K_0), \\ &\quad (p_i \in \mathbf{Z},\ p_1 \geq p_2, p_3 \geq p_4 \geq 0) \\ \Sigma^+(K) &= \{y_1 - y_2, y_1 + y_2, 2y_1, 2y_2, y_3 - y_4, y_3 + y_4, 2y_3, 2y_4\} \\ 2\delta_K &= 4y_1 + 2y_2 + 4y_3 + 2y_4 \end{aligned}$$

$$\begin{aligned} \Lambda' &= q_1y_1 + q_2y_2 + q_3y_3 + q_4y_4 \in D(K_1, K_0) \\ &\quad (q_i \in \mathbf{Z},\ q_i \geq 0,\ 1 \leq i \leq 4), q_1 = q_3,\ q_2 = q_4, \\ \Sigma^+(K_1) &= \{2y_i, 1 \leq i \leq 4\} \\ 2\delta_{K_1} &= 2(y_1 + y_2 + y_3 + y_4) \end{aligned}$$

$$
\begin{aligned}
c_\Lambda =&\langle \Lambda + 2\delta_K, \Lambda\rangle \\
=&p_1^2+p_2^2+p_3^2+p_4^2+4p_1+2p_2+4p_3+2p_4, \\
c_{\Lambda'} =&\langle \Lambda' + 2\delta_{K_1}, \Lambda'\rangle \\
=&q_1^2+q_2^2+q_3^2+q_4^2+2(q_1+q_2+q_3+q_4) \\
=&2(q_1^2+q_2^2+2q_1+2q_2), \quad (\because q_1=q_3, q_2=q_4),
\end{aligned}
$$

with respect to the inner product $\langle X, Y\rangle := -\frac{1}{2}\mathrm{Tr}(XY)$ for any $X, Y \in sp(4) \subset u(8)$. Hence, we have the following eigenvalue formula.

$$
\begin{aligned}
c &= c_{K/K_0} - \frac{1}{2}c_{K_1/K_0} \\
&= (\sum_{i=1}^{4} p_i^2 + 4p_1 + 2p_2 + 4p_3 + 2p_4) - (q_1^2 + q_2^2 + 2q_1 + 2q_2)
\end{aligned}
$$

Denote $\Lambda = p_1y_1+p_2y_2+p_3y_3+p_4y_4 \in D(K, K_0)$ by $\Lambda = (p_1, p_2, p_3, p_4)$. Then

$$
\begin{aligned}
&\{\ \Lambda \in D(K, K_0) \mid -c_L \leq 14\ \} \\
=&\{\ 0, (1,1,0,0), (0,0,1,1), (1,0,1,0), (1,1,1,1), \\
&(1,1,2,0), (2,0,1,1)\ \}.
\end{aligned}
$$

Suppose that $\Lambda = (1,1,0,0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 5$. It follows from the branching laws of $(Sp(2), Sp(1) \times Sp(1))$ that $(q_1, q_2) = (0,0) \oplus (1,1)$ and $(q_3, q_4) = (0,0)$. Then $(q_1, q_2, q_3, q_4) = (0,0,0,0) \in D(K_1, K_0)$. Hence,

$$
c_\Lambda = 8, \quad c_{\Lambda'} = 0, \quad c_L = c_\Lambda - \frac{1}{2}c_{\Lambda'} = 8 < 14.
$$

On the other hand, there is a double covering $\pi : Sp(2) \to SO(5)$, and $\pi(Sp(1) \times Sp(1)) = SO(4)$. Let λ_5 denote the standard representation of $SO(5)$ and 1 the trivial representation of $SO(5)$. Then the complex representation of $K = Sp(2) \times Sp(2)$ with the highest weight $(1,1,0,0)$ is $(\lambda_5 \otimes 1) \otimes \mathbf{C}$ and $V_\Lambda = \mathbf{C}^5$. It is easy to see that

$(V_\Lambda)_{K_0} = \mathbf{C}\mathbf{e}_1$, where $\mathbf{e}_1 = (1,0,0,0,0)^t \in \mathbf{C}^5$. But for

$$a = \begin{pmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & 0 & 1 & & \\ & & & & 1 & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix} \in K_{[\mathfrak{a}]} \subset K, \quad a \notin K_0,$$

$\pi(a) = \mathrm{diag}(-1,1,-1,-1,-1) \notin SO(4)$ and $\pi(a)\mathbf{e}_1 = -\mathbf{e}_1 \neq \mathbf{e}_1$. Therefore $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$ and $\Lambda = (1,1,0,0) \notin D(K, K_{[\mathfrak{a}]})$. Similarly, $\Lambda = (0,0,1,1) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (1,0,1,0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 16$. The corresponding representation with the highest weight Λ is just the complexified isotropy representation $\mathrm{Ad}_{\mathfrak{p}}(K)^{\mathbf{C}}$. Hence $\Lambda \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (1,1,1,1)$. Then $\dim_{\mathbf{C}} V_\Lambda = 25$. It follows from the branching laws of $(Sp(2), Sp(1) \times Sp(1))$ that $(q_1, q_2) = (1,1) \oplus (0,0)$ and $(q_3, q_4) = (1,1) \oplus (0,0)$. Then $(q_1, q_2, q_3, q_4) = (1,1,1,1) \oplus (0,0,0,0) \in D(K_1, K_0)$. When $(q_1, q_2, q_3, q_4) = (1,1,1,1)$, $c_L = 10 < 14$; When $(q_1, q_2, q_3, q_4) = (0,0,0,0)$, $c_L = 16 > 14$.

On the other hand,

$$V_{(1,1,1,1)} = \mathbf{C}^5 \boxtimes \mathbf{C}^5 \cong M(5, \mathbf{C}).$$

There is a double covering

$$\begin{aligned} \pi : K = Sp(2) \times Sp(2) &\longrightarrow SO(5) \times SO(5) \\ \pi|_{K_1} : K_1 = Sp(1) \times Sp(1) \times Sp(1) \times Sp(1) &\longrightarrow SO(4) \times SO(4) \\ K_0 = Sp(1) \times Sp(1) &\longrightarrow SO(4) \end{aligned}$$

The representation of K on V_Λ is realized as the action of $\pi(K) = SO(5) \times SO(5)$ on $M(5, \mathbf{C})$ in the following way: for any $(A, B) \in SO(5) \times SO(5)$, $X \in M(5, \mathbf{C})$ is mapped to $AXB^{-1} \in M(5, \mathbf{C})$. Then as a K_1-module,

$$\begin{aligned} M(5, \mathbf{C}) &= \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \\ &= W_{(1,1,0,0)} \oplus W_{(0,0,1,1)} \oplus W_{(0,0,0,0)} \oplus W_{(1,1,1,1)}. \end{aligned}$$

K_0 acts on $M(5, \mathbf{C})$ by the adjoint action as a diagonal subgroup of K_1. Hence,

$$(M(5, \mathbf{C}))_{K_0} = \left\{ \begin{pmatrix} x & 0 \\ 0 & yI_4 \end{pmatrix} \mid x, y \in \mathbf{C} \right\},$$
$$(M(5, \mathbf{C}))_{K_{[\mathfrak{a}]}} = \mathbf{C} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = W(0,0,0,0).$$

Although $\Lambda = (1,1,1,1) \in D(K, K_{[\mathfrak{a}]})$, by the previous calculation, we know that the element in $(M(5, \mathbf{C}))_{K_{[\mathfrak{a}]}}$ gives eigenvalue $c_L = 16 > 14$.

Suppose that $\Lambda = (1,1,2,0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 50$. It follows from the branching laws of $(Sp(2), Sp(1) \times Sp(1))$ that $(q_1, q_2) = (1,1) \oplus (0,0)$ and $(q_3, q_4) = (0,2) \oplus (1,1) \oplus (2,0)$. Then

$$\begin{aligned} V_\Lambda =& (W_{(1,1)} \boxtimes U_{(0,2)}) \oplus (W_{(1,1)} \boxtimes U_{(1,1)}) \oplus (W_{(1,1)} \boxtimes U_{(2,0)}) \\ & \oplus (W_{(0,0)} \boxtimes U_{(0,2)}) \oplus (W_{(0,0)} \boxtimes U_{(1,1)}) \oplus (W_{(0,0)} \boxtimes U_{(2,0)}). \end{aligned}$$

Only $(q_1, q_2, q_3, q_4) = (1,1,1,1) \in D(K_1, K_0)$ and the corresponding eigenvalue $c_L = 14$.

On the other hand, the representation of K with highest weight $\Lambda = (1,1,2,0)$ is $(\lambda_5 \boxtimes \mathrm{Ad}) \otimes \mathbf{C}$. Denote $\Lambda_1 = (p_1, p_2) = (1,1) \in D(Sp(2))$. Then

$$V_{\Lambda_1} \cong \mathbf{C}^5 = \mathbf{C}\mathbf{e}_1 \oplus \mathrm{span}_{\mathbf{C}}\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\} = W_{(0,0)} \oplus W_{(1,1)}.$$

Using the definition

$$sp(2) = \{X \in M(2, \mathbf{H}) \mid X^* + X = 0\},$$

we can chose the following basis in $sp(2)$.

$$E_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, E_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, E_3 := \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, E_4 := \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix},$$
$$E_5 := \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, E_6 := \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, E_7 := \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix},$$
$$E_8 := \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, E_9 := \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, E_{10} := \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix},$$

where $\{i, j, k\}$ denote the unit pure quaternions.

Denote $\Lambda_2 = (p_3, p_4) = (2,0) \in D(Sp(2))$. Then

$$\begin{aligned} V_{\Lambda_2} &= \mathrm{span}_{\mathbf{C}}\{E_1, E_2, E_3, E_4\} \oplus \mathrm{span}_{\mathbf{C}}\{E_5, E_6, E_7\} \oplus \mathrm{span}_{\mathbf{C}}\{E_8, E_9, E_{10}\} \\ &= W_{(1,1)} \oplus W_{(2,0)} \oplus W_{(0,2)} \end{aligned}$$

By straightforward computation, one can get

$$\begin{aligned} (V_\Lambda)_{K_0} =& \mathrm{span}_{\mathbf{C}}\{\mathbf{e}_2 \otimes E_1 + \mathbf{e}_3 \otimes E_2 + \mathbf{e}_4 \otimes E_3 + \mathbf{e}_5 \otimes E_4\} \\ =& (V_\Lambda)_{K_{[\mathfrak{a}]}} \subset W_{(1,1)} \otimes U_{(1,1)}. \end{aligned}$$

Therefore, $\Lambda = (1,1,2,0) \in D(K, K_{[\mathfrak{a}]})$, which gives eigenvalue 14.

Suppose that $\Lambda = (2,0,1,1)$. Then $\dim_{\mathbf{C}} V_\Lambda = 50$. It is similar to show that $\Lambda \in D(K, K_{[\mathfrak{a}]})$ and it gives eigenvalue 14. Therefore,

$$\begin{aligned}&\dim_{\mathbf{C}} V_{(1,1,2,0)} + \dim_{\mathbf{C}} V_{(2,0,1,1)} = 100\\ &= \dim SO(16) - \dim Sp(2) \times Sp(2) = n_{hk}(\mathcal{G}),\end{aligned}$$

$\mathcal{G}(N^{14}) \subset Q_{14}(\mathbf{C})$ is strictly Hamiltonian stable.

10.9. **Eigenvalue computation when $m \geq 3$.**

$$\Lambda = p_1 y_1 + p_2 y_2 + p_3 y_3 + \cdots + p_{m+2} y_{m+2} \in D(K, K_0),$$
$$(p_i \in \mathbf{Z},\ p_1 \geq p_2, p_3 \geq p_4 \geq \cdots \geq p_{m+2} \geq 0)$$
$$\Sigma^+(K) = \{y_1 - y_2, y_1 + y_2, 2y_1, 2y_2, y_i - y_j, y_i + y_j, 2y_i (3 \leq i < j \leq m+2)\}$$
$$2\delta_K = 4y_1 + 2y_2 + 2my_3 + (2m-2)y_4 + \cdots + 4y_{m+1} + 2y_{m+2}$$

$$\Lambda' = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 + q_5 y_5 + \cdots + q_{m+2} y_{m+2} \in D(K_2, K_0)$$
$$(q_i \in \mathbf{Z},\ q_1 \geq q_2 \geq 0, q_3 \geq q_4 \geq 0, q_5 \geq \cdots \geq q_{m+2} \geq 0,$$
$$q_1 = p_1, q_2 = p_2)$$
$$\begin{aligned}\Sigma^+(K_2) = \{&y_i - y_j, y_i + y_j ((i,j) = (1,2) \text{ or } (3,4) \text{ or } 5 \leq i < j \leq m+2),\\ &2y_i, 1 \leq i \leq m+2\}\end{aligned}$$
$$2\delta_{K_2} = 4y_1 + 2y_2 + 4y_3 + 2y_4 + 2(m-2)y_5 + \cdots 2y_{m+2}$$

$$\Lambda'' = k_1 y_1 + k_2 y_2 + k_3 k_3 + k_4 y_4 + k_5 y_5 + \cdots + k_{m+2} y_{m+2} \in D(K_1, K_0)$$
$$(k_i \in \mathbf{Z},\ k_i \geq 0, (1 \leq i \leq 4), k_5 \geq k_6 \geq \cdots \geq k_{m+2} \geq 0,$$
$$k_i = q_j \text{ for } 5 \leq j \leq m+2)$$
$$\Sigma^+(K_1) = \{y_i - y_j, y_i + y_j \text{ for } 5 \leq i < j \leq m+2), 2y_i \text{ for } 1 \leq i \leq m+2\}$$
$$2\delta_{K_1} = 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2(m-2)y_5 + \cdots + 2y_{m+2}$$

$$\begin{aligned}c_\Lambda =& \langle \Lambda + 2\delta_K, \Lambda \rangle\\ =& p_1^2 + \cdots + p_{m+2}^2 + 4p_1 + 2p_2 + 2mp_3 + (2m-2)p_4 + \cdots + 2p_{m+2},\\ c_{\Lambda'} =& \langle \Lambda' + 2\delta_{K_2}, \Lambda' \rangle\\ =& q_1^2 + \cdots + q_{m+2}^2 + 4q_1 + 2q_2 + 4q_3 + 2q_4 + (2m-4)q_5 + \cdots + 2q_{m+2},\\ c_{\Lambda''} =& \langle \Lambda'' + 2\delta_{K_1}, \Lambda'' \rangle\\ =& k_1^2 + \cdots + k_{m+2}^2 + 2k_1 + 2k_2 + 2k_3 + 2k_4 + (2m-4)k_5 + \cdots + 2k_{m+2},\end{aligned}$$

with respect to the inner product $\langle X, Y \rangle := -\frac{1}{2}\mathrm{Tr}(XY)$ for any $X, Y \in sp(m+2) \subset u(2m+4)$. Hence, we have the following eigenvalue formula.

$$
\begin{aligned}
c &= 2\mathcal{C}_{K/K_0} - \mathcal{C}_{K_2/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0} \\
&= 2(\sum_{i=1}^{m+2} p_i^2 + 4p_1 + 2p_2 + 2mp_3 + (2m-2)p_4 + \cdots + 2p_{m+2}) \\
&\quad - (\sum_{i=1}^{m+2} q_i^2 + 4q_1 + 2q_2 + 4q_3 + 2q_4 + (2m-4)q_5 + \cdots + 2q_{m+2}) \\
&\quad - \frac{1}{2}(\sum_{i=1}^{m+2} k_i^2 + 2k_1 + 2k_2 + 2k_3 + 2k_4 + (2m-4)k_5 + \cdots + 2k_{m+2})
\end{aligned}
$$

where $q_i = k_i$ for $5 \leq i \leq m+2$, $p_1 = q_1$, $p_2 = q_2$ and $k_1 = k_3$, $k_2 = k_4$.

Suppose that $\Lambda = (p_1, p_2, \cdots, p_{m+2}) = (2, 2, 0, \cdots, 0) \in D(K, K_0)$. Then $\Lambda' = (q_1, q_2, \cdots, q_{m+2}) = (2, 2, 0, \cdots, 0) \in D(K_2, K_0)$. It follows from the branching law of $(Sp(2), Sp(1) \times Sp(1))$ that $\Lambda'' = (k_1, k_2, \cdots, k_{m+2}) = (0, 0, 0, \cdots, 0) \in D(K_1, K_0)$. Hence the eigenvalue is $c_L = 20 < 8m - 2$ for $m \geq 3$.

On the other hand, the representation of K with highest weight $\Lambda = (2, 2, 0, \cdots, 0)$ is a 14-dimensional irreducible representation, $\rho_{\mathrm{Sym}_0^2(\mathbf{C}^5)} \boxtimes I$, where $\rho_{\mathrm{Sym}_0^2(\mathbf{C}^5)}$ is the composition of the natural surjective homomorphism $Sp(2) \to SO(5)$ and the traceless symmetric product representation of $SO(5)$ on $\mathrm{Sym}_0^2(\mathbf{C}^5) := \{X \in M(5; \mathbf{C}) | X^t = X, \mathrm{tr} X = 0\}$. Notice that an element $A \in SO(5)$ acts on $\mathrm{Sym}_0^2(\mathbf{C}^5)$ by $X \mapsto AXA^t$ for any $X \in \mathrm{Sym}_0^2(\mathbf{C}^5)$. So

$$
\begin{aligned}
\mathrm{Sym}_0(\mathbf{C}^5) &= \mathbf{C} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4}I_4 \end{pmatrix} \oplus \left\{ \begin{pmatrix} 0 & 0 \\ 0 & X' \end{pmatrix} \mid X' \in \mathrm{Sym}_0(\mathbf{C}^4) \right\} \\
&\quad \oplus \left\{ \begin{pmatrix} 0 & Z \\ Z^t & 0 \end{pmatrix} \mid Z \in M(1, 4; \mathbf{C}) \right\} \\
&= \mathbf{C} \oplus \mathrm{Sym}_0(\mathbf{C}^4) \oplus \mathbf{C}^4
\end{aligned}
$$

and

$$
(\mathrm{Sym}_0(\mathbf{C}^5))_{SO(4)} = \mathbf{C} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4}I_4 \end{pmatrix} \cong \mathbf{C}.
$$

Under the natural surjective homomorphism $Sp(2)(\subset SU(4)) \to SO(5)$, the element $\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \in Sp(2)$ corresponds to $\mathrm{diag}(-1, 1, -1, -1, -1) \in SO(5)$, denoted by Q'. By direct computation, we know that $(\mathrm{Sym}_0(\mathbf{C}^5))_{Q' \cdot SO(4)} \cap$

$(\mathrm{Sym}_0(\mathbf{C}^5))_{SO(4)} = (\mathrm{Sym}_0(\mathbf{C}^5))_{SO(4)}$. Thus,

$$(V_{\Lambda=(2,2,0,\cdots,0)})_{K_0} = \mathbf{C}\cdot\begin{pmatrix}1 & 0\\ 0 & -\frac{1}{4}I_4\end{pmatrix}\boxtimes\mathbf{C}$$

and moreover,

$$(V_{\Lambda=(2,2,0,\cdots,0)})_{K_{[\mathfrak{a}]}} = \mathbf{C}\cdot\begin{pmatrix}1 & 0\\ 0 & -\frac{1}{4}I_4\end{pmatrix}\boxtimes\mathbf{C}.$$

This means that $\Lambda = (2,2,0,\cdots,0) \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1, which corresponds to eigenvalue $20 < 8m-2$. Therefore, $\mathcal{G}(N^{8m-2}) \to Q_{8m-2}$ is not Hamiltonian stable.

Theorem 10.1. $L = \frac{Sp(2)\times Sp(m)}{(Sp(1)\times Sp(1)\times Sp(m-2))\cdot\mathbf{Z}_4} \longrightarrow Q_{8m-2}(\mathbf{C})$ $(m \geq 2)$ *is not Hamiltonian stable if and only if* $m \geq 3$*. If* $m = 2$*, it is strictly Hamiltonian stable.*

11. The case $(U, K) = (E_6, U(1)\cdot Spin(10))$

(U, K) is of BC_2-type.

$$\begin{aligned}
U = E_6 &\supset K = (U(1)\times Spin(10))/\mathbb{Z}_4\\
&\supset K_2 = (U(1)\times(Spin(2)\cdot Spin(8)))/\mathbb{Z}_4 = (S^1\times(Spin(2)\cdot Spin(8)))/\mathbb{Z}_4\\
&\supset K_1 = (S^1\times(Spin(2)\cdot(Spin(2)\cdot Spin(6)))/\mathbb{Z}_4\\
&\supset K_0 = (S^1\times Spin(6))/\mathbb{Z}_2
\end{aligned}$$

$$\begin{aligned}
U = E_6 \supset F_4 \supset Spin(9) \supset Spin(8)\\
F_4\cap K = F_4\cap Spin(10) = Spin(9)
\end{aligned}$$

(see [44])

11.1. **Cayley algebra.** Let $\mathbb{K}$ be the real Cayley algebra and $\{c_0 = 1, c_1, \cdots, c_7\}$ the standard units of $\mathbb{K}$. They satisfy the following relations:

$$\begin{aligned}
&c_ic_{i+1} = -c_{i+1}c_i = c_{i+3}, \quad c_{i+1}c_{i+3} = -c_{i+3}c_{i+1} = c_i,\\
&c_{i+3}c_i = -c_ic_{i+3} = c_{i+1}, \quad c_i^2 = -1 \text{ for } i\in\mathbb{Z}_7.
\end{aligned}$$

Let $x \mapsto \bar{x}$ be the canonical involution of $\mathbb{K}$, $(\ ,\ )$ the canonical inner product of $\mathbb{K}$. We extend them $\mathbb{C}$-linearly to the complexified algebra $\mathbb{K}^{\mathbb{C}}$ of $\mathbb{K}$ and denote them by the same notions $x \mapsto \bar{x}$ and $(\ ,\ )$ respectively.

11.2. **Exceptional Jordan algebra.** The exceptional Jordan algebra $H_3(\mathbb{K})$ is defined as the set

$$H_3(\mathbb{K}) = \{u \in M_3(\mathbb{K}) | u^* = u\},$$

with Jordan product

$$u \circ v = \frac{1}{2}(uv + vu), \qquad \text{for} \quad u, v \in H_3(\mathbb{K})$$

and the Freudenthal product

$$u \times v := \frac{1}{2}(2u \circ v - \mathrm{tr}(u)v - \mathrm{tr}(v)u + (\mathrm{tr}(u)\mathrm{tr}(v) - (u, v))\mathrm{I}_3),$$

where the standard real positive inner product $(\ , \ )$ of $H_3(\mathbb{K})$ is defined by

$$(u, v) := \mathrm{tr}(u \circ v)$$

for each $u, v \in H_3(\mathbb{K})$. In $H_3(\mathbb{K})$, a trilinear form (u, v, w) and the determinant $\det u$ are defined respectively by

$$(u, v, w) = (u, v \times w), \quad \det u = \frac{1}{3}(u, u, u).$$

The real dimension of $H_3(\mathbb{K})$ is 27 and a typical element

$$u = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \qquad \xi_i \in \mathbb{R}, x_i \in \mathbb{K} \tag{11.1}$$

of $H_3(\mathbb{K})$ will be denoted by

$$u = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + x_1 u_1 + x_2 u_2 + x_3 u_3.$$

Let

$$H_3(\mathbb{K})^{\mathbf{C}} := H_3(\mathbb{K}) + \sqrt{-1} H_3(\mathbb{K})$$

be the complexification of $H_3(\mathbb{K})$. Then the standard Hermitian inner product $\langle \ , \ \rangle$ of $H_3(\mathbb{K})^{\mathbf{C}}$ is defined by

$$\langle u, v \rangle := (\tau u, v)$$

for each $u, v \in H_3(\mathbb{K})^{\mathbf{C}}$, where τ denotes the complex conjugation in $H_3(\mathbb{K})^{\mathbf{C}}$ with respect to $H_3(\mathbb{K})$. Then $H_3(\mathbb{K})^{\mathbb{C}}$ is canonically identified with

$$H_3(\mathbb{K}^{\mathbb{C}}) = \{u \in M_3(\mathbb{K}^{\mathbb{C}}) | u^* = u\}.$$

An element $u \in H_3(\mathbb{K}^{\mathbb{C}})$ of the form (11.1), with $\xi_i \in \mathbb{C}$, $x_i \in \mathbb{K}^{\mathbb{C}}$, is still denoted by $u = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + x_1 u_1 + x_2 u_2 + x_3 u_3$. Put

$$SH_3(\mathbb{K}) = \{u \in M_3(\mathbb{K}) | u^* = -u, \mathrm{Tr} u = 0\}.$$

Denote the complexification of $SH_3(\mathbb{K})$ by $SH_3(\mathbb{K})^{\mathbb{C}}$. Then $SH_3(\mathbb{K})^{\mathbb{C}}$ is identified with

$$SH_3(\mathbb{K}^{\mathbb{C}}) = \{u \in M_3(\mathbb{K}^{\mathbb{C}}) | u^* = -u, \mathrm{Tr} u = 0\}.$$

An element $u \in SH_3(\mathbb{K})^{\mathbb{C}}$ of the form

$$u = \begin{pmatrix} z_1 & x_3 & -\bar{x}_2 \\ -\bar{x}_3 & z_2 & x_1 \\ x_2 & -\bar{x}_1 & z_3 \end{pmatrix}, \quad z_i, x_i \in \mathbb{K}^{\mathbb{C}}, \bar{z}_i = -z_i, \Sigma z_i = 0$$

is denoted by $u = z_1 e_1 + z_2 e_2 + z_3 e_3 + x_1 \bar{u}_1 + x_2 \bar{u}_2 + x_3 \bar{u}_3$.

Injective linear maps $R : H_3(\mathbb{K})^{\mathbf{C}} \to gl(H_3(\mathbb{K})^{\mathbf{C}})$ and $D : SH_3(\mathbb{K})^{\mathbf{C}} \to gl(H_3(\mathbb{K})^{\mathbf{C}})$ are defined as

$$\begin{cases} R(u)v = u \circ v = \frac{1}{2}(uv + vu), & \text{for}\, u, v \in H_3(\mathbb{K})^{\mathbf{C}}, \\ D(u)v = \frac{1}{2}[u, v] = \frac{1}{2}(uv - vu), & \text{for}\, u \in SH_3(\mathbb{K})^{\mathbf{C}}, v \in H_3(\mathbb{K})^{\mathbf{C}}. \end{cases} \tag{11.2}$$

Let $\mathfrak{D}_0$ denote the subalgebra of $gl(H_3(\mathbb{K}))$ generated by the set $\{D(\Sigma z_i e_i) | z_i \in \mathbb{K}, \bar{z}_i = -z_i, \Sigma z_i = 0\}$, and let

$$\begin{aligned} \mathfrak{D}_i &= \{D(x\bar{u}_i) | x \in \mathbb{K}\} \quad \text{for}\, i = 1, 2, 3, \\ \mathfrak{R}_0 &= \{R(\sum \xi_i e_i) | \xi_i \in \mathbb{R}, \sum \xi_i = 0\}, \\ \mathfrak{R}_i &= \{R(x u_i) | x \in \mathbb{K}\} \quad \text{for}\, i = 1, 2, 3. \end{aligned}$$

Remark that $\dim \mathfrak{D}_1 = \dim \mathfrak{D}_2 = \dim \mathfrak{D}_3 = 8$, $\dim \mathfrak{R}_0 = 2$ and $\dim \mathfrak{R}_1 = \dim \mathfrak{R}_2 = \dim \mathfrak{R}_3 = 8$.

Using Ise's notions ([18], p.82), put

$$D_{i,r} = D(c_r(-e_j + e_k)), \quad (1 \le i \le 3, 1 \le r \le 7),$$

and

$$D_{i,pq} = [D_{i,p}, D_{i,q}], \quad (1 \le i \le 3, 1 \le p, q \le 7), \tag{11.3}$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. Then from

$$D(\sum_{i=1}^{3} z_i e_i)(v) = \frac{1}{2} \sum_{\{i,j,k\}} (z_j x_i - x_i z_k) u_i,$$

we can obtain

$$\begin{cases} D_{i,r}(xu_i) = \begin{cases} -c_r u_i, & \text{if}\, x = c_0 \\ c_0 u_i, & \text{if}\, x = c_r \\ 0, & \text{if}\, x = c_q (q \neq r), \end{cases} \\ D_{i,r}(xu_j) = \frac{1}{2}(c_r x) u_j, \\ D_{i,r}(xu_k) = \frac{1}{2}(x c_r) u_k, \end{cases}$$

and

$$\begin{cases} D_{i,pq}(xu_i) = \begin{cases} c_q u_i, & \text{if } x = c_p \\ -c_p u_i, & \text{if } x = c_q \\ 0, & \text{if } x = c_r (r \leq 0, \neq p, q), \end{cases} \\ \\ D_{i,pq}(xu_j) = \frac{1}{2}\{c_p(c_q x)\}u_j, \\ \\ D_{i,pq}(xu_k) = \frac{1}{2}\{(xc_q)c_p\}u_k. \end{cases}$$

These mean that every $D_{i,r}, D_{i,pq}$ leave $\mathfrak{T}_i = \{xu_i | x \in \mathbb{K}\}$ invariant $(1 \leq i \leq 3, 1 \leq p, q, r \leq 7)$ and identifying $\mathfrak{T}_i$ with $\mathbb{K}$, it represents a skew-symmetric matrix with respect to the basis $\{c_0, c_1, \cdots, c_7\}$; namely $D_{i,r} = E_{0r} - E_{r0}$ and $D_{i,pq} = E_{qp} - E_{pq}$, where E_{pq} denotes the matrix of degree 8 with all 0-components except (p, q)-component, 1. Moreover,

$$\begin{aligned} [D_{i,r}, D_{i,pq}] &= D_{i,p}\delta_{qr} - D_{i,q}\delta_{rp}, & (11.4) \\ [D_{i,pq}, D_{i,rs}] &= D_{i,pr}\delta_{sq} + D_{i,qs}\delta_{pr} + D_{i,rq}\delta_{sp} + D_{i,sp}\delta_{rq}, & (11.5) \end{aligned}$$

where $1 \leq i \leq 3$ and $1 \leq p, q, r, s \leq 7$. Particulary, we have

$$[D_{i,r}, D_{i,pq}] = 0, \quad [D_{i,pq}, D_{i,rs}] = 0,$$

if p, q, r, s are all different each other. Denote the real linear space spanned by all $D_{i,r}, D_{i,pq}$ $(1 \leq p, q, r \leq 7)$ by $\mathfrak{D}_{i,0}$. Then all $\mathfrak{D}_{i,0} (1 \leq i \leq 3)$ are isomorphic to each other, and they are isomorphic to the Lie algebra $\mathfrak{o}(8)$. Here we use $\mathfrak{D}_0 = \mathfrak{D}_{1,0}$, $\dim \mathfrak{D}_0 = 28$. Notice that $\mathfrak{D}_0 = \{D \in \mathfrak{D} \mid D(e_i) = 0 \ (i = 1, 2, 3)\}$.

11.3. **From the group level.** The automorphism group of the Jordan algebra $H_3(\mathbb{K})$

$$F_4 := \{a \in GL_{\mathbf{C}}(H_3(\mathbb{K})) \mid a(u \circ v) = u \circ v\}$$

is known to be a connected, simply connected, compact Lie group of type $\mathfrak{f}_4$. Its Lie algebra $\mathfrak{f}_4$ is given by

$$\mathfrak{f}_4 := \{\delta \in gl_{\mathbf{C}}(H_3(\mathbb{K})) \mid \delta(u \circ v) = \delta u \circ v + u \circ \delta v\}.$$

The Lie subalgebra $\mathfrak{d}_4$ of $\mathfrak{f}_4$

$$\mathfrak{d}_4 = \{\delta \in \mathfrak{f}_4 \mid \delta e_i = 0, i = 1, 2, 3\}$$

is isomorphic to the Lie algebra $\mathfrak{o}(8)$ ([44], p.39). Any $\delta \in \mathfrak{f}_4$ can be uniquely expressed as

$$\delta = D_0 + D(u),$$

where $D_0 \in \mathfrak{d}_4$ and $u \in SH(\mathbb{K})$ with $\operatorname{diag}(u) = 0$. In another words, $\mathfrak{D} = \mathfrak{D}_0 + \mathfrak{D}_1 + \mathfrak{D}_2 + \mathfrak{D}_3$ is a subalgebra of $gl(H_3(\mathbb{K}))$ and a compact simple Lie algebra of type $\mathfrak{f}_4$.

$$\{a \in F_4 \mid ae_1 = e_1\} \cong Spin(9),$$
$$\{a \in F_4 \mid ae_i = e_i \ (i = 1, 2, 3)\} \cong Spin(8).$$

The groups $E_6^{\mathbf{C}}$ and E_6 are defined by

$$E_6^{\mathbf{C}} := \{\alpha \in GL_{\mathbf{C}}(H_3(\mathbb{K})^{\mathbf{C}}) \mid \det(\alpha u) = \det(u) \text{ for each } u \in H_3(\mathbb{K})^{\mathbf{C}}\},$$
$$E_6 := \{\alpha \in GL_{\mathbf{C}}(H_3(\mathbb{K})^{\mathbf{C}}) \mid \det(\alpha u) = \det u, \langle \alpha u, \alpha v\rangle = \langle u, v\rangle\},$$

respectively. The Lie algebras of $E_6^{\mathbf{C}}$ and E_6 are given by

$$\mathfrak{e}_6^{\mathbf{C}} := \{\phi \in \mathfrak{gl}_{\mathbf{C}}(H_3(\mathbb{K})^{\mathbf{C}}) \mid (\phi u, u \times u) = 0 \text{ for each } u \in H_3(\mathbb{K})^{\mathbf{C}}\},$$
$$\mathfrak{e}_6 := \{\phi \in \mathfrak{gl}_{\mathbf{C}}(H_3(\mathbb{K})^{\mathbf{C}}) \mid (\phi u, u \times u) = 0, \langle \phi u, v\rangle + \langle u, \phi v\rangle = 0\}.$$

E_6 is known to be a connected, simply connected, compact Lie group of type $\mathfrak{e}_6$.

Any element $\phi \in \mathfrak{e}_6^{\mathbf{C}}$ is uniquely expressed as

$$\phi = \delta + R(u), \quad \delta \in \mathfrak{f}_4^{\mathbf{C}}, u \in H_3(\mathbb{K})^{\mathbf{C}},$$

i.e., $\mathfrak{g}^{\mathbb{C}} = \mathfrak{D}^{\mathbb{C}} + \mathfrak{R}^{\mathbb{C}}$ is a subalgebra of $gl(H_3(\mathbb{K})^{\mathbb{C}})$ and a complex simple Lie algebra of type $\mathfrak{e}_6$, where $\mathfrak{D}^{\mathbb{C}}$ and $\mathfrak{R}^{\mathbb{C}}$ denote the complexifications of $\mathfrak{D}$ and $\mathfrak{R}$ respectively.

Any element $\phi \in \mathfrak{e}_6$ is uniquely expressed as

$$\phi = \delta + \sqrt{-1}R(u), \quad \delta \in \mathfrak{f}_4, u \in H_3(\mathbb{K}).$$

Equivalently, $\mathfrak{u} := \mathfrak{D} + \sqrt{-1}\mathfrak{R}$ is a compact simple Lie algebra of type $\mathfrak{e}_6$.

Consider a $\mathbf{C}$-linear transformation σ of $H_3(\mathbb{K})^{\mathbf{C}}$ defined by

$$\sigma \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}.$$

Then $\sigma \in E_6$ and $\sigma^2 = 1$. It follows from the fact that the subgroup $(E_6)_{e_1}$ of E_6 defined by

$$(E_6)_{e_1} = \{\alpha \in E_6 \mid \alpha e_1 = e_1\} \cong Spin(10)$$

that the subgroup $(E_6)^\sigma$ of E_6 defined by

$$(E_6)^\sigma = \{\alpha \in E_6 \mid \sigma\alpha = \alpha\sigma\} \cong U(1)\cdot Spin(10) = (U(1)\times Spin(10))/\mathbf{Z}_4.$$

Explicitly,

$$(E_6)_{e_1} := \{\alpha \in E_6 \mid \alpha e_1 = e_1\} \cong Spin(10),$$
$$\phi(\theta) := \exp(t\sqrt{-1}R(2e_1 - e_2 - e_3)) \in GL_{\mathbf{C}}(H_3(\mathbb{K})), \theta = e^{\frac{it}{2}},$$
$$\{\phi(\theta) | \theta \in \mathbf{C}, |\theta| = 1\} \cong U(1).$$

Define a mapping $p : \tilde{K} = U(1) \times Spin(10) \to K = (E)^\sigma$ by

$$(\theta, \alpha) \mapsto \phi(\theta)\alpha.$$

$$U(1) \cap Spin(10) = \{1 = \phi(1), \phi(-1), \phi(\sqrt{-1}), \phi(-\sqrt{-1})\}.$$

$$\begin{aligned} \ker p &= \{(1, \phi(1)), (-1, \phi(-1)), (\sqrt{-1}, \phi(-\sqrt{-1})), (-\sqrt{-1}, \phi(\sqrt{-1}))\} \\ &\cong \mathbf{Z}_4. \end{aligned}$$

Hence $K = \tilde{K}/\mathbf{Z}_4 = U(1) \times Spin(10)/\mathbf{Z}_4$.

$$\tilde{K}_2 = U(1) \times Spin(2) \times Spin(8) \subset \tilde{K} = U(1) \times Spin(10),$$

where $Spin(2) \subset Spin(10) \cong (E_6)_{e_1}$ is generated by

$$\alpha_{23}(t) := \exp(t\sqrt{-1}R(e_2 - e_3)) :$$

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 & e^{\frac{t\sqrt{-1}}{2}} x_3 & e^{-\frac{t\sqrt{-1}}{2}} \bar{x}_2 \\ e^{\frac{t\sqrt{-1}}{2}} \bar{x}_3 & e^{t\sqrt{-1}} \xi_2 & x_1 \\ e^{-\frac{t\sqrt{-1}}{2}} x_2 & \bar{x}_1 & e^{-t\sqrt{-1}} \xi_3 \end{pmatrix},$$

and

$$Spin(8) = (E_6)_{e_1, e_2, e_3} = \{\alpha \in E_6 \mid \alpha e_i = e_i (i = 1, 2, 3)\}.$$

Remark that the Lie algebra of $(E_6)_{e_1, e_2, e_3}$ is just $\mathfrak{D}_0$.

Therefore,

$$Spin(2) \cap Spin(8) = \{\alpha_{23}(t) \mid e^{t\sqrt{-1}} = 1\} = \{\alpha_{23}(0), \alpha_{23}(2\pi)\}.$$

Then the natural projection

$$\begin{aligned} p_2 : \quad & Spin(2) \times Spin(8) \to K_2' \\ & (\alpha_{23}(t), \beta) \mapsto \alpha_{23}(t)\beta \end{aligned}$$

has a kernel

$$\begin{aligned} \ker p_2 &= \{(\alpha_{23}(t), \alpha_{23}(t)^{-1}) \mid t = 2k\pi, k \in \mathbf{Z}\} \\ &= \{(\alpha_{23}(0), \alpha_{23}(0)), (\alpha_{23}(2\pi), \alpha_{23}(2\pi))\} \cong \mathbf{Z}_2. \end{aligned}$$

Hence $K_2' \cong (Spin(2) \times Spin(8))/\mathbf{Z}_2$.

On the other hand, we also have

$$\tilde{K}_2 = S^1 \times Spin(2) \times Spin(8),$$

where S^1 is generated by

$$\exp(t\sqrt{-1}R(e_1 - 2e_2 + e_3)) :$$

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} e^{t\sqrt{-1}} \xi_1 & e^{-\frac{t\sqrt{-1}}{2}} x_3 & e^{t\sqrt{-1}} \bar{x}_2 \\ e^{-\frac{t\sqrt{-1}}{2}} \bar{x}_3 & e^{-2t\sqrt{-1}} \xi_2 & e^{-\frac{t\sqrt{-1}}{2}} x_1 \\ e^{t\sqrt{-1}} x_2 & e^{-\frac{t\sqrt{-1}}{2}} \bar{x}_1 & e^{t\sqrt{-1}} \xi_3 \end{pmatrix},$$

$Spin(2) \subset E_6$ is generated by

$$\alpha_{13}(t) := \exp(t\sqrt{-1}R(e_1 - e_3)) :$$

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} e^{t\sqrt{-1}}\xi_1 & e^{\frac{t\sqrt{-1}}{2}}x_3 & \bar{x}_2 \\ e^{\frac{t\sqrt{-1}}{2}}\bar{x}_3 & \xi_2 & e^{-\frac{t\sqrt{-1}}{2}}x_1 \\ x_2 & e^{-\frac{t\sqrt{-1}}{2}}\bar{x}_1 & e^{-t\sqrt{-1}}\xi_3 \end{pmatrix}.$$

and $Spin(8) = (E_6)_{e_1,e_2,e_3}$. Here $Spin(2) \times Spin(8) \subset (E_6)_{e_2} \cong Spin(10)$. Similarly, here

$$Spin(2) \cap Spin(8) = \{\alpha_{13}(t) \mid e^{t\sqrt{-1}} = 1\} = \{\alpha_{13}(0), \alpha_{13}(2\pi)\}.$$

Then the natural projection

$$\begin{aligned} p_2' : \quad & Spin(2) \times Spin(8) \to K_2' \\ & (\alpha_{13}(t), \beta) \mapsto \alpha_{13}(t)\beta \end{aligned}$$

has a kernel

$$\begin{aligned} \ker p_2' &= \{(\alpha_{13}(t), \alpha_{13}(t)^{-1}) \mid t = 2k\pi, k \in \mathbf{Z}\} \\ &= \{(\alpha_{13}(0), \alpha_{13}(0)), (\alpha_{13}(2\pi), \alpha_{13}(2\pi))\} \cong \mathbf{Z}_2. \end{aligned}$$

Thus,

$$\begin{aligned} K_2 = & \quad (S^1 \times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4, \\ Spin(2) \cdot Spin(8) = & \quad (Spin(2) \times Spin(8))/\mathbf{Z}_2. \end{aligned}$$

Furthermore,

$$Spin(8) \supset Spin(2) \cdot Spin(6) \cong (Spin(2) \times Spin(6))/\mathbf{Z}_2,$$

where

$$\begin{aligned} Spin(2) &:= \{(\alpha_1, \alpha_2, \alpha_3) \in Spin(8) \mid \alpha_2(c_i) = c_i, \text{ if } i \neq 0, 4\}, \\ Spin(6) &:= \{(\alpha_1, \alpha_2, \alpha_3) \in Spin(8) \mid \alpha_2(1) = 1, \alpha_2(c_4) = c_4\}, \end{aligned}$$

and

$$\begin{aligned} Spin(2) \cap Spin(6) &= \{(\mathrm{Id}, \mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id})\}, \\ \mathbf{Z}_2 = \{((\mathrm{Id}, \mathrm{Id}, \mathrm{Id}), &(\mathrm{Id}, \mathrm{Id}, \mathrm{Id})), ((-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id}), (-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id}))\}. \end{aligned}$$

Thus,

$$K_1 = (S^1 \times (Spin(2) \cdot (Spin(2) \cdot Spin(6))))/\mathbf{Z}_4.$$

Moreover,

$$S^1 \cap Spin(6) = \{(\mathrm{Id}, \mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id})\},$$

$$\mathbf{Z}_2 = \{((\mathrm{Id}, \mathrm{Id}, \mathrm{Id}), (\mathrm{Id}, \mathrm{Id}, \mathrm{Id})), ((-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id}), (-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id}))\}.$$

Hence,

$$K_0 = (S^1 \times Spin(6))/\mathbf{Z}_2.$$

11.4. **From the Lie algebra level.** Consider a subalgebra $(\mathfrak{e}_6)_{e_1}$ of $\mathfrak{e}_6$:

$$(\mathfrak{e}_6)_{e_1} := \{\phi \in \mathfrak{e}_6 \mid \phi e_1 = 0\}.$$

Since for any $\phi \in \mathfrak{e}_6$, there exist $u \in SH_3(\mathbb{K})$ and $v \in H_3(\mathbb{K})$ such that

$$\phi e_1 = D(u)(e_1) + \sqrt{-1}R(v)(e_1),$$

$\phi e_1 = 0$ if and only if

$u = z_1e_1 + z_2e_2 + z_3e_3 + a_1\bar{u}_1 \in SH_3(\mathbb{K}),\ v = \xi_2e_2 + \xi_3e_3 + x_1u_1 \in H_3(\mathbb{K}),$

where $a_1, x_1 \in \mathbb{K}$, $\xi_2, \xi_3 \in \mathbf{R}$ with $\xi_2 + \xi_3 = 0$. Hence

$$\begin{aligned}(\mathfrak{e}_6)_{e_1} &= \{\phi \in \mathfrak{e}_6 \mid \phi e_1 = 0\}\\ &= \mathfrak{D}_0 + \mathfrak{D}_1 + \mathbf{R}\sqrt{-1}R(e_2 - e_3) + \sqrt{-1}\mathfrak{R}_1\\ &\cong \mathfrak{d}_5.\end{aligned}$$

If $Z = (e_1 - e_2) + (e_1 - e_3) = 2e_1 - e_2 - e_3 \in SH_3(\mathbb{K})$, then $\sqrt{-1}R(Z)(e_1) = 2\sqrt{-1}e_1$. Therefore,

$$\begin{aligned}\mathfrak{e}_6 &= (\mathfrak{e}_6)_{e_1} + \mathbf{R}\sqrt{-1}R(Z) + \mathfrak{D}_2 + \mathfrak{D}_3 + \sqrt{-1}\mathfrak{R}_2 + \sqrt{-1}\mathfrak{R}_3,\\ \mathfrak{k} &= \mathfrak{D}_0 + \mathfrak{D}_1 + \sqrt{-1}\mathfrak{R}_0 + \sqrt{-1}\mathfrak{R}_1 = (\mathfrak{e}_6)_{e_1} + \mathbf{R}\sqrt{-1}R(Z),\\ \mathfrak{p} &= \mathfrak{D}_2 + \mathfrak{D}_3 + \sqrt{-1}\mathfrak{R}_2 + \sqrt{-1}\mathfrak{R}_3,\end{aligned}$$

where

$$\begin{aligned}\mathfrak{k} &= (\mathfrak{e}_6)_{e_1} + \mathbf{R}\sqrt{-1}R(2e_1 - e_2 - e_3)\\ &= (\mathfrak{d}_4 + \mathfrak{D}_1 + \mathbf{R}\sqrt{-1}R(e_2 - e_3) + \sqrt{-1}\mathfrak{R}_1) + \mathbf{R}\sqrt{-1}R(2e_1 - e_2 - e_3)\\ &= (\mathfrak{d}_3 + \mathbf{R}(D_{1,4} + D_{1,12} - D_{1,36} + D_{1,57})) + \mathfrak{D}_1 + \mathbf{R}\sqrt{-1}R(e_2 - e_3)\\ &\quad + \sqrt{-1}\mathfrak{R}_1 + \mathbf{R}\sqrt{-1}R(2e_1 - e_2 - e_3)\\ &= \mathfrak{d}_3 + \mathbf{R}(D_{1,4} + D_{1,12} - D_{1,36} + D_{1,57}) + \mathbf{R}\sqrt{-1}R(e_3 - e_1)\\ &\quad + \mathbf{R}\sqrt{-1}R(e_1 - 2e_2 + e_3) + \mathfrak{D}_1 + \sqrt{-1}\mathfrak{R}_1\\ &= (\mathfrak{d}_3 + \mathbf{R}\sqrt{-1}R(e_1 - 2e_2 + e_3)) + \mathbf{R}(D_{1,4} + D_{1,12} - D_{1,36} + D_{1,57})\\ &\quad + \mathbf{R}\sqrt{-1}R(e_3 - e_1) + \mathfrak{D}_1 + \sqrt{-1}\mathfrak{R}_1.\end{aligned}$$

Here

$$\begin{aligned}\mathfrak{k}_2 &= \mathfrak{d}_4 + \mathbf{R}\sqrt{-1}R(e_2 - e_3) + \mathbf{R}\sqrt{-1}R(2e_1 - e_2 - e_3),\\ \mathfrak{k}_1 &= \mathfrak{d}_3 + \mathbf{R}(D_{1,4} + D_{1,12} - D_{1,36} + D_{1,57}) + \mathbf{R}\sqrt{-1}R(e_2 - e_3) + \mathbf{R}\sqrt{-1}R(2e_1 - e_2 - e_3),\\ \mathfrak{k}_0 &= \mathfrak{d}_3 + \mathbf{R}\sqrt{-1}R(e_1 - 2e_2 + e_3).\end{aligned}$$

11.5. **Realization of $\mathfrak{e}_6$ and EIII.** Put

$$\begin{aligned}\mathfrak{D} &= \mathfrak{D}_0 + \mathfrak{D}_1 + \mathfrak{D}_2 + \mathfrak{D}_3,\\ \mathfrak{R} &= \mathfrak{R}_0 + \mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3.\end{aligned}$$

Then $\mathfrak{D}$ is a subalgebra of $gl(H_3(\mathbb{K}))$ and a compact simple Lie algebra of type $\mathfrak{f}_4$. Denoting by $\mathfrak{D}^{\mathbb{C}}$ and $\mathfrak{R}^{\mathbb{C}}$ the complexifications of $\mathfrak{D}$ and $\mathfrak{R}$ respectively, we put

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{D}^{\mathbb{C}} + \mathfrak{R}^{\mathbb{C}}.$$

Then $\mathfrak{g}^{\mathbb{C}}$ is a subalgebra of $gl(H_3(\mathbb{K})^{\mathbb{C}})$ and a complex simple Lie algebra of type $\mathfrak{e}_6$. The inclusion $\phi : \mathfrak{g}^{\mathbb{C}} \subset gl(H_3(\mathbb{K})^{\mathbb{C}})$ is a 27-dimensional irreducible representation of $\mathfrak{g}^{\mathbb{C}}$.

Lemma 11.1. *For $v = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + x_1 u_1 + x_2 u_2 + x_3 u_3 \in H_3(\mathbb{K})^{\mathbb{C}}$, we have*

$$\begin{aligned} R(\sum \eta_l e_l)v &= \eta_1\xi_1 e_1 + \eta_2\xi_2 e_2 + \eta_3\xi_3 e_3 + \frac{1}{2}(\eta_2+\eta_3)x_1u_1\\ &\qquad + \frac{1}{2}(\eta_3+\eta_1)x_2u_2 + \frac{1}{2}(\eta_1+\eta_2)x_3u_3,\\ D(\sum z_l e_l)v &= \frac{1}{2}(z_2x_1 - x_1z_3)u_1 + \frac{1}{2}(z_3x_2 - x_2z_1)u_2 + \frac{1}{2}(z_1x_3 - x_3z_2)u_3,\\ D(x\bar{u}_i)v &= (x,x_i)(e_j - e_k) + \frac{1}{2}(\xi_k - \xi_j)xu_i - \frac{1}{2}(\bar{x}\bar{x}_k)u_j + \frac{1}{2}(\bar{x}_j\bar{x})u_k,\\ R(xu_i)v &= (x,x_i)(e_j + e_k) + \frac{1}{2}(\xi_j + \xi_k)xu_i + \frac{1}{2}\bar{x}\bar{x}_k u_j + \frac{1}{2}\bar{x}_j\bar{x}u_k,\end{aligned}$$

where $\{i,j,k\}$ is a cyclic permutation of $\{1,2,3\}$.

Define a real form $\mathfrak{u}$ of $\mathfrak{g}^{\mathbb{C}}$ by

$$\mathfrak{u} = \mathfrak{k} + \mathfrak{p},$$

where

$$\begin{aligned}\mathfrak{k} &= \mathfrak{D}_0 + \mathfrak{D}_1 + \sqrt{-1}\mathfrak{R}_0 + \sqrt{-1}\mathfrak{R}_1,\\ \mathfrak{p} &= \mathfrak{D}_2 + \mathfrak{D}_3 + \sqrt{-1}\mathfrak{R}_2 + \sqrt{-1}\mathfrak{R}_3.\end{aligned}$$

Then $\mathfrak{u}$ is a compact simple Lie algebra of type EIII and the above decomposition is a Cartan decomposition of $\mathfrak{u}$, where $\mathfrak{k}$ is isomorphic to $\mathfrak{u}(1) + \mathfrak{o}(10)$,

$$[\mathfrak{k},\mathfrak{k}] = \mathfrak{D}_0 + \mathfrak{D}_1 + \sqrt{-1}\mathbb{R}R(e_2 - e_3) + \sqrt{-1}\mathfrak{R}_1$$

is isomorphic to $\mathfrak{o}(10)$ and the center of $\mathfrak{k}$ is spanned by

$$Z = \sqrt{-1}R(2e_1 - e_2 - e_3).$$

11.6. **Commutation rules for $\mathfrak{g}^{\mathbb{C}}$ of type $\mathfrak{e}_6$.** The relations (11.3), (11.4), (11.5) and the following list give commutation rules for $\mathfrak{g}^{\mathbb{C}}$. Here, $x, y, z_i \in \mathbb{K}^{\mathbb{C}}$, $\bar{z}_i = -z_i$ for $i = 1, 2, 3$, $\sum_i z_i = 0$, and $\xi_1, \xi_2, \xi_3 \in \mathbb{C}$ with $\sum_l \xi_l = 0$. In formulae (11.6) (11.14), (i, j, k) is a cyclic permutation of $(1, 2, 3)$. In formulae (11.16) and (11.17), $i = 1, 2, 3$.

$$[R(xu_i), R(yu_j)] = -(1/2)D(\overline{xy}\,\bar{u}_k), \tag{11.6}$$
$$[R(xu_i), D(yu_j)] = [D(x\bar{u}_i), R(yu_j)] = (1/2)R(\overline{xy}\,\bar{u}_k), \tag{11.7}$$
$$[D(x\bar{u}_i), D(y\bar{u}_j)] = -(1/2)D(\overline{xy}\bar{u}_k), \tag{11.8}$$
$$[D(x\bar{u}_i), R(y\bar{u}_i)] = (x, y)R(e_j - e_k), \tag{11.9}$$
$$\Big[R(\sum \xi_l e_l), R(x\bar{u}_i)\Big] = (1/2)(\xi_j - \xi_k)D(x\bar{u}_i), \tag{11.10}$$
$$\Big[R(\sum \xi_l e_l), D(x\bar{u}_i)\Big] = (1/2)(\xi_j - \xi_k)R(x\bar{u}_i), \tag{11.11}$$
$$\Big[D(\sum z_l e_l), D(x\bar{u}_i)\Big] = (1/2)D((z_j x - x z_k)\bar{u}_i), \tag{11.12}$$
$$\Big[D(\sum z_l e_l), R(x\bar{u}_i)\Big] = (1/2)R((z_j x - x z_k)u_i), \tag{11.13}$$
$$\begin{aligned}[R(xu_i), R(yu_i)] &= -[D(x\bar{u}_i), D(y\bar{u}_i)] \\ &= D((\frac{y+\bar{y}}{2}\frac{x-\bar{x}}{2} - \frac{x+\bar{x}}{2}\frac{y-\bar{y}}{2})(e_j - e_k)) \\ &\quad -[D(\frac{x-\bar{x}}{2}(e_j - e_k)), D(\frac{y-\bar{y}}{2}(e_j - e_k))],\end{aligned} \tag{11.14}$$
$$\big[\mathfrak{R}_0^{\mathbb{C}}, \mathfrak{R}_0^{\mathbb{C}} + \mathfrak{D}_0^{\mathbb{C}}\big] = \{0\}, \tag{11.15}$$
$$[R(xu_i), [R(xu_i), R(yu_i)]] = R(((x, x)y - (x, y)x)u_i), \tag{11.16}$$
$$[D(x\bar{u}_i), [D(x\bar{u}_i), D(y\bar{u}_i)]] = D(((x, y)x - (x, x)y)\bar{u}_i). \tag{11.17}$$

The following lemma gives the Killing-Cartan form B of $\mathfrak{g}^{\mathbb{C}}$.

Lemma 11.2 ([18], p.88 or [44], p.74)**.** *The Killing-Cartan form of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of type E_6 is given by*

$$B(X, Y) = 4\mathrm{Tr}XY, \qquad X, Y \in \mathfrak{g}^{\mathbb{C}} \subset gl(H_3(\mathbb{K})^{\mathbb{C}}). \tag{11.18}$$

11.7. **Isotropy representation of** $(E_6, U(1) \cdot Spin(10))$**.**

$$\begin{aligned}U/K &= \mathrm{EIII} \\ &= \{X \in H_3(\mathbb{K})^{\mathbf{C}} \mid X \times X = 0, X \neq 0\}/\mathbf{C}^* \\ &\cong E_6/(U(1) \cdot Spin(10))\end{aligned}$$

(cf. [1, p74-75])

The tangent vector space $T_o(U/K)$ at $o = [e_1]$ is

$$\begin{aligned}
T_o(U/K) = T_o(\mathrm{EIII}) \\
&= (H_3(\mathbb{K})^{\mathbf{C}})_{-\sigma} \\
&= \{X \in H_3(\mathbb{K})^{\mathbf{C}} \mid X \times e_1 = 0, \langle X, e_1\rangle = 0\} \\
&= \{x_2u_2 + x_3u_3 \mid x_2, x_3 \in \mathbb{K}^{\mathbf{C}}\}.
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{u} = \mathfrak{e}_6, \\
&\mathfrak{k} = (\mathfrak{e}_6)_\sigma = \{\phi \in \mathfrak{e}_6 \mid \sigma_*\phi = \phi\}, \\
&\mathfrak{p} = (\mathfrak{e}_6)_{-\sigma} = \{\phi \in \mathfrak{e}_6 \mid \sigma_*\phi = -\phi\} \\
&\quad = \mathfrak{D}_2 + \mathfrak{D}_3 + \sqrt{-1}\mathfrak{R}_2 + \sqrt{-1}\mathfrak{R}_3.
\end{aligned}$$

The differential of the natural projection $p : U = E_6 \to U/K = \mathrm{EIII}$ at e induces a linear isomorphism $p_* : \mathfrak{p} \to T_o(\mathrm{EIII})$. Then $p_*(\phi) = \phi(e_1)$ and

$$\begin{aligned}
&p_*(2(D(x_2\bar{u}_2) + D(x_3\bar{u}_3)) + 2\sqrt{-1}(R(x_2'u_2) + R(x_3'u_3))) \\
=&(x_2 + \sqrt{-1}x_2')u_2 + (x_3 + \sqrt{-1}x_3')u_3.
\end{aligned}$$

Lemma 11.3. (1) *For each $a \in K$ and each $\xi \in \mathfrak{p}$,*

$$p_*(\mathrm{Ad}(a)\xi) = (\mathrm{Ad}(a)\xi)(e_1) = (a \circ \xi \circ a^{-1})(e_1).$$

(2) *For each $T \in \mathfrak{k}$ and each $\xi \in \mathfrak{p}$,*

$$\begin{aligned}
p_*(\mathrm{ad}(T)\xi) =&p_*([T, \xi]) \\
=&([T, \xi])(e_1) \\
=&(T \circ \xi)(e_1) - (\xi \circ T)(e_1) \\
=&T(p_*(\xi)) - \xi(T(e_1)).
\end{aligned}$$

The restriction $(\rho_K, V = H_3(\mathbb{K}^{\mathbf{C}}))$ of Cheally-Schafer's representation $(\tilde{\rho}, H_3(\mathbb{K}^{\mathbf{C}}))$ can be decomposed into 3 irreducible representations

$$(\rho_K, V) = (\rho_1, V_1) \oplus (\rho_3, V_3) \oplus (\rho_2, V_2),$$

where V_1, V_3 and V_2 are given as follows:

$$\begin{aligned}
V_1 &= \{\xi e_1 \mid \xi \in \mathbf{C}\}, \\
V_3 &= H_2(\mathbb{K}^{\mathbf{C}}) \\
&= \{\xi_2 e_2 + \xi_3 e_3 + x_1 u_1 \mid x_1 \in \mathbb{K}^{\mathbf{C}}, \xi_2, \xi_3 \in \mathbf{C}\}, \\
V_2 &= (H_3(\mathbb{K}^{\mathbf{C}}))_{-\sigma} \\
&= \{x_2u_2 + x_3u_3 \mid x_2, x_3 \in \mathbb{K}^{\mathbf{C}}\},
\end{aligned}$$

and $V_1 \oplus V_3 = (H_3(\mathbb{K}^{\mathbf{C}}))_\sigma$. ρ_1 is a scalar representation, the restriction of ρ_2 to $Spin(10)$ is equivalent to one of the half-spin representations of $Spin(10, \mathbf{C})$, called Δ_{10}^+, and the restriction of ρ_3 to $Spin(10)$ is equivalent to the standard representation of $Spin(10, \mathbf{C})$.

The maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is given as follows :

$$\begin{aligned}\mathfrak{a} &= \mathbf{R}H_1^u \oplus \mathbf{R}H_2^u \\ &= \mathbf{R}(D(\mathbf{c}_2\bar{u}) + \sqrt{-1}R(\mathbf{c}_1 u_2)) \oplus \mathbf{R}(D(\mathbf{c}_1\bar{u}_1) - \sqrt{-1}R(\mathbf{c}_2 u_2)).\end{aligned}$$

and

$$p_*(\mathfrak{a}) = \mathbf{R}(\mathbf{c}_2 + \sqrt{-1}\mathbf{c}_1)u_2 \oplus \mathbf{R}(\mathbf{c}_1 - \sqrt{-1}\mathbf{c}_2)u_2.$$

11.7.1. *Isotropy action of* $\phi(\theta) = \exp(t\sqrt{-1}R(2e_1 - e_2 - e_3)) \in C(K) \subset K$. For an element $\phi(\theta) = \exp(t\sqrt{-1}R(2e_1 - e_2 - e_3)) : H_3(\mathbb{K}^{\mathbf{C}}) \to H_3(\mathbb{K}^{\mathbf{C}})$ lying in the center $U(1)$ of K,

$$\phi(\theta)\begin{pmatrix}\xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3\end{pmatrix} = \begin{pmatrix}\theta^4\xi_1 & \theta x_3 & \theta\bar{x}_2 \\ \theta\bar{x}_3 & \theta^{-2}\xi_2 & \theta^{-2}x_1 \\ \theta x_2 & \theta^{-2}\bar{x}_1 & \theta^{-2}\xi_3\end{pmatrix},$$

where $t = e^{it/2}$. Since

$$\begin{aligned}&p_*(\mathrm{Ad}(\phi(\theta))D(x_2\bar{u}_2)) \\ =&(\phi(\theta) \circ D(x_2\bar{u}_2) \circ \phi(\theta)^{-1})(e_1) = (\phi(\theta) \circ D(x_2\bar{u}_2))(\phi(\theta)^{-1}(e_1)) \\ =&\phi(\theta) \circ D(x_2\bar{u}_2)(\theta^{-4}e_1) = \phi(\theta)(\theta^{-4}D(x_2\bar{u}_2)(e_1)) \\ =&\theta^{-4}\phi(\theta)(\frac{1}{2}x_2u_2) = \theta^{-4}\theta \cdot \frac{1}{2}x_2u_2 \\ =&\theta^{-3}(\frac{1}{2}x_2u_2) = \theta^{-3}(D(x_2\bar{u}_2))(e_1) \\ =&\theta^{-3}p_*(D(x_2\bar{u}_2))\end{aligned}$$

and

$$\begin{aligned}&p_*(\mathrm{Ad}(\phi(\theta))R(x_2u_2)) \\ =&(\phi(\theta) \circ R(x_2u_2) \circ \phi(\theta)^{-1})(e_1) = (\phi(\theta) \circ R(x_2u_2))(\phi(\theta)^{-1}(e_1) \\ =&(\phi(\theta) \circ R(x_2u_2))(\theta^{-4}e_1) = \theta^{-4}\phi(\theta)(R(x_2u_2)(e_1)) \\ =&\theta^{-4}\phi(\theta)(\frac{1}{2}x_2u_2) = \theta^{-4}\theta\frac{1}{2}x_2u_2 \\ =&\theta^{-3}\frac{1}{2}x_2u_2 = \theta^{-3}R(x_2u_2)(e_1) \\ =&\theta^{-3}p_*(R(x_2u_2)).\end{aligned}$$

Similarly,

$$\begin{aligned} p_*(Ad(\phi(\theta))D(x_3\bar{u}_3)) &= \theta^{-3}p_*(D(x_3\bar{u}_3)), \\ p_*(Ad(\phi(\theta))R(x_3u_3)) &= \theta^{-3}p_*(R(x_3u_3)). \end{aligned}$$

Therefore, the linear isotropy representation of $(E_6, U(1)\cdot Spin(10))$ is $(\mu_3 \otimes_{\mathbf{C}} \triangle_{10}^{+})_{\mathbf{R}}$.

The maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is given as follows :

$$\begin{aligned} \mathfrak{a} &= \mathbf{R}H_1^u \oplus \mathbf{R}H_2^u \\ &= \mathbf{R}(D(\mathbf{c}_2\bar{u}_2) + \sqrt{-1}R(\mathbf{c}_1u_2)) \oplus \mathbf{R}(D(\mathbf{c}_1\bar{u}_2) - \sqrt{-1}R(\mathbf{c}_2u_2)). \end{aligned}$$

and

$$p_*(\mathfrak{a}) = \mathbf{R}(\mathbf{c}_2 + \sqrt{-1}\mathbf{c}_1)u_2 \oplus \mathbf{R}(\mathbf{c}_1 - \sqrt{-1}\mathbf{c}_2)u_2.$$

11.8. **The Subgroup $K_{[\mathfrak{a}]}$.** The Cayley algebra $\mathbb{K}$ naturally contains the field $\mathbb{H}$ of quaternions as

$$\mathbb{H} = \{x_0 + x_2c_2 + x_3c_3 + x_5c_5 | x_i \in \mathbf{R}\}.$$

Any element $x \in \mathbb{K}$ is expressed by

$$\begin{aligned} x &= x_0 + x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 + x_5c_5 + x_6c_6 + x_7c_7 \\ &= (x_0 + x_2c_2 + x_3c_3 + x_5c_5) + (x_4 + x_1c_2 + x_6c_3 - x_7c_5)c_4 \\ &=: m + a\mathbf{e} \in \mathbb{H} \oplus \mathbb{H}\mathbf{e} = \mathbb{K}, \end{aligned}$$

where $m := x_0+x_2c_2+x_3c_3+x_5c_5 \in \mathbb{H}$, $a := x_4+x_1c_2+x_6c_3-x_7c_5 \in \mathbb{H}$ and $\mathbf{e} := c_4$. In $\mathbb{H} \oplus \mathbb{H}\mathbf{e}$, we define a multiplication by

$$(m + a\mathbf{e})(n + b\mathbf{e}) = (mn - \bar{b}a) + (a\bar{n} + bm)\mathbf{e}.$$

More explicitly,

$$(a\mathbf{e})n = (a\bar{n})\mathbf{e}, \quad m(b\mathbf{e}) = (bm)\mathbf{e}, \quad (a\mathbf{e})(b\mathbf{e}) = -\bar{b}a.$$

We can also define a conjugation and an $\mathbf{R}$-linear transformation γ on $\mathbb{H} \oplus \mathbb{H}\mathbf{e}$ respectively by

$$\begin{aligned} \overline{m + a\mathbf{e}} &= \bar{m} - a\mathbf{e}, \\ \gamma(m + a\mathbf{e}) &= m - a\mathbf{e}. \end{aligned}$$

Thus $\gamma \in G_2 = \{\alpha \in \mathrm{Iso}(\mathbb{K}) \mid \alpha(xy) = \alpha(x)\alpha(y)\}$. Consider an $\mathbf{R}$-linear transformation of $H_3(\mathbb{K})$, denoted still by γ, defined by

$$\gamma \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} := \begin{pmatrix} \xi_1 & \gamma x_3 & \overline{\gamma x_2} \\ \overline{\gamma x_3} & \xi_2 & \gamma x_1 \\ \gamma x_2 & \overline{\gamma x_1} & \xi_3 \end{pmatrix},$$

for $x_i \in \mathbb{K}$ $(i = 1, 2, 3)$. Thus $\gamma \in F_4 = \{\alpha \in \mathrm{Iso}_{\mathbf{R}}(H_3(\mathbb{K})) \mid \alpha(X \circ Y) = \alpha(X) \circ \alpha(Y)\}$. Any element

$$X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & a_3\mathbf{e} & -a_2\mathbf{e} \\ -a_3\mathbf{e} & 0 & a_1\mathbf{e} \\ a_2\mathbf{e} & -a_1\mathbf{e} & 0 \end{pmatrix},$$

of $H_3(\mathbb{K})$, where $x_i = m_i + a_i\mathbf{e} \in \mathbb{H} \oplus \mathbb{H}\mathbf{e} = \mathbb{K}$ and $\xi_i \in \mathbf{R}$, can be identified with the element

$$\begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + (a_1, a_2, a_3)$$

in $H_3(\mathbb{H}) \oplus \mathbb{H}^3$. Hereafter, there exists an identification $H_3(\mathbb{K}) \cong H_3(\mathbb{H}) \oplus \mathbb{H}^3$.

Let $H_3(\mathbb{K}^{\mathbf{C}}) = \{X_1 + iX_2 \mid X_1, X_2 \in H_3(\mathbb{K})\}$ be the complexification of the Jordan algebra $H_3(\mathbb{K})$. $H_3(\mathbb{K}^{\mathbf{C}})$ has two complex conjugations, namely,

$$\overline{X_1 + iX_2} = \bar{X}_1 + i\bar{X}_2, \quad \tau(X_1 + iX_2) = X_1 - iX_2, \quad X_i \in H_3(\mathbb{K}).$$

Let the $\mathbf{C}$-linear mapping $\gamma : H_3(\mathbb{K}^{\mathbf{C}}) \to H_3(\mathbb{K}^{\mathbf{C}})$ be the complexification of $\gamma \in G_2 \subset F_4$. Then $\gamma \in E_6$ and $\gamma^2 = 1$.

Consider an involutive complex conjugate linear transformation $\tau\gamma$ of $H_3(\mathbb{K}^{\mathbf{C}})$ and the following subgroup $(E_6)^{\tau\gamma}$ of E_6:

$$(E_6)^{\tau\gamma} = \{\alpha \in E_6 \mid \tau\gamma\alpha = \alpha\tau\gamma\}.$$

Correspondingly, $H_3(\mathbb{K}^{\mathbf{C}})$ can be decomposed into the following two $\mathbf{R}$-vector subspaces:

$$H_3(\mathbb{K}^{\mathbf{C}}) = (H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma} \oplus (H_3(\mathbb{K}^{\mathbf{C}}))_{-\tau\gamma},$$

where

$$\begin{aligned}
&(H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma} := \{X \in H_3(\mathbb{K}^{\mathbf{C}}) \mid \tau\gamma X = X\} \\
&= \{\begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} 0 & a_3e & -a_2e \\ -a_3e & 0 & a_1e \\ a_2e & -a_1e & 0 \end{pmatrix} \mid \xi_i \in \mathbf{R}, m_i, a_i \in \mathbb{H}\} \\
&= H_3(\mathbb{H}) \oplus \sqrt{-1}\mathbb{H}^3,
\end{aligned}$$

$$\begin{aligned}
&(H_3(\mathbb{K}^{\mathbf{C}}))_{-\tau\gamma} := \{X \in H_3(\mathbb{K}^{\mathbf{C}}) \mid \tau\gamma X = -X\} \\
&= \{\sqrt{-1}\begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & a_3e & -a_2e \\ -a_3e & 0 & a_1e \\ a_2e & -a_1e & 0 \end{pmatrix} \mid \xi_i \in \mathbf{R}, m_i, a_i \in \mathbb{H}\} \\
&= \sqrt{-1}H_3(\mathbb{H}) \oplus \mathbb{H}^3.
\end{aligned}$$

In particular, $H_3(\mathbb{K}^{\mathbf{C}}) = ((H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma})^{\mathbf{C}}$.

Let $H_4(\mathbb{H})_0 := \{P \in H_4(\mathbb{H}) \mid \mathrm{Tr}P = 0\}$. Define a $\mathbf{C}$-linear mapping $g : H_3(\mathbb{K}^{\mathbf{C}}) = H_3(\mathbb{H}^{\mathbf{C}}) \oplus (\mathbb{H}^3)^{\mathbf{C}} \to H_4(\mathbb{H})_0$ by

$$g(M + \mathbf{a}) := \begin{pmatrix} \frac{1}{2}\mathrm{tr}(M) & \sqrt{-1}\mathbf{a} \\ \sqrt{-1}\mathbf{a}^* & M - \frac{1}{2}\mathrm{tr}(M)\mathrm{I} \end{pmatrix},$$

for $M + \mathbf{a} \in H_3(\mathbb{K}^{\mathbf{C}})$. Then we have

$$\begin{aligned} g(H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma}) &= H_4(\mathbb{H})_0, \\ g(H_3(\mathbb{K}^{\mathbf{C}}))_{-\tau\gamma}) &= \sqrt{-1}H_4(\mathbb{H})_0. \end{aligned}$$

The mapping $\varphi : Sp(4) \longrightarrow (E_6)^{\tau\gamma} \subset E_6$, defined by $\varphi(A)X := g^{-1}(A(gX)A^*)$ for each $X \in H_3(\mathbb{K}^{\mathbf{C}})$, is a surjective Lie group homomorphism and $\mathrm{Ker}(\varphi) = \{\mathrm{I}, -\mathrm{I}\} \cong \mathbf{Z}_2$. Therefore we obtain

$$Sp(4)/\mathbf{Z}_2 \cong (E_6)^{\tau\gamma}.$$

Consider $\mathbf{R}$-vector subspaces $(H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma,\sigma}$, $(H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma}$ of $H_3(\mathbb{K}^{\mathbf{C}})_{\tau\gamma}$ and $(H_3(\mathbb{K}^{\mathbf{C}}))_{-\tau\gamma,\sigma}$, $(H_3(\mathbb{K}^{\mathbf{C}}))_{-\tau\gamma,-\sigma}$ of $(H_3(\mathbb{K}^{\mathbf{C}}))_{-\tau\gamma}$, which are eigenspaces of σ, respectively given by

$$\begin{aligned} &(H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma,\sigma} = \{X \in H_3(\mathbb{K}^{\mathbf{C}}) \mid \tau\gamma X = X, \sigma X = X\} \\ =&\{\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & m_1 \\ 0 & \bar{m}_1 & \xi_3 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_1\mathbf{e} \\ 0 & -a_1\mathbf{e} & 0 \end{pmatrix} \mid \xi_i \in \mathbf{R}, m_1, a_1 \in \mathbb{H}\}, \end{aligned}$$

$$\begin{aligned} &(H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma} = \{X \in H_3(\mathbb{K}^{\mathbf{C}}) \mid \tau\gamma X = X, \sigma X = -X\} \\ =&\{\begin{pmatrix} 0 & m_3 & \bar{m}_2 \\ \bar{m}_3 & 0 & 0 \\ m_2 & 0 & 0 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} 0 & a_3\mathbf{e} & -a_2\mathbf{e} \\ -a_3\mathbf{e} & 0 & 0 \\ a_2\mathbf{e} & 0 & 0 \end{pmatrix} \mid m_2, m_3, a_2, a_3 \in \mathbb{H}\}, \end{aligned}$$

$$\begin{aligned} &(H_3(\mathbb{K}^{\mathbf{C}}))_{-\tau\gamma,\sigma} = \{X \in H_3(\mathbb{K}^{\mathbf{C}}) \mid \tau\gamma X = -X, \sigma X = X\} \\ =&\{\sqrt{-1}\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & m_1 \\ 0 & \bar{m}_1 & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_1\mathbf{e} \\ 0 & -a_1\mathbf{e} & 0 \end{pmatrix} \mid \xi_i \in \mathbf{R}, m_1, a_1 \in \mathbb{H}\}, \end{aligned}$$

$$\begin{aligned} &(H_3(\mathbb{K}^{\mathbf{C}}))_{-\tau\gamma,-\sigma} = \{X \in H_3(\mathbb{K}^{\mathbf{C}}) \mid \tau\gamma X = -X, \sigma X = -X\} \\ =&\{\sqrt{-1}\begin{pmatrix} 0 & m_3 & \bar{m}_2 \\ \bar{m}_3 & 0 & 0 \\ m_2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_3\mathbf{e} & -a_2\mathbf{e} \\ -a_3\mathbf{e} & 0 & 0 \\ a_2\mathbf{e} & 0 & 0 \end{pmatrix} \mid m_2, m_3, a_2, a_3 \in \mathbb{H}\}. \end{aligned}$$

Thus we have the following decompositions

$$\begin{aligned}(H_3(\mathbb{K})^{\mathbf{C}})_\sigma &= (H_3(\mathbb{K})^{\mathbf{C}})_{\tau\gamma,\sigma} \oplus (H_3(\mathbb{K})^{\mathbf{C}})_{-\tau\gamma,\sigma}, \\ (H_3(\mathbb{K})^{\mathbf{C}})_{-\sigma} &= (H_3(\mathbb{K})^{\mathbf{C}})_{\tau\gamma,-\sigma} \oplus (H_3(\mathbb{K})^{\mathbf{C}})_{-\tau\gamma,-\sigma}.\end{aligned}$$

Note that the images of $(H_3(\mathbb{K})^{\mathbf{C}})_{\tau\gamma,\sigma}$ and $(H_3(\mathbb{K})^{\mathbf{C}})_{\tau\gamma,-\sigma}$ of the mapping g defined above can be expressed explicitly as follows:

$$\begin{aligned}&g((H_3(\mathbb{K})^{\mathbf{C}})_{\tau\gamma,\sigma}) \\ =&\{\begin{pmatrix} \frac{1}{2}(\xi_1+\xi_2+\xi_3) & -a_1 & 0 & 0 \\ -\bar{a}_1 & \frac{1}{2}(\xi_1-\xi_2-\xi_3) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(-\xi_1+\xi_2-\xi_3) & m_1 \\ 0 & 0 & \bar{m}_1 & \frac{1}{2}(-\xi_1-\xi_2+\xi_3) \end{pmatrix} \\ &\mid \xi_1,\xi_2,\xi_3 \in \mathbf{R}, a_1, m_1 \in \mathbb{H}\}\end{aligned}$$

$$g((H_3(\mathbb{K})^{\mathbf{C}})_{\tau\gamma,-\sigma}) = \{\begin{pmatrix} 0 & 0 & -a_2 & -a_3 \\ 0 & 0 & m_3 & \bar{m}_2 \\ -\bar{a}_2 & \bar{m}_3 & 0 & 0 \\ -\bar{a}_3 & m_2 & 0 & 0 \end{pmatrix} \mid a_2, a_3, m_2, m_3 \in \mathbb{H}\}.$$

The restriction of the map φ to the subgroup $Sp(2)\times Sp(2)$ of $Sp(4)$, we have

$$\varphi : Sp(2)\times Sp(2) \longrightarrow (E_6)^{\tau\gamma,\sigma} \subset (E_6)^\sigma \cong U(1)\cdot Spin(10).???$$

Next, the restriction of φ to the subgroup $Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)$ gives

$$\begin{aligned}\varphi : Sp(1)\times Sp(1)\times Sp(1)\times Sp(1) \longrightarrow&\{\alpha \in E_6 \mid \alpha(E_i) = E_i \ (i=1,2,3)\} \\ =&\{\alpha \in F_4 \mid \alpha(E_i) = E_i \ (i=1,2,3)\} \\ \cong& Spin(8).????\end{aligned}$$

And the group $Sp(1)\times Sp(1)$ can be considered as the diagonal subgroup of $Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)$, namely, each $(a,b)\in Sp(1)\times Sp(1)$ corresponds to $(a,b,a,b) \in Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)$. Thus the restriction of φ to $Sp(1)\times Sp(1)$ is mapped to a subgroup $K_0 = S^1\cdot Spin(6)$ of $K = E^\sigma = U(1)\cdot Spin(10)$. In fact, for a 2-dimensional $\mathbf{R}$-vector subspace

$$\tilde{\mathfrak{a}} := \{\begin{pmatrix} 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & m_2 \\ a_2 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix} \mid a_2, m_2 \in \mathbf{R}\} \subset g((H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma}),$$

it follows from

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \begin{pmatrix} 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & m_2 \\ a_2 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} a^* & 0 & 0 & 0 \\ 0 & b^* & 0 & 0 \\ 0 & 0 & a^* & 0 \\ 0 & 0 & 0 & b^* \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & m_2 \\ a_2 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix}.$$

that $\tilde{\mathfrak{a}}$ corresponds to the subspace

$$\{ \begin{pmatrix} 0 & 0 & m_2 - \sqrt{-1}a_2\mathbf{e} \\ 0 & 0 & 0 \\ m_2 + \sqrt{-1}a_2\mathbf{e} & 0 & 0 \end{pmatrix} \mid m_2, a_2 \in \mathbf{R}\} \subset \ (H_3(\mathbb{K})^{\mathbf{C}})_{\tau\gamma,-\sigma},$$

which corresponds to a maximal abelian subspace of $\mathfrak{p}$. It implies that φ maps the subgroup $\check{K}_0 = Sp(1)\times Sp(1)$ for the exceptional symmetric space $(E_6, Sp(4)/\mathbf{Z}_2)$ of type EI to the subgroup $K_0 = S^1 \times Spin(6)$ of the exceptional symmetric space $(E_6, U(1)\cdot Spin(10))$ of type EIII.

Recall that

$$\check{k} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \check{K}_{[\check{\mathfrak{a}}]} = (Sp(1)\times Sp(1))\cdot \mathbf{Z}_4$$

is a generator of $\mathbf{Z}_4$. Its adjoint actions on $g((H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma,\sigma})$ and $g((H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma})$ are given in the following:

$$\check{k}\begin{pmatrix} \frac{1}{2}(\xi_1+\xi_2+\xi_3) & -a_1 & 0 & 0 \\ -\bar{a}_1 & \frac{1}{2}(\xi_1-\xi_2-\xi_3) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(-\xi_1+\xi_2-\xi_3) & m_1 \\ 0 & 0 & \bar{m}_1 & \frac{1}{2}(-\xi_1-\xi_2+\xi_3) \end{pmatrix}\check{k}^{-1}$$
$$= \begin{pmatrix} \frac{1}{2}(\xi_1-\xi_2-\xi_3) & -\bar{a}_1 & 0 & 0 \\ -a_1 & \frac{1}{2}(\xi_1+\xi_2+\xi_3) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(-\xi_1-\xi_2+\xi_3) & -\bar{m}_1 \\ 0 & 0 & -m_1 & -\frac{1}{2}(-\xi_1+\xi_2-\xi_3) \end{pmatrix},$$

$$\check{k}\begin{pmatrix} 0 & 0 & -a_2 & -a_3 \\ 0 & 0 & m_3 & \bar{m}_2 \\ -\bar{a}_2 & \bar{m}_3 & 0 & 0 \\ -\bar{a}_3 & m_2 & 0 & 0 \end{pmatrix}\check{k}^{-1} = \begin{pmatrix} 0 & 0 & -\bar{m}_2 & m_3 \\ 0 & 0 & a_3 & -a_2 \\ -m_2 & \bar{a}_3 & 0 & 0 \\ \bar{m}_3 & -\bar{a}_2 & 0 & 0 \end{pmatrix}.$$

Taking $(H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma} = (H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma,\sigma} \oplus (H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma}$ and $H_3(\mathbb{K}^{\mathbf{C}}) = ((H_3(\mathbb{K}^{\mathbf{C}}))_{\tau\gamma})^{\mathbf{C}}$ into account, together with the above computation, we know that any element

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & m_3 + \sqrt{-1}a_3\mathbf{e} & \bar{m}_2 - \sqrt{-1}a_2\mathbf{e} \\ \bar{m}_3 - \sqrt{-1}a_3\mathbf{e}_3 & \xi_3 & m_1 + \sqrt{-1}a_1\mathbf{e} \\ m_2 + \sqrt{-1}a_2\mathbf{e} & \bar{m}_1 - \sqrt{-1}a_1\mathbf{e} & \xi_3 \end{pmatrix}$$

in $H_3(\mathbb{K}^{\mathbf{C}})$ is mapped by the adjoint action of $\check{k}$ up to isomorphism to the element

$$\begin{aligned} &\begin{pmatrix} \xi_1 & a_3 - \sqrt{-1}m_3\mathbf{e} & -a_2 - \sqrt{-1}\bar{m}_2\mathbf{e} \\ \bar{a}_3 + \sqrt{-1}m_3\mathbf{e} & -\xi_2 & -\bar{m}_1 + \sqrt{-1}\bar{a}_1\mathbf{e} \\ -\bar{a}_2 + \sqrt{-1}\bar{m}_2\mathbf{e} & -m_1 - \sqrt{-1}\bar{a}_1\mathbf{e} & -\xi_3 \end{pmatrix} \\ = &\begin{pmatrix} \xi_1 & \sqrt{-1}(-m_3 + \sqrt{-1}a_3\mathbf{e})\mathbf{e} & -\sqrt{-1}(\bar{m}_2 + \sqrt{-1}a_2\mathbf{e})\mathbf{e} \\ \sqrt{-1}(m_3 + \sqrt{-1}\bar{a}_3\mathbf{e})\mathbf{e} & -\xi_2 & -(\bar{m}_1 + \sqrt{-1}\overline{\bar{a}_1\mathbf{e}}) \\ -\sqrt{-1}(-\bar{m}_2 + \sqrt{-1}\bar{a}_2\mathbf{e})\mathbf{e} & -(m_1 + \sqrt{-1}\bar{a}_1\mathbf{e}) & -\xi_3 \end{pmatrix} \\ = \ &\alpha_{23}(\pi)(\begin{pmatrix} \xi_1 & (-m_3 + \sqrt{-1}a_3\mathbf{e})\mathbf{e} & (\bar{m}_2 + \sqrt{-1}a_2\mathbf{e})\mathbf{e} \\ (m_3 + \sqrt{-1}\bar{a}_3\mathbf{e})\mathbf{e} & \xi_2 & -(\bar{m}_1 + \sqrt{-1}\overline{\bar{a}_1\mathbf{e}}) \\ (-\bar{m}_2 + \sqrt{-1}\bar{a}_2\mathbf{e})\mathbf{e} & -(m_1 + \sqrt{-1}\bar{a}_1\mathbf{e}) & \xi_3 \end{pmatrix}). \end{aligned} \tag{11.19}$$

Define $\alpha_1, \alpha_2, \alpha_3 \in SO(\mathbb{K}) \cong SO(8)$ by

$$\begin{aligned} \alpha_1(m_1 + a_1\mathbf{e}) &:= -(\bar{m}_1 - \bar{a}_1\mathbf{e}), \\ \alpha_2(m_2 + a_2\mathbf{e}) &:= -\bar{a}_2 - \bar{m}_2\mathbf{e}, \\ \alpha_3(m_3 + a_3\mathbf{e}) &:= -a_3 - m_3\mathbf{e}. \end{aligned}$$

for each $x_1 = m_1 + a_1\mathbf{e}$, $x_2 = m_2 + a_2\mathbf{e}$, $x_3 = m_3 + a_3\mathbf{e} \in \mathbb{K} = \mathbb{H} \oplus \mathbb{H}\mathbf{e}$. By simple computation,

$$\begin{aligned} \alpha_1(m_1 + a_1\mathbf{e})\ \alpha_2(m_2 + a_2\mathbf{e}) &= (\bar{m}_1 - \bar{a}_1\mathbf{e})(\bar{a}_2 + \bar{m}_2\mathbf{e}) \\ &= (\bar{m}_1\bar{a}_2 + m_2\bar{a}_1) + (-\bar{a}_1a_2 + \bar{m}_2\bar{m}_1)\mathbf{e}, \end{aligned}$$

$$\begin{aligned} \overline{\alpha_3(\overline{(m_1 + a_1\mathbf{e})(m_2 + a_2\mathbf{e})})} &= \overline{\alpha_3(\overline{(m_1m_2 - \bar{a}_2a_1) + (a_1\bar{m}_2 + a_2m_1)\mathbf{e}})} \\ &= (m_2\bar{a}_1 + \bar{m}_1\bar{a}_2) + (\bar{m}_2\bar{m}_1 - \bar{a}_1a_2)\mathbf{e}, \end{aligned}$$

we obtain $\alpha_1(m_1 + a_1\mathbf{e})\ \alpha_2(m_2 + a_2\mathbf{e}) = \overline{\alpha_3(\overline{(m_1 + a_1\mathbf{e})(m_2 + a_2\mathbf{e})})}$. Hence, $(\alpha_1, \alpha_2, \alpha_3) \in Spin(8)$. moreover, (11.19) can be expressed as

$$\alpha_{23}(\pi) \circ (\alpha_1, \alpha_2, \alpha_3)(\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}).$$

Notice that

$$\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3)(u_2) = \alpha_{23}(\alpha_2(u_2)) = \alpha_{23}(\pi)(-\mathbf{e}u_2) = \sqrt{-1}\mathbf{e}u_2,$$
$$\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3)(\sqrt{-1}\mathbf{e}u_2) = \alpha_{23}(\pi)(\alpha_2(\sqrt{-1}\mathbf{e}u_2)) = \alpha_{23}(\pi)(\sqrt{-1}u_2) = -u_2.$$

It implies that

$$\begin{aligned}\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3) \in Spin(2) \cdot spin(8) \subset (U(1) \times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4 \\ = K_2\end{aligned}$$

induces an isometry of the maximal abelian subspace $\mathfrak{a}$ of order 4 which is a $\pi/2$-rotation of $\mathfrak{a}$, we obtain

$$\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3) \in K_{[\mathfrak{a}]}$$

and it is a generator of $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

11.9. **Representation of the Casimir operator.** Denote $\langle u, v\rangle := -\mathrm{tr}(uv)$ for $u, v \in \mathfrak{e}_6 \subset gl(H_3(\mathbb{K})^{\mathbf{C}})$. Then the Casimir operator $\mathcal{C}_L$ with respect to the induced metric $\mathcal{G}^* g^{\mathrm{std}}_{Q_{30}(\mathbf{C})}$ is

$$\mathcal{C}_L = 12 C_{K/K_0} - 6 C_{K_2/K_0} - 3 C_{K_1/K_0},$$

where C_{K/K_0}, C_{K_2/K_0} and C_{K_1/K_0} are the Casimir operators of homogeneous spaces K/K_0, K_2/K_0 and K_1/K_0 with respect to the K_0-invariant metrics induced from the metric $\langle\, ,\, \rangle$ of E_6.

11.10. $D(SO(10))$ **and** $D(Spin(10))$**.** The maximal torus of $SO(2n)$ ($n = 5$) is described as follows :

$$\begin{aligned} T^5 :=&\left\{ \begin{pmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & \begin{pmatrix} \cos\theta_5 & -\sin\theta_5 \\ \sin\theta_5 & \cos\theta_5 \end{pmatrix} \end{pmatrix} \mid \right. \\ &\qquad\qquad \left. \theta_i \in \mathbf{R}\ (i = 1,2,3,4,5) \right\} \subset SO(10), \\ \mathfrak{t}^5 :=&\{\xi = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \mid \theta_i \in \mathbf{R}\ (i = 1,2,3,4,5)\} \subset \mathfrak{o}(10). \end{aligned}$$

$$\begin{aligned}
&\Gamma(SO(10))\\
:=&\{\xi \in \mathfrak{t} \mid \exp(\xi) = e\}\\
=&\{\xi = 2\pi(k_1, k_2, k_3, k_4, k_5) \mid k_i \in \mathbf{Z}(i = 1, 2, 3, 4, 5)\},\\
&Z(SO(10))\\
:=&\{\Lambda \in \mathfrak{t}^* \mid \Lambda(\xi) \in 2\pi\mathbf{Z} \text{ for each } \xi \in \Gamma(SO(10))\}\\
=&\{\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^* \mid p_i \in \mathbf{Z}\ (i = 1, 2, 3, 4, 5)\}\\
\cong& \mathbf{Z}^5,
\end{aligned}$$

where $y_i : \mathfrak{t} \ni \xi \mapsto \theta_i \in \mathbf{R}$ $(i = 1, 2, 3, 4, 5)$. Here we use

$$\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5$$

and

$$\Lambda(\xi) = \langle\Lambda, \xi\rangle = 2\pi(p_1k_1 + p_2k_2 + p_3k_3 + p_4k_4 + p_5k_5).$$

Then

$$\begin{aligned}
&D(SO(10))\\
=&\{\Lambda \in Z(SO(10)) \mid \langle\Lambda, \alpha\rangle \geq 0 \text{ for each } \alpha \in \Sigma^+(SO(10))\}\\
=&\{\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^*\\
&\mid k_i \in \mathbf{Z}\ (i = 1, 2, 3, 4, 5),\ p_1 \geq p_2 \geq p_3 \geq p_4 \geq |p_5|\},
\end{aligned}$$

where the fundamental root system of $\mathfrak{o}(10)$ is given by

$$\begin{aligned}
&\Pi(SO(10))\\
:=&\{\alpha_1 = y_1 - y_2, \alpha_2 = y_2 - y_3, \alpha_3 = y_3 - y_4, \alpha_4 = y_4 - y_5, \alpha_5 = y_4 + y_5\}.
\end{aligned}$$

The standard universal $\mathbf{Z}_2$-covering map $p : Spin(10) \to SO(10)$ is defined by

$$(p(\alpha))\mathbf{x} := \alpha \cdot \mathbf{x} \cdot {}^t\alpha \in \mathbf{R}^{10} \subset Cl(\mathbf{R}^{10})$$

for each $\alpha \in Spin(10)$ and each $\mathbf{x} \in \mathbf{R}^{10}$.

The maximal torus of $Spin(10)$ is

$$\begin{aligned}
&\tilde{T}^5\\
:=&\{\tilde{t} = \tilde{t}_1(\theta_1)\tilde{t}_2(\theta_2)\tilde{t}_3(\theta_3)\tilde{t}_4(\theta_4)\tilde{t}_5(\theta_5) \in Cl(\mathbf{R}^{10})\\
&\mid \tilde{t}_i(\theta_i) = \cos\theta_i - e_{2i-1}e_{2i}\sin\theta_i, \theta_i \in \mathbf{R}\ (i = 1, 2, 3, 4, 5)\}\\
=&\{(\cos\theta_1 - e_1e_2\sin\theta_1) \cdot (\cos\theta_2 - e_3e_4\sin\theta_2) \cdot (\cos\theta_3 - e_5e_6\sin\theta_3)\\
&\cdot (\cos\theta_4 - e_7e_8\sin\theta_4) \cdot (\cos\theta_5 - e_9e_{10}\sin\theta_5)\\
&\mid \theta_i \in \mathbf{R}\ (i = 1, 2, 3, 4, 5)\}.
\end{aligned}$$

The double covering map

$$p : \tilde{T}^5 \longrightarrow T^5$$

is given by

$$\begin{aligned}
&(\cos\theta_1 - e_1e_2\sin\theta_1)\cdot(\cos\theta_2 - e_3e_4\sin\theta_2)\cdot(\cos\theta_3 - e_5e_6\sin\theta_3)\\
&\cdot(\cos\theta_4 - e_7e_8\sin\theta_4)\cdot(\cos\theta_5 - e_9e_{10}\sin\theta_5)\\
&\longmapsto \quad \begin{pmatrix} \begin{pmatrix}\cos 2\theta_1 & -\sin 2\theta_1\\ \sin 2\theta_1 & \cos 2\theta_1\end{pmatrix} & 0 & 0\\ 0 & \cdots & 0\\ 0 & 0 & \begin{pmatrix}\cos 2\theta_5 & -\sin 2\theta_5\\ \sin 2\theta_5 & \cos 2\theta_5\end{pmatrix}\end{pmatrix}.
\end{aligned}$$

In fact,

$$\begin{aligned}
&(p(\cos\theta_1 - e_1e_2\sin\theta_1))(e_1)\\
=&(\cos\theta_1 - e_1e_2\sin\theta_1)e_1{}^t(\cos\theta_1 - e_1e_2\sin\theta_1)\\
=&(\cos\theta_1 - e_1e_2\sin\theta_1)e_1(\cos\theta_1 - e_2e_1\sin\theta_1)\\
=&(\cos\theta_1e_1 - e_1e_2e_1\sin\theta_1)(\cos\theta_1 - e_2e_1\sin\theta_1)\\
=&\cos\theta_1\cos\theta_1e_1 - e_1e_2e_1\sin\theta_1\cos\theta_1 - \cos\theta_1e_1e_2e_1\sin\theta_1 + e_1e_2e_1e_2e_1\sin\theta_1\sin\theta_1\\
=&\cos^2\theta_1e_1 - e_1e_2e_1\sin\theta_1\cos\theta_1 - e_1e_2e_1\cos\theta_1\sin\theta_1 + e_1e_2e_1e_2e_1\sin^2\theta_1\\
=&\cos^2\theta_1e_1 - e_2\sin\theta_1\cos\theta_1 - e_2\cos\theta_1\sin\theta_1 - e_1\sin^2\theta_1\\
=&(\cos^2\theta_1 - \sin^2\theta_1)e_1 - e_2(2\sin\theta_1\cos\theta_1)\\
=&(\cos 2\theta_1)e_1 - e_2(\sin 2\theta_1).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\tilde{\mathfrak{t}} = \mathfrak{t} = \{(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) \mid \theta_i \in \mathbf{R}\ (i=1,2,3,4,5)\}\\
&\longrightarrow\\
&\tilde{T} = \{\ (\cos(\theta_1/2) - e_1e_2\sin(\theta_1/2))\cdot(\cos(\theta_2/2) - e_3e_4\sin(\theta_2/2))\\
&\qquad\cdot(\cos(\theta_3/2) - e_5e_6\sin(\theta_3/2))\cdot(\cos(\theta_4/2) - e_7e_8\sin(\theta_4/2))\\
&\qquad\cdot(\cos(\theta_5/2) - e_9e_{10}\sin(\theta_5/2))\\
&\qquad\mid \theta_i \in \mathbf{R}\ (i=1,2,3,4,5)\} \subset Spin(10)
\end{aligned}$$

$$
\begin{aligned}
&\Gamma(Spin(10))\\
=&\{\xi = (\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) \in \tilde{\mathfrak{t}} \mid \ \exp(\xi) = e\}\\
=&\{\xi = 2\pi\ (k_1,k_2,k_3,k_4,k_5) \mid k_i \in \mathbf{Z}\ (i=1,2,3,4,5), \sum_{i=1}^{5} k_i \in 2\mathbf{Z}\}\\
\subset&\Gamma(SO(10)),\\
&Z(Spin(10))\\
:=&\{\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^*\\
&\qquad\qquad \mid \Lambda(\xi) \in 2\pi\mathbf{Z} \text{ for each } \xi \in \Gamma(Spin(10))\}\\
=&\{\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^*\\
&\qquad \mid p_i \in \mathbf{Z}\ (i=1,2,3,4,5) \text{ or } p_i + \frac{1}{2} \in \mathbf{Z}\ (i=1,2,3,4,5)\} \supset Z(SO(10))\\
\cong&\mathbf{Z}^5 \oplus \varepsilon(1,1,1,1,1), \quad \text{ where } \varepsilon = 0 \text{ or } \frac{1}{2}\ .
\end{aligned}
$$

$$
\begin{aligned}
&D(Spin(10))\\
:=&\{\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^*\\
&\mid \langle \Lambda, \alpha\rangle \geq 0 \text{ for each } \alpha \in \Sigma^+(Spin(10))\}\\
=&\{\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^*\\
&\mid (p_1,\cdots,p_5) \in \mathbf{Z}_5 + \varepsilon(1,1,1,1,1), \text{ where } \varepsilon = 0 \text{ or } \frac{1}{2},\\
&\ p_1 \geq p_2 \geq p_3 \geq p_4 \geq |p_5|\}\\
&\supset D(SO(10)),
\end{aligned}
$$

where the fundamental root system of $\mathfrak{o}(10)$ is given by

$$
\begin{aligned}
&\Pi(Spin(10))\\
:=&\{\alpha_1 = y_1 - y_2, \alpha_2 = y_2 - y_3, \alpha_3 = y_3 - y_4, \alpha_4 = y_4 - y_5, \alpha_5 = y_4 + y_5\}.
\end{aligned}
$$

11.11. **Branching Laws.**

11.11.1. *Branching laws of* $(SO(10), SO(2)\times SO(8))$ *and* $(Spin(10), Spin(2)\cdot Spin(8))$. $Spin(10)$ and $Spin(2)\cdot Spin(8)$ have the common maximal torus

$$\tilde{T} \subset Spin(2)\cdot Spin(8) \subset Spin(10).$$

$SO(10)$ and $SO(2)\cdot SO(8)$ have the common maximal torus

$$T \subset SO(2)\times SO(8) \subset SO(10).$$

And $\tilde{\mathfrak{t}} = \mathfrak{t}$. Hence

$$\begin{aligned}
&\Gamma(Spin(2)\cdot Spin(8)) = \Gamma(Spin(10))\\
=&\{\xi = (\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) \in \tilde{\mathfrak{t}} \mid \exp(\xi) = e\}\\
=&\{\xi = 2\pi\ (k_1,k_2,k_3,k_4,k_5) \mid k_i \in \mathbf{Z}\ (i=1,2,3,4,5), \sum_{i=1}^{5} k_i \in 2\mathbf{Z}\}\\
\subset&\Gamma(SO(2)\times SO(10)),
\end{aligned}$$

$$\begin{aligned}
&Z(Spin(2)\cdot Spin(8))\\
:=&\{\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^*\\
&\qquad \mid \Lambda(\xi) \in 2\pi\mathbf{Z} \text{ for each } \xi \in \Gamma(Spin(2)\cdot Spin(8))\}\\
=&\{\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^*\\
&\mid p_i \in \mathbf{Z}\ (i=1,2,3,4,5) \text{ or } p_i + \frac{1}{2} \in \mathbf{Z}\ (i=1,2,3,4,5)\} \supset Z(SO(2)\times SO(8))\\
\cong&\mathbf{Z}^5 \oplus \varepsilon(1,1,1,1,1), \quad \text{where } \varepsilon = 0 \text{ or } \frac{1}{2}\ .
\end{aligned}$$

$$\begin{aligned}
&D(Spin(2)\cdot Spin(8))\\
:=&\{\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^*\\
&\mid \langle\Lambda,\alpha\rangle \geq 0 \text{ for each } \alpha \in \Sigma^+(Spin(2)\times Spin(8))\}\\
=&\{\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^*\\
&\mid (p_1,\cdots,p_5) \in \mathbf{Z}_5 + \varepsilon(1,1,1,1,1),\ \text{where } \varepsilon = 0 \text{ or } \frac{1}{2},\\
&p_2 \geq p_3 \geq p_4 \geq |p_5|\}\\
&\supset D(SO(2)\times SO(8)).
\end{aligned}$$

Theorem (Branching Laws of $(SO(10), SO(2)\times SO(8))$, [40]). *For each*

$$\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + \epsilon p_5y_5 \in D(SO(10))$$

with $\epsilon = 1$ *or* -1 *and* $(p_1,p_2,p_3,p_4,p_5) \in \mathbf{Z}^5$ *satisfying* $p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5 \geq 0$*,* V_Λ *contains an irreducible* $SO(2)\times SO(8)$*-module with the highest weight*

$$\Lambda' = q_1y_1 + q_2y_2 + q_3y_3 + q_4y_4 + \epsilon' p_5y_5 \in D(SO(2)\times SO(8))$$

with $\epsilon' = 1$ *or* -1 *and*

$$(q_1,q_2,q_3,q_4,q_5) \in \mathbf{Z}^5,$$
$$q_2 \geq q_3 \geq q_4 \geq q_5 \geq 0$$

if and only if $\Lambda' = q_1y_1+q_2y_2+q_3y_3+q_4y_4+\epsilon' q_5y_5 \in D(SO(2)\times SO(8))$ *satisfying the following conditions :*

(1)

$$\begin{aligned} &p_1 \geq q_2 \geq p_3, \\ &p_2 \geq q_3 \geq p_4, \\ &p_3 \geq q_4 \geq p_5, \\ &p_4 \geq q_5 \geq 0. \end{aligned}$$

(2) *The coefficient of* X^{q_1}

$$X^{\epsilon\epsilon'\ell_5}(\prod_{i=1}^{4} \frac{X^{\ell_i+1} - X^{-\ell_i-1}}{X - X^{-1}})$$

does not vanish. Here the integers $\ell_1, \ell_2, \cdots, \ell_5$ *are defined by*

$$\begin{aligned} &\ell_1 := p_1 - \max\{p_2, q_2\}, \\ &\ell_2 := \min\{p_2, q_2\} - \max\{p_3, q_3\}, \\ &\ell_3 := \min\{p_3, q_3\} - \max\{p_4, q_4\}, \\ &\ell_4 := \min\{p_4, q_4\} - \max\{p_5, q_5\}, \\ &\ell_5 := \min\{p_5, q_5\}. \end{aligned}$$

Moreover its multiplicity is equal to the coefficient of X^{q_1}.

Theorem (Branching Laws of $(Spin(10), Spin(2)\cdot Spin(8))$)**.** *For each*

$$\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + \delta p_5y_5 \in D(Spin(10)),$$

with $\delta = 1$ *or* -1 *and*

$$\begin{aligned} &(p_1, p_2, p_3, p_4, p_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 1 \text{ or } \frac{1}{2} \\ &p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5 \geq 0, \end{aligned}$$

V_Λ *contains an irreducible* $Spin(2)\cdot Spin(8)$*-module with the highest weight*

$$\Lambda' = q_1y_1 + q_2y_2 + q_3y_3 + q_4y_4 + \delta' q_5y_5 \in D(Spin(2)\cdot Spin(8))$$

with $\delta' = 1$ *or* -1 *and*

$$\begin{aligned} &(q_1, q_2, q_3, q_4, q_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 1 \text{ or } \frac{1}{2} \\ &q_2 \geq q_3 \geq q_4 \geq q_5 \geq 0, \end{aligned}$$

if and only if $\Lambda' = q_1y_1+q_2y_2+q_3y_3+q_4y_4+\delta' q_5y_5 \in D(Spin(2)\cdot Spin(8))$ *satisfies the following conditions :*

(1)

$$\begin{aligned}
p_1 + 1 &> q_2 > p_3 - 1, \\
p_2 + 1 &> q_3 > p_4 - 1, \\
p_3 + 1 &> q_4 > p_5 - 1, \\
p_4 + 1 &> q_5 \geq 0.
\end{aligned}$$

(2) *The coefficient of* X^{q_1} *in the following power series expansion in* X *of*

$$X^{\delta\delta'\ell_5}\Big(\prod_{i=1}^{4} \frac{X^{\ell_i+1} - X^{-\ell_i-1}}{X - X^{-1}}\Big)$$

does not vanish. Here

$$\begin{aligned}
\ell_1 &:= p_1 - \max\{p_2, q_2\}, \\
\ell_2 &:= \min\{p_2, q_2\} - \max\{p_3, q_3\}, \\
\ell_3 &:= \min\{p_3, q_3\} - \max\{p_4, q_4\}, \\
\ell_4 &:= \min\{p_4, q_4\} - \max\{p_5, q_5\}, \\
\ell_5 &:= \min\{p_5, q_5\}.
\end{aligned}$$

Moreover its multiplicity is equal to the coefficient of X^{q_1}.

11.11.2. *Branching Laws of* $(SO(8), SO(2)\times SO(6))$ *and* $(Spin(8), Spin(2)\cdot Spin(6))$. $Spin(8)$ and $Spin(2)\cdot Spin(6)$ have the common maximal torus

$$\tilde{T} \subset Spin(2) \cdot Spin(6) \subset Spin(8).$$

$SO(8)$ and $SO(2)\cdot SO(6)$ have the common maximal torus

$$T \subset SO(2) \times SO(6) \subset SO(8).$$

And $\tilde{\mathfrak{t}} = \mathfrak{t}$. Hence

$$\begin{aligned}
&\Gamma(Spin(2)\cdot Spin(6)) = \Gamma(Spin(8)) \\
=&\{\xi = (\theta_2, \theta_3, \theta_4, \theta_5) \in \tilde{\mathfrak{t}} \mid \ \exp(\xi) = e\} \\
=&\{\xi = 2\pi\ (k_2, k_3, k_4, k_5) \mid k_i \in \mathbf{Z}\ (i = 2, 3, 4, 5), \sum_{i=2}^{5} k_i \in 2\mathbf{Z}\} \\
\subset&\Gamma(SO(2)\times SO(6)),
\end{aligned}$$

$$\begin{aligned}
&Z(Spin(2)\cdot Spin(6)) \\
:=&\{\Lambda = p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^* \\
&\qquad\qquad \mid \Lambda(\xi) \in 2\pi\mathbf{Z} \text{ for each } \xi \in \Gamma(Spin(2)\cdot Spin(6))\} \\
=&\{\Lambda = p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^* \\
&\mid p_i \in \mathbf{Z}\ (i = 2, 3, 4, 5) \text{ or } p_i + \frac{1}{2} \in \mathbf{Z}\ (i = 2, 3, 4, 5)\} \supset Z(SO(2)\times SO(6)) \\
\cong&\mathbf{Z}^5 \oplus \varepsilon(1, 1, 1, 1), \quad \varepsilon = 0 \text{ or } \frac{1}{2}\ .
\end{aligned}$$

$$\begin{aligned}
&D(Spin(2)\cdot Spin(6)) \\
:=&\{\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^* \\
&\mid \langle\Lambda, \alpha\rangle \geq 0 \text{ for each } \alpha \in \Sigma^+(Spin(2)\cdot Spin(6))\} \\
=&\{\Lambda = p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^* \\
&\mid (p_2, p_3, p_4, p_5) \in \mathbf{Z}_4 + \varepsilon(1, 1, 1, 1), \text{ where } \varepsilon = 0 \text{ or } \frac{1}{2}, \ , \\
&p_2 \geq p_3 \geq p_4 \geq |p_5|\} \\
&\supset D(SO(2)\times SO(6))
\end{aligned}$$

Theorem (Branching Laws of $(SO(8), SO(2)\times SO(6))$)**.** *For each*

$$\Lambda = p_2y_2 + p_3y_3 + p_4y_4 + \epsilon p_5y_5 \in D(SO(8))$$

with $\epsilon = 1$ *or* -1 *and* $(p_2, p_3, p_4, p_5) \in \mathbf{Z}^4$ *satisfying* $p_2 \geq p_3 \geq p_4 \geq p_5 \geq 0$, V_Λ *contains an irreducible* $SO(2)\times SO(6)$*-module with the highest weight*

$$\Lambda' = q_2y_2 + q_3y_3 + q_4y_4 + \epsilon' p_5y_5 \in D(SO(2)\times SO(6))$$

with $\epsilon' = 1$ *or* -1 *and*

$$\begin{aligned}
&(q_2, q_3, q_4, q_5) \in \mathbf{Z}^4, \\
&q_3 \geq q_4 \geq q_5 \geq 0
\end{aligned}$$

if and only if $\Lambda' = q_2y_2 + q_3y_3 + q_4y_4 + \epsilon' q_5y_5 \in D(SO(2) \times SO(6))$ *satisfying the following conditions :*

(1)

$$\begin{aligned} &p_2 \geq q_3 \geq p_4, \\ &p_3 \geq q_4 \geq p_5, \\ &p_4 \geq q_5 \geq 0. \end{aligned}$$

(2) *The coefficient of* X^{q_2}

$$X^{\epsilon\epsilon'\ell_5}\left(\prod_{i=2}^{4} \frac{X^{\ell_i+1} - X^{-\ell_i-1}}{X - X^{-1}}\right)$$

does not vanish. Here

$$\begin{aligned} &\ell_2 := p_2 - \max\{p_3, q_3\}, \\ &\ell_3 := \min\{p_3, q_3\} - \max\{p_4, q_4\}, \\ &\ell_4 := \min\{p_4, q_4\} - \max\{p_5, q_5\}, \\ &\ell_5 := \min\{p_5, q_5\}. \end{aligned}$$

Moreover its multiplicity is equal to the coefficient of X^{q_2}.

Theorem (Branching Laws of $(Spin(8), Spin(2) \cdot Spin(6))$). *For each*

$$\Lambda = p_2y_2 + p_3y_3 + p_4y_4 + \delta p_5y_5 \in D(Spin(8)),$$

with $\epsilon = 1$ *or* -1 *and*

$$\begin{aligned} &(p_2, p_3, p_4, p_5) \in \mathbf{Z}^4 + \varepsilon(1, 1, 1, 1), \ \varepsilon = 1 \ \textit{or} \ \frac{1}{2} \\ &p_2 \geq p_3 \geq p_4 \geq p_5 \geq 0, \end{aligned}$$

V_Λ *contains an irreducible* $Spin(2) \cdot Spin(6)$*-module with the highest weight*

$$\Lambda' = q_2y_2 + q_3y_3 + q_4y_4 + \delta' p_5y_5 \in D(Spin(2) \cdot Spin(6))$$

with $\epsilon' = 1$ *or* -1 *and*

$$\begin{aligned} &(q_2, q_3, q_4, q_5) \in \mathbf{Z}^4 + \varepsilon(1, 1, 1, 1), \ \varepsilon = 1 \ \textit{or} \ \frac{1}{2} \\ &q_3 \geq q_4 \geq q_5 \geq 0. \end{aligned}$$

if and only if $\Lambda' = q_2y_2 + q_3y_3 + q_4y_4 + \epsilon' q_5y_5 \in D(SO(2) \times SO(6))$ *satisfies the following conditions :*

(1)

$$\begin{aligned}
&p_2+1>q_3>p_4-1,\\
&p_3+1>q_4>p_5-1,\\
&p_4+1>q_5\geq 0.
\end{aligned}$$

(2) *The coefficient of* X^{q_2}

$$X^{\delta\delta'\ell_5}(\prod_{i=2}^{4}\frac{X^{\ell_i+1}-X^{-\ell_i-1}}{X-X^{-1}})$$

does not vanish. Here

$$\begin{aligned}
&\ell_2:=p_2-\max\{p_3,q_3\},\\
&\ell_3:=\min\{p_3,q_3\}-\max\{p_4,q_4\},\\
&\ell_4:=\min\{p_4,q_4\}-\max\{p_5,q_5\},\\
&\ell_5:=\min\{p_5,q_5\}.
\end{aligned}$$

Moreover its multiplicity is equal to the coefficient of X^{q_2}*.*

11.12. $D(K)=D((U(1)\times Spin(10))/\mathbf{Z}_4)$.

11.12.1. *Descriptions of* $\Gamma(K)$*,* $Z(K)$ *and* $D(K)$*.* The maximal torus T_K of $K=(U(1)\times Spin(10))/\mathbf{Z}_4$ is given as follows :

$$\begin{aligned}
T_K=&\{(e^{\sqrt{-1}\theta_0},(\cos\frac{\theta_1}{2}-e_1e_2\sin\frac{\theta_1}{2})(\cos\frac{\theta_1}{2}-e_3e_4\sin\frac{\theta_1}{2})\\
&(\cos\frac{\theta_1}{2}-e_5e_6\sin\frac{\theta_1}{2})(\cos\frac{\theta_1}{2}-e_7e_8\sin\frac{\theta_1}{2})(\cos\frac{\theta_1}{2}-e_9e_{10}\sin\frac{\theta_1}{2}))\\
&\mid \theta_0,\theta_1\in\mathbf{R}\}/\mathbf{Z}_4,
\end{aligned}$$

where $t_0=2\theta_0$, $t_1=\theta_1$, $U(1)=\{\exp(t_0\sqrt{-1}R(2e_1-e_2-e_3))\mid t_0\in\mathbf{R}\}$, $Spin(2)=\{\exp(t_1\sqrt{-1}R(e_2-e_3))\mid t_1\in\mathbf{R}\}$ and

$$\mathbf{Z}_4:=\{(1,1),(-1,-1),(\sqrt{-1},-e_1e_2\cdots e_{10}),(-\sqrt{-1},e_1e_2\cdots e_{10})\}.$$

The corresponding maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is

$$\mathfrak{t}=\{(\theta_0,\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)\mid\theta_i\in\mathbf{R}\ (i=0,1,2,3,4,5)\}.$$

Then

$$\begin{aligned}\Gamma(K) =&\{\xi = 2\pi(\frac{k_0}{2}, k_1, k_2, k_3, k_4, k_5) + \pi\varepsilon(\frac{1}{2}, 1, 1, 1, 1, 1)\\ &= 2\pi(\frac{k_0}{2}, k_1, k_2, k_3, k_4, k_5) + 2\pi\varepsilon(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\\ &\mid k_0, k_1, k_2, k_3, k_4, k_5 \in \mathbf{Z},\ \varepsilon = 0 \text{ or } 1,\ \sum_{\alpha=0}^{5} k_\alpha \in 2\mathbf{Z}\},\end{aligned}$$

$$\begin{aligned}Z(K) =&Z((U(1) \times Spin(10))/\mathbf{Z}_4)\\ =&\{\Lambda = p_0y_0 + p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^* \mid\\ &\frac{1}{2}p_0 + p_1 + p_2 + p_3 + p_4 + p_5 \in 2\mathbf{Z},\\ &p_0 \in \mathbf{Z},\ (p_1, p_2, p_3, p_4, p_5) \in \mathbf{Z}^5 + \varepsilon(1, 1, 1, 1, 1),\ \varepsilon = 0 \text{ or } \frac{1}{2}\}\end{aligned}$$

and

$$\begin{aligned}\triangle(K) &= \triangle((U(1) \times Spin(10))/\mathbf{Z}_4)\\ &= \{y_1 - y_2, y_2 - y_3, y_3 - y_4, y_4 - y_5, y_4 + y_5\},\end{aligned}$$

$$\begin{aligned}D(K) =&D((U(1) \times Spin(10))/\mathbf{Z}_4)\\ =&\{\Lambda = p_0y_0 + p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in Z(K) \mid\\ &\quad \langle\Lambda, \alpha\rangle \geq 0 \text{ for each } \alpha \in \Sigma^+(K)\}\\ =&\{\Lambda = p_0y_0 + p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^* \mid\\ &\frac{1}{2}p_0 + p_1 + p_2 + p_3 + p_4 + p_5 \in 2\mathbf{Z},\ p_0 \in \mathbf{Z},\\ &(p_1, p_2, p_3, p_4, p_5) \in \mathbf{Z}^5 + \varepsilon(1, 1, 1, 1, 1),\ \varepsilon = 0 \text{ or } \frac{1}{2},\\ &\quad p_1 \geq p_2 \geq p_3 \geq p_4 \geq |p_5|\ \}.\end{aligned}$$

11.13. **Descriptions of $D(K_2)$, $D(K_1)$ and $D(K_0)$.** For $K_2 = (U(1)\times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4 \subset K$, $T_{K_2} = T_K$, where $t_1 = \theta_1$, $Spin(2) = \{\exp(t_1\sqrt{-1}R(e_2 - e_3)) \mid t_1 \in \mathbf{R}\}$. Hence, $\Gamma(K_2) = \Gamma(K)$, $Z(K_2) = Z(K)$ and

$$
\begin{aligned}
D(K_2) =& D((U(1) \times Spin(2) \cdot Spin(8))/\mathbf{Z}_4) \\
=& \{\Lambda = p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + p_5 y_5 \in \mathfrak{t}^* \mid \\
& \frac{1}{2} p_0 + p_1 + p_2 + p_3 + p_4 + p_5 \in 2\mathbf{Z},\ p_0 \in \mathbf{Z}, \\
& (p_1, p_2, p_3, p_4, p_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2}, \\
& \quad p_2 \geq p_3 \geq p_4 \geq |p_5| \ \}.
\end{aligned}
$$

On the other hand, $K_2 = (S^1 \times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4$, where $S^1 = \{\exp(\hat{t}_0 \sqrt{-1} R(-e_1 + 2e_2 - e_3)) \mid \hat{t}_0 \in \mathbf{R}\}$, $Spin(2) = \{\exp(\hat{t}_1 \sqrt{-1} R(e_3 - e_1)) \mid \hat{t}_1 \in \mathbf{R}\}$ and here $Spin(2) \cdot Spin(8) \subset (E_6)_{e_2} \cong Spin(10)$. Since

$$
\begin{aligned}
&\exp(t_0 \sqrt{-1} R(2e_1 - e_2 - e_3)) \cdot \exp(t_1 \sqrt{-1} R(e_2 - e_3)) \\
&\quad = \exp(-\frac{t_0 - t_1}{2} \sqrt{-1} R(-e_1 + 2e_2 - e_3)) \cdot \exp(-\frac{3t_0 + t_1}{2} \sqrt{-1} R(e_3 - e_1)),
\end{aligned}
$$

one can take

$$
\hat{t}_0 = -\frac{t_0 - t_1}{2}, \quad \hat{t}_1 = -\frac{3t_0 + t_1}{2}.
$$

Set

$$
\begin{aligned}
\hat{y}_0 = \hat{\theta}_0 = \frac{\hat{t}_0}{2} &:= -\frac{2y_0 - y_1}{4} = -\frac{1}{2} y_0 + \frac{1}{4} y_1, \\
\hat{y}_1 = \hat{\theta}_1 = \hat{t}_1 &:= -\frac{6y_0 + y_1}{2} = -3y_0 - \frac{1}{2} y_1, \\
\hat{y}_2 = \hat{\theta}_2 &:= \frac{1}{2}(y_2 + y_3 + y_4 + y_5), \\
\hat{y}_3 = \hat{\theta}_3 &:= \frac{1}{2}(y_2 + y_3 - y_4 - y_5), \\
\hat{y}_4 = \hat{\theta}_4 &:= \frac{1}{2}(y_2 - y_3 + y_4 - y_5), \\
\hat{y}_5 = \hat{\theta}_5 &:= \frac{1}{2}(-y_2 + y_3 + y_4 - y_5).
\end{aligned}
\tag{11.20}
$$

Correspondingly,

$$\begin{aligned}
y_0 &= -\frac{2\hat{y}_0 + \hat{y}_1}{4} = -\frac{1}{2}\hat{y}_0 - \frac{1}{4}\hat{y}_1,\\
y_1 &= \frac{6\hat{y}_0 - \hat{y}_1}{2} = 3\hat{y}_0 - \frac{1}{2}\hat{y}_1,\\
y_2 &= \frac{1}{2}(\hat{y}_2 + \hat{y}_3 + \hat{y}_4 - \hat{y}_5),\\
y_3 &= \frac{1}{2}(\hat{y}_2 + \hat{y}_3 - \hat{y}_4 + \hat{y}_5),\\
y_4 &= \frac{1}{2}(\hat{y}_2 - \hat{y}_3 + \hat{y}_4 + \hat{y}_5),\\
y_5 &= \frac{1}{2}(\hat{y}_2 - \hat{y}_3 - \hat{y}_4 - \hat{y}_5),
\end{aligned}$$

Thus

$$\begin{aligned}
\Lambda &= \hat{p}_0\hat{y}_0 + \hat{p}_1\hat{y}_1 + \hat{p}_2\hat{y}_2 + \hat{p}_3\hat{y}_3 + \hat{p}_4\hat{y}_4 + \hat{p}_5\hat{y}_5\\
&\in D(K_2) = D((S^1 \times Spin(2) \cdot Spin(8))/\mathbf{Z}_4),
\end{aligned}$$

where

$$\begin{aligned}
\hat{p}_0 &= -\frac{1}{2}p_0 + 3p_1,\\
\hat{p}_1 &= -\frac{1}{4}p_0 - \frac{1}{2}p_1,\\
\hat{p}_2 &= \frac{1}{2}(p_2 + p_3 + p_4 + p_5),\\
\hat{p}_3 &= \frac{1}{2}(p_2 + p_3 - p_4 - p_5),\\
\hat{p}_4 &= \frac{1}{2}(p_2 - p_3 + p_4 - p_5),\\
\hat{p}_5 &= \frac{1}{2}(-p_2 + p_3 + p_4 - p_5).
\end{aligned}$$

Then $D(K_2)$ has the following another expression:

$$\begin{aligned}
D(K_2) =& D((S^1 \times Spin(2) \cdot Spin(8))/\mathbf{Z}_4)\\
=& \{\Lambda = \hat{p}_0\hat{y}_0 + \hat{p}_1\hat{y}_1 + \hat{p}_2\hat{y}_2 + \hat{p}_3\hat{y}_3 + \hat{p}_4\hat{y}_4 + \hat{p}_5\hat{y}_5 \in \mathfrak{t}^* \mid\\
& \frac{1}{2}\hat{p}_0 + \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4 + \hat{p}_5 \in 2\mathbf{Z},\ \hat{p}_0 \in \mathbf{Z},\\
& (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2},\\
& \quad \hat{p}_2 \geq \hat{p}_3 \geq \hat{p}_4 \geq |\hat{p}_5|\ \}.
\end{aligned}$$

The maximal torus of $K_1 = (S^1 \times (Spin(2) \cdot (Spin(2) \cdot Spin(6))))/\mathbf{Z}_4$ is given as follows:

$$
\begin{aligned}
T_{K_1} = \{ & (e^{\sqrt{-1}\hat{\theta}_0}, (\cos\frac{\hat{\theta}_1}{2} - e_1e_2\sin\frac{\hat{\theta}_1}{2})(\cos\frac{\hat{\theta}_1}{2} - e_3e_4\sin\frac{\hat{\theta}_1}{2}) \\
& (\cos\frac{\hat{\theta}_1}{2} - e_5e_6\sin\frac{\hat{\theta}_1}{2})(\cos\frac{\hat{\theta}_1}{2} - e_7e_8\sin\frac{\hat{\theta}_1}{2})(\cos\frac{\hat{\theta}_1}{2} - e_9e_{10}\sin\frac{\hat{\theta}_1}{2})) \\
& \mid \hat{\theta}_0, \hat{\theta}_1 \in \mathbf{R}\}/\mathbf{Z}_4 \\
& = \hat{T}_{K_2} = T_{K_2} = T_K.
\end{aligned}
$$

and the corresponding maximal abelian subalgebra $\mathfrak{t}_{\mathfrak{k}_1}$ of $\mathfrak{k}_1$ is

$$
\begin{aligned}
\mathfrak{t}_{\mathfrak{k}_1} &= \hat{\mathfrak{t}}_{\mathfrak{k}_2} = \{(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5) \mid \hat{\theta}_i \in \mathbf{R} \ (i = 0,1,2,3,4,5)\} \\
&= \mathfrak{t}_{\mathfrak{k}_2} = \mathfrak{t}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
D(K_1) = & D((S^1 \times (Spin(2) \cdot (Spin(2) \cdot Spin(6))))/\mathbf{Z}_4) \\
= & \{\Lambda = \hat{p}_0\hat{y}_0 + \hat{p}_1\hat{y}_1 + \hat{p}_2\hat{y}_2 + \hat{p}_3\hat{y}_3 + \hat{p}_4\hat{y}_4 + \hat{p}_5\hat{y}_5 \in \mathfrak{t}^*_{\mathfrak{k}_1} = \mathfrak{t}^* \mid \\
& \frac{1}{2}\hat{p}_0 + \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4 + \hat{p}_5 \in 2\mathbf{Z}, \ \hat{p}_0 \in \mathbf{Z}, \\
& (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1), \ \varepsilon = 0 \text{ or } \frac{1}{2}, \\
& \quad \hat{p}_3 \geq \hat{p}_4 \geq |\hat{p}_5| \ \}.
\end{aligned}
$$

The maximal torus of $K_0 = (S^1 \times Spin(6))/\mathbf{Z}_2$ is given as follows :

$$
\begin{aligned}
T_{K_0} = & \{(e^{\sqrt{-1}\hat{\theta}_0}, (\cos\frac{\hat{\theta}_3}{2} - e_5e_6\sin\frac{\hat{\theta}_3}{2})(\cos\frac{\hat{\theta}_4}{2} - e_7e_8\sin\frac{\hat{\theta}_4}{2}) \\
& \quad (\cos\frac{\hat{\theta}_5}{2} - e_9e_{10}\sin\frac{\hat{\theta}_5}{2})) \mid \hat{\theta}_i \in \mathbf{R} \ (i = 0,3,4,5)\}/\mathbf{Z}_2 \\
& \subset \hat{T}_{K_2} = T_K
\end{aligned}
$$

and the corresponding maximal abelian subalgebra of $\mathfrak{k}_0$ is

$$
\begin{aligned}
\mathfrak{t}_{\mathfrak{k}_0} = & \{(\hat{\theta}_0, 0, 0, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5) \mid \hat{\theta}_i \in \mathbf{R} \ (i = 0,3,4,5)\} \\
& \subset \mathfrak{t}_{\mathfrak{k}_2} = \mathfrak{t}.
\end{aligned}
$$

Then

$$\begin{aligned}
D(K_0) =& D((S^1 \times Spin(6))/\mathbf{Z}_2) \\
=& \{\Lambda = \hat{q}_0\hat{y}_0 + \hat{q}_3\hat{y}_3 + \hat{q}_4\hat{y}_4 + \hat{q}_5\hat{y}_5 \in \mathfrak{t}^*_{\mathfrak{k}_0} \quad | \\
& \frac{1}{2}\hat{q}_0 + \hat{q}_3 + \hat{q}_4 + \hat{q}_5 \in 2\mathbf{Z},\ \hat{q}_0 \in \mathbf{Z}, \\
& (\hat{q}_3, \hat{q}_4, \hat{q}_5) \in \mathbf{Z}^3 + \varepsilon(1,1,1), \varepsilon = 0 \text{ or } \frac{1}{2} \\
& \quad \hat{q}_3 \geq \hat{q}_4 \geq |\hat{q}_5|\ \}.
\end{aligned}$$

11.14. **Branching Laws of** $K \supset K_2 \supset K_1 \supset K_0$**.**

11.14.1. *Branching laws of* (K, K_2).

$(K, K_2) = ((U(1) \times Spin(10))/\mathbf{Z}_4, (U(1) \times Spin(2) \cdot Spin(8))/\mathbf{Z}_4)$.

Let

$$\Lambda = p_0y_0 + p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + \epsilon p_5y_5 \in D(K),$$

where

$$\begin{aligned}
& \frac{1}{2}p_0 + p_1 + p_2 + p_3 + p_4 + \epsilon p_5 \in 2\mathbf{Z}, p_0 \in \mathbf{Z}, \\
& (p_1, p_2, p_3, p_4, p_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2}, \\
& p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5 \geq 0,\ \epsilon = 1 \text{ or } -1.
\end{aligned}$$

Let

$$\Lambda' = p'_0y_0 + p'_1y_1 + p'_2y_2 + p'_3y_3 + p'_4y_4 + \epsilon' p'_5y_5 \in D(K_2),$$

where

$$\begin{aligned}
& \frac{1}{2}p'_0 + p'_1 + p'_2 + p'_3 + p'_4 + \epsilon' p'_5 \in 2\mathbf{Z},\ p'_0 = p_0 \in \mathbf{Z}, \\
& (p'_1, p'_2, p'_3, p'_4, p'_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2}, \\
& p'_2 \geq p'_3 \geq p'_4 \geq p'_5 \geq 0,\ \epsilon' = 1 \text{ or } -1.
\end{aligned}$$

Then V_Λ contains an irreducible K_2-module $W_{\Lambda'}$ with the highest weight Λ' if and only if

(1)

$$\begin{aligned}
p_1 &\geq p'_2 \geq p_3, \\
p_2 &\geq p'_3 \geq p_4, \\
p_3 &\geq p'_4 \geq p_5, \\
p_4 &\geq p'_5 \geq 0;
\end{aligned}$$

(2) The coefficient of $X^{p'_1}$ in the following (finite) power series expansion in X

$$X^{\epsilon\epsilon'\ell_5}\prod_{i=1}^{4}\frac{X^{\ell_i+1}-X^{-\ell_i-1}}{X-X^{-1}}$$

does not vanish. Here

$$\begin{aligned}
\ell_1 &:= p_1 - \max\{p_2, p'_2\},\\
\ell_2 &:= \min\{p_2, p'_2\} - \max\{p_3, p'_3\},\\
\ell_3 &:= \min\{p_3, p'_3\} - \max\{p_4, p'_4\},\\
\ell_4 &:= \min\{p_4, p'_4\} - \max\{p_5, p'_5\},\\
\ell_5 &:= \min\{p_5, p'_5\}.
\end{aligned}$$

Moreover, its multiplicity is equal to the coefficient of $X^{p'_1}$.

11.14.2. *Branching laws of* (K_2, K_1).

$$\begin{aligned}
(K_2, K_1) =&((U(1)\times Spin(2)\cdot Spin(8))/\mathbf{Z}_4 = (S^1\times Spin(2)\cdot Spin(8))/\mathbf{Z}_4,\\
&(S^1\times Spin(2)\cdot(Spin(2)\cdot Spin(6)))/\mathbf{Z}_4).
\end{aligned}$$

Let

$$\begin{aligned}
\Lambda' &= p'_0y_0 + p'_1y_1 + p'_2y_2 + p'_3y_3 + p'_4y_4 + \epsilon'p'_5y_5\\
&\in D(K_2) = D((U(1)\times Spin(2)\cdot Spin(8))/\mathbf{Z}_4)
\end{aligned}$$

with

$$\begin{aligned}
&\frac{1}{2}p'_0 + p'_1 + p'_2 + p'_3 + p'_4 + p'_5 \in 2\mathbf{Z},\ p'_0\in\mathbf{Z},\\
&(p'_1, p'_2, p'_3, p'_4, p'_5)\in\mathbf{Z}^5 + \varepsilon(1,1,1,1,1), \varepsilon = 0 \text{ or } \frac{1}{2},\\
&p'_2\geq p'_3\geq p'_4\geq p'_5\geq 0,\ \epsilon' = 1 \text{ or } -1.
\end{aligned}$$

Then

$$\begin{aligned}
\Lambda' &= \hat{p}'_0\hat{y}_0 + \hat{p}'_1\hat{y}_1 + \hat{p}'_2\hat{y}_2 + \hat{p}'_3\hat{y}_3 + \hat{p}'_4\hat{y}_4 + \hat{\epsilon}'\hat{p}'_5\hat{y}_5\\
&\in D(K_2) = D((S^1\times Spin(2)\cdot Spin(8))/\mathbf{Z}_4),
\end{aligned}$$

where

$$
\begin{aligned}
\hat{p}'_0 &= -\frac{1}{2}p'_0 + 3p'_1,\\
\hat{p}'_1 &= -\frac{1}{4}p'_0 - \frac{1}{2}p'_1,\\
\hat{p}'_2 &= \frac{1}{2}(p'_2 + p'_3 + p'_4 + \epsilon' p'_5),\\
\hat{p}'_3 &= \frac{1}{2}(p'_2 + p'_3 - p'_4 - \epsilon' p'_5),\\
\hat{p}'_4 &= \frac{1}{2}(p'_2 - p'_3 + p'_4 - \epsilon' p'_5),\\
\hat{\epsilon}'\hat{p}'_5 &= \frac{1}{2}(-p'_2 + p'_3 + p'_4 - \epsilon' p'_5),
\end{aligned}
$$

with $\hat{p}'_5 \geq 0$, $\hat{\epsilon}' = 1$ or -1 and $\hat{y}_i$ $(i = 0,1,2,3,4,5)$ are expressed in terms of y_i's by (11.20). Here

$$
\begin{aligned}
&\frac{1}{2}\hat{p}'_0 + \hat{p}'_1 + \hat{p}'_2 + \hat{p}'_3 + \hat{p}'_4 + \hat{\epsilon}'\hat{p}'_5 \in 2\mathbf{Z},\ \hat{p}'_0 \in \mathbf{Z},\\
&(\hat{p}'_1, \hat{p}'_2, \hat{p}'_3, \hat{p}'_4, \hat{p}'_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1), \varepsilon = 0 \text{ or } \frac{1}{2},\\
&\hat{p}'_2 \geq \hat{p}'_3 \geq \hat{p}'_4 \geq \hat{p}'_5 \geq 0,\ \hat{\epsilon}' = 1 \text{ or } -1.
\end{aligned}
$$

Let $\Lambda'' = \hat{p}''_0\hat{y}_0 + \hat{p}''_1\hat{y}_1 + \hat{p}''_2\hat{y}_2 + \hat{p}''_3\hat{y}_3 + \hat{p}''_4\hat{y}_4 + \hat{\epsilon}''\hat{p}''_5\hat{y}_5 \in D(K_1)$, with

$$
\begin{aligned}
&\frac{1}{2}\hat{p}''_0 + \hat{p}''_1 + \hat{p}''_2 + q''_3 + \hat{p}''_4 + \hat{p}''_5 \in 2\mathbf{Z},\ \hat{p}''_0 \in \mathbf{Z},\\
&(\hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2},\\
&\hat{p}''_3 \geq \hat{p}''_4 \geq \hat{p}''_5 \geq 0,\ \hat{\epsilon}'' = 1 \text{ or } -1.
\end{aligned}
$$

Then $W_{\Lambda'}$ contains an irreducible K_1-module $U_{\Lambda''}$ with the highest weight Λ'' if and only if

(1)

$$
\begin{aligned}
\hat{p}''_0 &= \hat{p}'_0,\\
\hat{p}''_1 &= \hat{p}'_1,\\
\hat{p}'_2 &\geq \hat{p}''_3 \geq \hat{p}'_4,\\
\hat{p}'_3 &\geq \hat{p}''_4 \geq \hat{p}'_5,\\
\hat{p}'_4 &\geq \hat{p}''_5 \geq 0.
\end{aligned}
$$

(2) The coefficient of $X^{\hat{p}_2''}$ in the (finite) power series expansion in X

$$X^{\hat{\epsilon}'\hat{\epsilon}''\ell_5}\prod_{i=2}^{4}\frac{X^{\ell_i+1}-X^{-\ell_i-1}}{X-X^{-1}}$$

does not vanish. Here

$$\begin{aligned}
\ell_2 &:= \hat{p}_2' - \max\{\hat{p}_3', \hat{p}_3''\},\\
\ell_3 &:= \min\{\hat{p}_3', \hat{p}_3''\} - \max\{\hat{p}_4', \hat{p}_4''\},\\
\ell_4 &:= \min\{\hat{p}_4', \hat{p}_4''\} - \max\{\hat{p}_5', \hat{p}_5''\},\\
\ell_5 &:= \min\{\hat{p}_5', \hat{p}_5''\}.
\end{aligned}$$

Moreover, its multiplicity is equal to the coefficient of $X^{\hat{p}_2''}$.

11.14.3. *Branching laws of* (K_1, K_0).

$(K_1, K_0) = ((S^1 \times Spin(2)\cdot(Spin(2)\cdot Spin(6)))/\mathbf{Z}_4, (S^1\cdot Spin(6))/\mathbf{Z}_2)$.

Let

$$\begin{aligned}
\Lambda'' &= \hat{p}_0''\hat{y}_0 + \hat{p}_1''\hat{y}_1 + \hat{p}_2''\hat{y}_2 + \hat{p}_3''\hat{y}_3 + \hat{p}_4''\hat{y}_4 + \hat{\epsilon}''\hat{p}_5''\hat{y}_5\\
&\in D(K_1) = D((U(1)\times Spin(2)\cdot(Spin(2)\cdot Spin(6)))/\mathbf{Z}_4)
\end{aligned}$$

with

$$\begin{aligned}
&\frac{1}{2}\hat{p}_0'' + \hat{p}_1'' + \hat{p}_2'' + \hat{p}_3'' + \hat{p}_4'' + \hat{p}_5'' \in 2\mathbf{Z},\ \hat{p}_0'' \in \mathbf{Z},\\
&(\hat{p}_1'', \hat{p}_2'', \hat{p}_3'', \hat{p}_4'', \hat{p}_5'') \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2},\\
&\hat{p}_3'' \geq \hat{p}_4'' \geq \hat{p}_5'' \geq 0,\ \hat{\epsilon}'' = 1 \text{ or } -1.
\end{aligned}$$

Let

$$\Lambda''' = \hat{p}_0'''\hat{y}_0 + \hat{p}_3'''\hat{y}_3 + \hat{p}_4'''\hat{y}_4 + \hat{\epsilon}'''\hat{p}_5'''\hat{y}_5 \in D(K_0) = D((S^1\cdot Spin(6))/\mathbf{Z}_2)$$

with

$$\begin{aligned}
&\hat{p}_0''' \in 2\mathbf{Z}, (\hat{p}_3''', \hat{p}_4''', \hat{p}_5''') \in \mathbf{Z}^3 + \varepsilon(1,1,1), \varepsilon = 0 \text{ or } \frac{1}{2}\\
&\hat{p}_3''' \geq \hat{p}_4''' \geq \hat{p}_5''' \geq 0,\ \hat{\epsilon}''' = 1 \text{ or } -1.
\end{aligned}$$

Then $U_{\Lambda''}$ contains an irreducible K_0-module $U_{\Lambda'''}$ with the highest weight Λ''' if and only if

$$\begin{aligned}
&\hat{p}_0''' = \hat{p}_0'',\\
&\hat{p}_3''' = \hat{p}_3'', \hat{p}_4''' = \hat{p}_4'', \hat{p}_5''' = \hat{p}_5'', \hat{\epsilon}'' = \hat{\epsilon}'''.
\end{aligned}$$

11.15. **Determination of** $D(K, K_0)$**.** Let

$$\begin{aligned}
\Lambda &= p_0y_0 + p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + \epsilon p_5y_5 \in D(K),\\
\Lambda' &= p_0'y_0 + p_1'y_1 + p_2'y_2 + p_3'y_3 + p_4'y_4 + \epsilon' p_5'y_5\\
&= \hat{p}_0'\hat{y}_0 + \hat{p}_1'\hat{y}_1 + \hat{p}_2'\hat{y}_2 + \hat{p}_3'\hat{y}_3 + \hat{p}_4'\hat{y}_4 + \hat{\epsilon}'\hat{p}_5'\hat{y}_5 \in D(K_2),\\
\Lambda'' &= \hat{p}_0''\hat{y}_0 + \hat{p}_1''\hat{y}_1 + \hat{p}_2''\hat{y}_2 + \hat{p}_3''\hat{y}_3 + \hat{p}_4''\hat{y}_4 + \hat{\epsilon}''\hat{p}_5''\hat{y}_5 \in D(K_1),\\
\Lambda''' &= \hat{p}_0'''\hat{y}_0 + \hat{p}_3'''\hat{y}_3 + \hat{p}_4'''\hat{y}_4 + \hat{\epsilon}'''\hat{p}_5'''\hat{y}_5 \in D(K_0).
\end{aligned}$$

Assume that the corresponding representation spaces satisfy

$$V_\Lambda \supset W_{\Lambda'} \supset U_{\Lambda''} = U_{\Lambda'''} \neq \{0\}.$$

Suppose that $U_{\Lambda'''} \neq \{0\}$ is a trivial representation of K_0, that is, $\Lambda''' = 0$. It must be $\dim_{\mathbf{C}} U_{\Lambda'''} = \dim_{\mathbf{C}} U_{\Lambda''} = 1$. Then we have

$$\begin{aligned}
\hat{p}_0''' &= \hat{p}_0'' = 0,\\
\hat{p}_3''' &= \hat{p}_3'' = 0,\\
\hat{p}_4''' &= \hat{p}_4'' = 0,\\
\hat{p}_5''' &= \hat{p}_5'' = 0\ .
\end{aligned}$$

Thus

$$\Lambda'' = \hat{p}_1''\hat{y}_1 + \hat{p}_2''\hat{y}_2 \in D(K_1)$$

with

$$\hat{p}_1'', \hat{p}_2'' \in \mathbf{Z}, \hat{p}_1'' + \hat{p}_2'' \in 2\mathbf{Z}.$$

By the branching laws of (K_2, K_1),

$$\begin{aligned}
\hat{p}_2' \geq \hat{p}_3'' &= 0 \geq \hat{p}_4'\\
\hat{p}_3' \geq \hat{p}_4'' &= 0 \geq \hat{p}_5'\\
\hat{p}_4' \geq \hat{p}_5'' &= 0 \geq 0
\end{aligned}$$

Thus $\hat{p}_4' = 0$ and $\hat{p}_5' = 0$. Hence $(\hat{p}_4', \hat{p}_5') = (0, 0)$ and $\hat{p}_2' \geq 0$, $\hat{p}_3' \geq 0$. Then

$$\begin{aligned}
\ell_2 &= \hat{p}_2' - \max\{\hat{p}_3', \hat{p}_3''\} = \hat{p}_2' - \max\{\hat{p}_3', 0\}\\
&= \hat{p}_2' - \hat{p}_3',\\
\ell_3 &= \min\{\hat{p}_3', \hat{p}_3''\} - \max\{\hat{p}_4', \hat{p}_4''\} = \min\{\hat{p}_3', 0\} - \max\{0, 0\}\\
&= 0 - 0 = 0,\\
\ell_4 &= \min\{\hat{p}_4', \hat{p}_4''\} - \max\{\hat{p}_5', \hat{p}_5''\} = \min\{0, 0\} - \max\{0, 0\}\\
&= 0 - 0 = 0,\\
\ell_5 &= \min\{\hat{p}_5', \hat{p}_5''\} = \min\{0, 0\} = 0.
\end{aligned}$$

Then the coefficient of $X^{\hat{p}''_2}$ in the (finite) power series expansion in X

$$\begin{aligned}
&X^{\hat{\epsilon}'\hat{\epsilon}''\ell_5}\prod_{i=2}^{4}\frac{X^{\ell_i+1}-X^{-\ell_i-1}}{X-X^{-1}}\\
=&X^0\frac{X^{\ell_2+1}-X^{-\ell_2-1}}{X-X^{-1}}\\
=&\frac{X^{\hat{p}'_2-\hat{p}'_3+1}-X^{-(\hat{p}'_2-\hat{p}'_3)-1}}{X-X^{-1}}\\
=&X^{\hat{p}'_2-\hat{p}'_3}+X^{\hat{p}'_2-\hat{p}'_3-2}+\cdots+X^{-(\hat{p}'_2-\hat{p}'_3)}.
\end{aligned}$$

is equal to its multiplicity. Hence we have

$$-(\hat{p}'_2-\hat{p}'_3)\leq\hat{p}''_2=\hat{p}'_2-\hat{p}'_3-2i\leq\hat{p}'_2-\hat{p}'_3$$

for some $i\in\mathbf{Z}$ with $0\leq i\leq\hat{p}'_2-\hat{p}'_3$. Moreover, $\hat{p}'_0=\hat{p}''_0=0,\hat{p}'_1=\hat{p}''_1$. Thus

$$\begin{aligned}
&\Lambda''=\hat{p}''_1\hat{y}_1+\hat{p}''_2\hat{y}_2\in D(K_1),\\
&\Lambda'=\hat{p}'_1\hat{y}_1+\hat{p}'_2\hat{y}_2+\hat{p}'_3\hat{y}_3\in D(K_2).
\end{aligned}$$

with

$$\begin{aligned}
&\hat{p}'_1=\hat{p}''_1,\hat{p}'_2,\hat{p}'_3,\hat{p}''_2\in\mathbf{Z},\ \hat{p}'_1+\hat{p}'_2+\hat{p}'_3\in2\mathbf{Z},\ \hat{p}''_1+\hat{p}''_2\in2\mathbf{Z},\\
&-(\hat{p}'_2-\hat{p}'_3)\leq\hat{p}''_2=\hat{p}'_2-\hat{p}'_3-2i\leq\hat{p}'_2-\hat{p}'_3
\end{aligned}$$

for some $i\in\mathbf{Z}$ with $0\leq i\leq\hat{p}'_2-\hat{p}'_3$. Therefore,

$$\Lambda'=p'_0y_0+p'_1y_1+p'_2y_2+p'_3y_3+p'_4y_4+\epsilon'p'_5y_5\in D(K_2)$$

with

$$\begin{aligned}
p'_0&=-\frac{1}{2}\hat{p}'_0-3\hat{p}'_1=-3\hat{p}'_1,\\
p'_1&=\frac{1}{4}\hat{p}'_0-\frac{1}{2}\hat{p}'_1=-\frac{1}{2}\hat{p}'_1,\\
p'_2&=\frac{1}{2}(\hat{p}'_2+\hat{p}'_3+\hat{p}'_4-\hat{\epsilon}'\hat{p}'_5)=\frac{1}{2}(\hat{p}'_2+\hat{p}'_3),\\
p'_3&=\frac{1}{2}(\hat{p}'_2+\hat{p}'_3-\hat{p}'_4+\hat{\epsilon}'\hat{p}'_5)=\frac{1}{2}(\hat{p}'_2+\hat{p}'_3),\\
p'_4&=\frac{1}{2}(\hat{p}'_2-\hat{p}'_3+\hat{p}'_4+\hat{\epsilon}'\hat{p}'_5)=\frac{1}{2}(\hat{p}'_2-\hat{p}'_3),\\
\epsilon'p'_5&=\frac{1}{2}(\hat{p}'_2-\hat{p}'_3-\hat{p}'_4-\hat{\epsilon}'\hat{p}'_5)=\frac{1}{2}(\hat{p}'_2-\hat{p}'_3).
\end{aligned}$$

In particular,

$$\epsilon' = 1,$$
$$p'_2 = p'_3 = \frac{1}{2}(\hat{p}'_2 + \hat{p}'_3),$$
$$p'_4 = p'_5 = \frac{1}{2}(\hat{p}'_2 - \hat{p}'_3).$$

By the branching laws of (K, K_2),

$$\begin{aligned}
&p_1 \geq p'_2 \geq p_3 \\
&p_2 \geq p'_3 = p'_2 \geq p_4 \\
&p_3 \geq p'_4 \geq p_5 \\
&p_4 \geq p'_5 = p'_4 \geq 0
\end{aligned}$$

Thus

$$p_1 \geq p_2 \geq p'_2 = p'_3 \geq p_3 \geq p_4 \geq p'_4 = p'_5 \geq p_5 \geq 0.$$

Then

$$\begin{aligned}
&\ell_1 = p_1 - \max\{p_2, p'_2\} = p_1 - p_2, \\
&\ell_2 = \min\{p_2, p'_2\} - \max\{p_3, p'_3\} = p'_2 - p'_3 = 0, \\
&\ell_3 = \min\{p_3, p'_3\} - \max\{p_4, p'_4\} = p_3 - p_4, \\
&\ell_4 = \min\{p_4, p'_4\} - \max\{p_5, p'_5\} = p'_4 - p'_5 = 0, \\
&\ell_5 = \min\{p_5, p'_5\} = p_5.
\end{aligned}$$

Then the coefficient of $X^{p'_1} = X^{-\frac{1}{2}\hat{p}'_1} = X^{-\frac{1}{2}\hat{p}''_1}$ in the (finite) power series expansion in X

$$\begin{aligned}
&X^{\epsilon\epsilon'\ell_5} \prod_{i=1}^{4} \frac{X^{\ell_i+1} - X^{-\ell_i-1}}{X - X^{-1}} \\
=&X^{\epsilon\epsilon' p_5} \frac{X^{p_1-p_2+1} - X^{-(p_1-p_2+1)}}{X - X^{-1}} \frac{X^{p_3-p_4+1} - X^{-(p_3-p_4+1)}}{X - X^{-1}} \\
=&X^{\epsilon\epsilon' p_5} (X^{p_1-p_2} + \cdots + X^{p_1-p_2-2i} + \cdots + X^{-(p_1-p_2)}) \\
&\qquad (X^{p_3-p_4} + \cdots + X^{p_3-p_4-2j} + \cdots + X^{-(p_3-p_4)}) \\
=&X^{\epsilon\epsilon' p_5} \sum_{i=0}^{p_1-p_2} X^{p_1-p_2-2i} \sum_{j=0}^{p_3-p_4} X^{p_3-p_4-2j} \\
=&X^{\epsilon\epsilon' p_5} \sum_{i=0}^{p_1-p_2} \sum_{j=0}^{p_3-p_4} X^{(p_1-p_2)+(p_3-p_4)-2(i+j)}
\end{aligned}$$

is equal to its multiplicity.

Then

$$\Lambda = p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + \epsilon p_5 y_5 \in D(K)$$

satisfies $p_0 = p_0' = -3\hat{p}_1' = 6p_1' \in 3\mathbf{Z}$.

11.16. **Eigenvalue formula of $\mathcal{C}_L$.**

$$\mathcal{C}_L = 12C_{K/K_0} - 6C_{K_2/K_0} - 3C_{K_1/K_0}.$$

Recall that the description of the maximal torus of K, with respect to the inner product $\langle A, B\rangle = -\mathrm{tr}AB$ for $A, B \in K \subset E_6$, the orthogonal basis

$$\mathbf{e}_0 = (1,0,0,0,0,0) \in \mathfrak{t} = \{(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \mid \theta_i \in \mathbf{R}\}$$

corresponds to $2\sqrt{-1}R(2e_1 - e_2 - e_3)$ and $\mathbf{e}_1 = (0,1,0,0,0,0) \in \mathfrak{t}$ corresponds to $\sqrt{-1}R(e_2 - e_3) \in spin(10)$. Moreover,

$$\begin{aligned}
\langle \mathbf{e}_0, \mathbf{e}_0\rangle &= \langle 2\sqrt{-1}R(2e_1 - e_2 - e_3), 2\sqrt{-1}R(2e_1 - e_2 - e_3)\rangle = 4 \times 18 = 72, \\
\langle \mathbf{e}_1, \mathbf{e}_1\rangle &= \langle \sqrt{-1}R(e_2 - e_3), \sqrt{-1}R(e_2 - e_3)\rangle = 6.
\end{aligned}$$

It follows that the inner products of the dual bases $\{y_0, y_1, y_2, y_3, y_4, y_5\}$ of $\mathfrak{t}^*$ of $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ of $\mathfrak{t}$ are given by

$$\begin{aligned}
&\langle y_\alpha, y_\beta\rangle = 0, \quad (0 \le \alpha \ne \beta \le 5), \\
&\langle y_0, y_0\rangle = \frac{1}{72}, \quad \langle y_i, y_j\rangle = \frac{1}{6}, \quad (1 \le i \ne j \le 5).
\end{aligned}$$

For

$$\begin{aligned}
\Lambda &= p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + \epsilon p_5 y_5 \in D(K, K_0), \\
\Lambda' &= p_0 y_0 + \frac{p_0}{6} y_1 + p_2' y_2 + p_2' y_3 + p_4' y_4 + p_4' y_5 \\
&= -\frac{p_0}{3}\hat{y}_1 + (p_2' + p_4')\hat{y}_2 + (p_2' - p_4')\hat{y}_3 \in D(K_2, K_0), \\
\Lambda'' &= -\frac{p_0}{3}\hat{y}_1 + \hat{p}_2''\hat{y}_2 \in D(K_1, K_0),
\end{aligned}$$

$$\begin{aligned}
&\Sigma^+(K) = \{y_i \pm y_j (1 \le i < j \le 5), y_i + y_5\} \\
&2\delta_K = \Sigma_{\alpha>0}\alpha = 8y_1 + 6y_2 + 4y_3 + 2y_4, \\
&C_K(\Lambda) = \langle \Lambda + 2\delta_K, \Lambda\rangle \\
&= \frac{1}{72}p_0^2 + \frac{1}{6}\{(p_1 + 8)p_1 + (p_2 + 6)p_2 + (p_3 + 4)p_3 + (p_4 + 2)p_4 + (p_5)^2\}
\end{aligned}$$

$$\begin{aligned}
&\Sigma^+(K_2) = \{y_i \pm y_j (2 \le i < j \le 5)\}\\
&2\delta_{K_2} = \Sigma_{\alpha>0}\alpha = 6y_2 + 4y_3 + 2y_4,\\
&C_{K_2}(\Lambda') = \langle \Lambda' + 2\delta_{K_2}, \Lambda' \rangle\\
=&\frac{1}{72}(p_0')^2 + \frac{1}{6}\{(p_1')^2 + (p_2'+6)p_2' + (p_3'+4)p_3' + (p_4'+2)p_4' + (p_5')^2\}\\
=&\frac{1}{72}(\hat{p}_0')^2 + \frac{1}{6}\{(\hat{p}_1')^2 + (\hat{p}_2'+6)\hat{p}_2' + (\hat{p}_3'+4)\hat{p}_3' + (\hat{p}_4'+2)\hat{p}_4' + (\hat{p}_5')^2\},
\end{aligned}$$

$$\begin{aligned}
&\Sigma^+(K_1) = \{y_i \pm y_j (3 \le i < j \le 5)\}\\
&2\delta_{K_1} = \Sigma_{\alpha>0}\alpha = 4y_3 + 2y_4,\\
&C_{K_1}(\Lambda'') = \langle \Lambda'' + 2\delta_{K_1}, \Lambda'' \rangle\\
=&\frac{1}{72}(\hat{p}_0'')^2 + \frac{1}{6}\{(\hat{p}_1'')^2 + (\hat{p}_2'')^2 + (\hat{p}_3''+4)\hat{p}_3'' + (\hat{p}_4''+2)\hat{p}_4'' + (\hat{p}_5'')^2\}.
\end{aligned}$$

Then for each $\Lambda \in D(K, K_0)$,

$$\begin{aligned}
C_L =&2\{(p_1+8)p_1 + (p_2+6)p_2 + (p_3+4)p_3 + (p_4+2)p_4 + (p_5)^2\}\\
&- \{(p_2'+6)p_2' + (p_2'+4)p_2' + (p_4'+2)p_4' + (p_4')^2\} - \frac{1}{2}(\hat{p}_2'')^2\\
=&2(p_1+8)p_1 + 2((p_2)^2 - (p_2')^2) + 12p_2 - 10p_2' + 2(p_3)^2 + 8p_3\\
&+ 2((p_4)^2 - (p_4')^2) + 4p_4 - 2p_4' + 2(p_5)^2 - \frac{1}{2}(\hat{p}_2'')^2\\
=&2(p_1+8)p_1 + 2((p_2)^2 - (p_2')^2) + 2p_2 + 10(p_2 - p_2') + 2(p_3)^2 + 8p_3\\
&+ 2((p_4)^2 - (p_4')^2) + 2p_4 + 2(p_4 - p_4') + 2(p_5)^2 - \frac{1}{2}(\hat{p}_2'')^2\\
\ge&2(p_1+8)p_1 + 2p_2 + 2(p_3)^2 - \frac{1}{2}(\hat{p}_2'')^2 + 8p_3 + 2p_4 + 2(p_5)^2\\
=&2(p_1+8)p_1 + 2p_2 + (2(p_5')^2 - \frac{1}{2}(\hat{p}_2'')^2) + 8p_3 + 2p_4 + 2(p_5)^2\\
\ge&2(p_1+8)p_1 + 2p_2 + 8p_3 + 2p_4 + 2(p_5)^2\\
&\quad (\text{in case } p_2 = p_2',\ p_4 = p_4',\ 2p_3 = 2p_4 = 2p_4' = 2p_5' = |\hat{p}_2''|).
\end{aligned}$$

Here

$$\begin{aligned}
&p_1 \ge p_2 \ge p_2' = p_3' \ge p_3 \ge p_4 \ge p_4' = p_5' \ge p_5 \ge 0,\\
&-2p_4' = -2p_5' = -(\hat{p}_2' - \hat{p}_3') \le \hat{p}_2'' \le \hat{p}_2' - \hat{p}_3' = 2p_5' = 2p_4'.
\end{aligned}$$

Notice that if $p_1 = 0$, then $C_L = 0$ and if $p_1 \geq 2$, then $C_L \geq 40 > 30$. In case $p_1 = \frac{3}{2}$, $(p_0, p_1, p_2, p_3, p_4, p_5) =$

$$(p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}), (p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2}), (p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}), (p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{1}{2}),$$
$$(p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}), (p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}), (p_0, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (p_0, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}),$$
$$(p_0, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (p_0, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}).$$

In these cases,

$$\begin{aligned} C_L \geq & 2(p_1 + 8)p_1 + 2p_2 + 8p_3 + 2p_4 + 2(p_5)^2 \\ \geq & 2 \cdot (\frac{3}{2} + 8) \cdot \frac{3}{2} + 2 \cdot \frac{1}{2} + 8 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} + 2 \cdot (\frac{1}{2})^2 \\ = & 35 > 30. \end{aligned}$$

Hence in order to decide the Hamiltonian stability, i.e., to compare the first eigenvalue C_L and 30, we can assume that $p_1 = \frac{1}{2}$ or 1 in the following.

11.16.1. *List of* $\Lambda \in D(K, K_0)$, $\Lambda' \in D(K_2, K_0)$, $\Lambda'' \in D(K_1, K_0)$ *with* $p_1 = \frac{1}{2}$ *or* 1. In case $p_1 = \frac{1}{2}$,

$$(p_0, p_1, p_2, p_3, p_4, p_5) = (p_0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \text{ or } (p_0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}).$$

If $(p_0, p_1, p_2, p_3, p_4, p_5) = (p_0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, then

$$\begin{aligned} &(p'_0, p'_1, p'_2, p'_3, p'_4, p'_5) = (3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ &(\hat{p}'_0, \hat{p}'_1, \hat{p}'_2, \hat{p}'_3, \hat{p}'_4, \hat{p}'_5) = (0, -1, 1, 0, 0, 0), \\ &(\hat{p}''_0, \hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) = (0, -1, 1, 0, 0, 0) \text{ or } (0, -1, -1, 0, 0, 0). \end{aligned}$$

If $(p_0, p_1, p_2, p_3, p_4, p_5) = (p_0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, then

$$\begin{aligned} &(p'_0, p'_1, p'_2, p'_3, p'_4, p'_5) = (-3, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \\ &(\hat{p}'_0, \hat{p}'_1, \hat{p}'_2, \hat{p}'_3, \hat{p}'_4, \hat{p}'_5) = (0, 1, 1, 0, 0, 0), \\ &(\hat{p}''_0, \hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) = (0, 1, 1, 0, 0, 0) \text{ or } (0, 1, -1, 0, 0, 0). \end{aligned}$$

In case $p_1 = 1$,

$$
\begin{aligned}
(p_0,p_1,p_2,p_3,p_4,p_5) =&(p_0,1,1,1,1,1),\\
&(p_0,1,1,1,1,-1),\\
&(p_0,1,1,1,1,0),\\
&(p_0,1,1,1,0,0),\\
&(p_0,1,1,0,0,0),\\
&(p_0,1,0,0,0,0).
\end{aligned}
$$

If $(p_0,p_1,p_2,p_3,p_4,p_5) = (p_0,1,1,1,1,1)$, then

$(p'_0,p'_1,p'_2,p'_3,p'_4,p'_5) = (6,1,1,1,1,1)$,

$(\hat{p}'_0,\hat{p}'_1,\hat{p}'_2,\hat{p}'_3,\hat{p}'_4,\hat{p}'_5) = (0,-2,2,0,0,0)$,

$(\hat{p}''_0,\hat{p}''_1,\hat{p}''_2,\hat{p}''_3,\hat{p}''_4,\hat{p}''_5) = (0,-2,2,0,0,0), (0,-2,-2,0,0,0)$ or $(0,-2,0,0,0,0)$.

If $(p_0,p_1,p_2,p_3,p_4,p_5) = (p_0,1,1,1,1,-1)$, then

$(p'_0,p'_1,p'_2,p'_3,p'_4,p'_5) = (6,-1,1,1,1,1)$,

$(\hat{p}'_0,\hat{p}'_1,\hat{p}'_2,\hat{p}'_3,\hat{p}'_4,\hat{p}'_5) = (0,2,2,0,0,0)$,

$(\hat{p}''_0,\hat{p}''_1,\hat{p}''_2,\hat{p}''_3,\hat{p}''_4,\hat{p}''_5) = (0,2,2,0,0,0), (0,2,-2,0,0,0)$ or $(0,2,0,0,0,0)$.

If $(p_0,p_1,p_2,p_3,p_4,p_5) = (p_0,1,1,1,1,0)$, then

$(p'_0,p'_1,p'_2,p'_3,p'_4,p'_5) = (0,0,1,1,0,0)$,

$(\hat{p}'_0,\hat{p}'_1,\hat{p}'_2,\hat{p}'_3,\hat{p}'_4,\hat{p}'_5) = (0,0,1,1,0,0)$,

$(\hat{p}''_0,\hat{p}''_1,\hat{p}''_2,\hat{p}''_3,\hat{p}''_4,\hat{p}''_5) = (0,0,0,0,0,0)$

or

$(p'_0,p'_1,p'_2,p'_3,p'_4,p'_5) = (0,0,1,1,1,1)$,

$(\hat{p}'_0,\hat{p}'_1,\hat{p}'_2,\hat{p}'_3,\hat{p}'_4,\hat{p}'_5) = (0,0,2,0,0,0)$,

$(\hat{p}''_0,\hat{p}''_1,\hat{p}''_2,\hat{p}''_3,\hat{p}''_4,\hat{p}''_5) = (0,0,2,0,0,0), (0,0,-2,0,0,0)$ or $(0,0,0,0,0,0)$.

If $(p_0,p_1,p_2,p_3,p_4,p_5) = (p_0,1,1,1,0,0)$, then

$$
\begin{aligned}
&(p'_0,p'_1,p'_2,p'_3,p'_4,p'_5) = (6,1,1,1,0,0),\\
&(\hat{p}'_0,\hat{p}'_1,\hat{p}'_2,\hat{p}'_3,\hat{p}'_4,\hat{p}'_5) = (0,-2,1,1,0,0),\\
&(\hat{p}''_0,\hat{p}''_1,\hat{p}''_2,\hat{p}''_3,\hat{p}''_4,\hat{p}''_5) = (0,-2,0,0,0,0)
\end{aligned}
$$

or

$$
\begin{aligned}
&(p'_0,p'_1,p'_2,p'_3,p'_4,p'_5) = (-6,-1,1,1,0,0),\\
&(\hat{p}'_0,\hat{p}'_1,\hat{p}'_2,\hat{p}'_3,\hat{p}'_4,\hat{p}'_5) = (0,2,1,1,0,0),\\
&(\hat{p}''_0,\hat{p}''_1,\hat{p}''_2,\hat{p}''_3,\hat{p}''_4,\hat{p}''_5) = (0,2,0,0,0,0).
\end{aligned}
$$

If $(p_0, p_1, p_2, p_3, p_4, p_5) = (p_0, 1, 1, 0, 0, 0)$, then

$$
\begin{aligned}
&(p'_0, p'_1, p'_2, p'_3, p'_4, p'_5) = (0, 0, 0, 0, 0, 0),\\
&(\hat{p}'_0, \hat{p}'_1, \hat{p}'_2, \hat{p}'_3, \hat{p}'_4, \hat{p}'_5) = (0, 0, 0, 0, 0, 0),\\
&(\hat{p}''_0, \hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) = (0, 0, 0, 0, 0, 0)\\
&\text{or}\\
&(p'_0, p'_1, p'_2, p'_3, p'_4, p'_5) = (0, 0, 1, 1, 0, 0),\\
&(\hat{p}'_0, \hat{p}'_1, \hat{p}'_2, \hat{p}'_3, \hat{p}'_4, \hat{p}'_5) = (0, 0, 1, 1, 0, 0),\\
&(\hat{p}''_0, \hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) = (0, 0, 0, 0, 0, 0).
\end{aligned}
$$

If $(p_0, p_1, p_2, p_3, p_4, p_5) = (p_0, 1, 0, 0, 0, 0)$, then

$$
\begin{aligned}
&(p'_0, p'_1, p'_2, p'_3, p'_4, p'_5) = (6, 1, 0, 0, 0, 0),\\
&(\hat{p}'_0, \hat{p}'_1, \hat{p}'_2, \hat{p}'_3, \hat{p}'_4, \hat{p}'_5) = (0, -2, 0, 0, 0, 0),\\
&(\hat{p}''_0, \hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) = (0, -2, 0, 0, 0, 0)\\
&\text{or}\\
&(p'_0, p'_1, p'_2, p'_3, p'_4, p'_5) = (-6, -1, 0, 0, 0, 0),\\
&(\hat{p}'_0, \hat{p}'_1, \hat{p}'_2, \hat{p}'_3, \hat{p}'_4, \hat{p}'_5) = (0, 2, 0, 0, 0, 0),\\
&(\hat{p}''_0, \hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) = (0, 2, 0, 0, 0, 0).
\end{aligned}
$$

By the direct computation we get the following small eigenvalues in the above cases.

Λ	Λ'	Λ''	C_L
$3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, -1, 1, 0, 0, 0$	15
$3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, -1, -1, 0, 0, 0$	15
$-3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	$-3, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, 1, 1, 0, 0, 0$	15
$-3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	$-3, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, 1, -1, 0, 0, 0$	15
$6, 1, 0, 0, 0, 0$	$6, 1, 0, 0, 0, 0$	$0, -2, 0, 0, 0, 0$	18
$6, 1, 0, 0, 0, 0$	$6, -1, 0, 0, 0, 0$	$0, 2, 0, 0, 0, 0$	18
$0, 1, 1, 0, 0, 0$	$0, 0, 0, 0, 0, 0$	$0, 0, 0, 0, 0, 0$	32
$0, 1, 1, 0, 0, 0$	$0, 0, 1, 1, 0, 0$	$0, 0, 0, 0, 0, 0$	20
$6, 1, 1, 1, 0, 0$	$6, 1, 1, 1, 0, 0$	$0, -2, 0, 0, 0, 0$	30
$-6, 1, 1, 1, 0, 0$	$-6, -1, -1, 1, 0, 0$	$0, 2, 0, 0, 0, 0$	30
$0, 1, 1, 1, 1, 0$	$0, 0, 1, 1, 0, 0$	$0, 0, 0, 0, 0, 0$	36
$0, 1, 1, 1, 1, 0$	$0, 0, 1, 1, 1, 1$	$0, 0, 0, 0, 0, 0$	32
$0, 1, 1, 1, 1, 0$	$0, 0, 1, 1, 1, 1$	$0, 0, 2, 0, 0, 0$	30
$0, 1, 1, 1, 1, 0$	$0, 0, 1, 1, 1, 1$	$0, 0, -2, 0, 0, 0$	30
$6, 1, 1, 1, 1, 1$	$6, 1, 1, 1, 1, 1$	$0, -2, 2, 0, 0, 0$	32
$6, 1, 1, 1, 1, 1$	$6, 1, 1, 1, 1, 1$	$0, -2, -2, 0, 0, 0$	32
$6, 1, 1, 1, 1, 1$	$6, 1, 1, 1, 1, 1$	$0, -2, 0, 0, 0, 0$	34
$6, 1, 1, 1, 1, -1$	$6, -1, 1, 1, 1, 1$	$0, 2, 2, 0, 0, 0$	32
$6, 1, 1, 1, 1, -1$	$6, -1, 1, 1, 1, 1$	$0, 2, -2, 0, 0, 0$	32
$6, 1, 1, 1, 1, -1$	$6, -1, 1, 1, 1, 1$	$0, 2, 0, 0, 0, 0$	34

Here, $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) \in D(K, K_0)$, $\Lambda' = (p'_0, p'_1, p'_2, p'_3, p'_4, p'_5) \in D(K_2, K_0)$, and $\Lambda'' = (\hat{p}''_0, \hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) \in D(K_1, K_0)$.

11.16.2. *On* $(p_0, p_1, p_2, p_3, p_4, p_5) = (0, 1, 1, 0, 0, 0)$. In case

$$\begin{aligned}
\Lambda &= (p_0, p_1, p_2, p_3, p_4, p_5) = (0, 1, 1, 0, 0, 0), \\
\Lambda' &= (p'_0, p'_1, p'_2, p'_3, p'_4, p'_5) = (0, 0, 1, 1, 0, 0), \\
&= (\hat{p}'_0, \hat{p}'_1, \hat{p}'_2, \hat{p}'_3, \hat{p}'_4, \hat{p}'_5) = (0, 0, 1, 1, 0, 0), \\
\Lambda'' &= (\hat{p}''_0, \hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) = (0, 0, 0, 0, 0, 0),
\end{aligned}$$

we have

$$\begin{aligned}
\rho'_{\Lambda'} =& \mathrm{Id} \boxtimes \mathrm{Id} \boxtimes \mathrm{Ad}^{\mathbf{C}}_{Spin(8)} = \mathrm{Id} \boxtimes \mathrm{Id} \boxtimes \mathrm{Ad}^{\mathbf{C}}_{SO(8)} \\
&\in \mathcal{D}(K_2) = \mathcal{D}((U(1) \times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4).
\end{aligned}$$

Notice that $\mathfrak{o}(8) = \mathfrak{o}(2) \oplus \mathfrak{o}(6) \oplus M(2, 6; \mathbf{R})$, $W_{\Lambda'} = \mathfrak{o}(8)^{\mathbf{C}} = \mathfrak{o}(2)^{\mathbf{C}} \oplus \mathfrak{o}(6)^{\mathbf{C}} \oplus M(2, 6; \mathbf{R})^{\mathbf{C}}$, and the subgroups $U(1)$ and $Spin(2)$ of $K_2 = (U(1) \times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4$ acts trivially on $\mathfrak{o}(8)^{\mathbf{C}}$. The subgroup $Spin(6)$ of $Spin(2) \cdot Spin(6)$ acts trivially on $\mathfrak{o}(2)^{\mathbf{C}}$, hence

$$(W_{\Lambda'})_{K_0} = \mathfrak{o}(2)^{\mathbf{C}},$$

that is, $\Lambda \in D(K, K_0)$.

$$\alpha_{23}(\pi)(\alpha_1,\alpha_2,\alpha_3) \in K_{[\mathfrak{a}]}$$

which corresponds to a generator of $\mathbf{Z}_4$. $\alpha_{23}(\pi) \in Spin(2)$ and $(\alpha_1,\alpha_2,\alpha_3)$ commute each other. $\alpha_{23}(\pi) \in Spin(2)$ acts trivially on $\mathfrak{o}(2)^{\mathbf{C}}$. α_2 of $(\alpha_1,\alpha_2,\alpha_3)$ acts on $\mathbf{R}1+\mathbf{R}e$ as $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and preserves the vector subspace orthogonally complementary to $\mathbf{R}1+\mathbf{R}e$ in $\mathbb{K} \cong \mathbf{R}^8$. Thus the $Spin(2)$-factor of $(\alpha_1,\alpha_2,\alpha_3)$ in $Spin(2)\cdot Spin(6)$ corresponds to

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in O(2).$$

Since the adjoint action of

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in SO(2)$$

on $\mathfrak{o}(2)^{\mathbf{C}}$ is $-\mathrm{Id}$, the adjoint action of $(\alpha_1,\alpha_2,\alpha_3) \in Spin(8)$ is not trivial on $\mathfrak{o}(2)^{\mathbf{C}}$. Hence

$$(W_{\Lambda'})_{K_{[\mathfrak{a}]}} = \{0\}$$

and in particular we obtain $\Lambda = y_1+y_2 \notin D(K,K_{[\mathfrak{a}]})$.

11.16.3. *On* $(p_0,p_1,p_2,p_3,p_4,p_5)=(6,1,0,0,0,0)$.

$$\begin{aligned}
&\Lambda = 6y_0+y_1 \in D(K,K_0),\\
&\Lambda' = 6y_0+y_1 = -2\hat{y}_1 \in D(K_2,K_0),\\
&\Lambda'' = -2\hat{y}_1 \in D(K_1,K_0),\\
&\Lambda''' = 0 \in D(K_0)
\end{aligned}$$

$C_L = 18 < 30$.

In this case, $V_\Lambda \cong \mathbf{C}\otimes\mathbf{C}^{10} \cong \mathbf{C}^{10}$ and $\rho_\Lambda = \mu_6 \boxtimes \sigma_{\mathbf{C}^{10}}$, where $\sigma_{\mathbf{C}^{10}}$ denotes the standard representation of $SO(10)$.

$$\begin{aligned}
V_\Lambda \cong &\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} \mid \xi_2,\xi_3 \in \mathbf{C}, x_1 \in \mathbb{K}^{\mathbf{C}} \right\} \cong \mathbf{C}^{10}\\
&\supset W_{\Lambda'} = U_{\Lambda''} = U_{\Lambda'''} = (V_\Lambda)_{K_0}
\end{aligned}$$

Note that $\frac{t_0}{2}=\theta_0$, $t_1=\theta_1$. For each $\phi(\theta) \in U(1)$,

$$\mu_6(\phi(\theta)) \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} = \theta^{-6} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix},$$

where $\theta = e^{\sqrt{-1}t_0/2}$. Since

$$\begin{aligned}&\exp(\hat{t}_0\sqrt{-1}R(e_1-2e_2+e_3))\\=&\exp(\hat{t}_0\frac{1}{2}\sqrt{-1}R(2e_1-e_2-e_3))\exp(-\hat{t}_0\frac{3}{2}\sqrt{-1}R(e_2-e_3)),\end{aligned}$$

we compute

$$\begin{aligned}
&\rho_\Lambda(\exp(\hat{t}_0\sqrt{-1}R(e_1-2e_2+e_3)))\begin{pmatrix}0&0&0\\0&\xi_2&x_1\\0&\bar{x}_1&\xi_3\end{pmatrix}\\
=&(\mu_6\boxtimes\sigma_{\mathbf{C}^{10}})(\exp(\hat{t}_0\sqrt{-1}R(e_1-2e_2+e_3)))\begin{pmatrix}0&0&0\\0&\xi_2&x_1\\0&\bar{x}_1&\xi_3\end{pmatrix}\\
=&(\mu_6\boxtimes\sigma_{\mathbf{C}^{10}})(\exp(\hat{t}_0\frac{1}{2}\sqrt{-1}R(2e_1-e_2-e_3))\exp(-\hat{t}_0\frac{3}{2}\sqrt{-1}R(e_2-e_3)))\begin{pmatrix}0&0&0\\0&\xi_2&x_1\\0&\bar{x}_1&\xi_3\end{pmatrix}\\
=&\mu_6(\exp(\hat{t}_0\frac{1}{2}\sqrt{-1}R(2e_1-e_2-e_3)))\sigma_{\mathbf{C}^{10}}(\exp(-\hat{t}_0\frac{3}{2}\sqrt{-1}R(e_2-e_3))\begin{pmatrix}0&0&0\\0&\xi_2&x_1\\0&\bar{x}_1&\xi_3\end{pmatrix}\\
=&\mu_6(\exp(\hat{t}_0\frac{1}{2}\sqrt{-1}R(2e_1-e_2-e_3)))\alpha_{23}(-\hat{t}_0\frac{3}{2})\begin{pmatrix}0&0&0\\0&\xi_2&x_1\\0&\bar{x}_1&\xi_3\end{pmatrix}\\
=&(e^{\sqrt{-1}\frac{1}{2}\hat{t}_0\frac{1}{2}})^{-6}\begin{pmatrix}0&0&0\\0&e^{-\sqrt{-1}\hat{t}_0\frac{3}{2}}\xi_2&x_1\\0&\bar{x}_1&e^{\sqrt{-1}\hat{t}_0\frac{3}{2}}\xi_3\end{pmatrix}\\
=&e^{-\sqrt{-1}\frac{3}{2}\hat{t}_0}\begin{pmatrix}0&0&0\\0&e^{-\sqrt{-1}\hat{t}_0\frac{3}{2}}\xi_2&x_1\\0&\bar{x}_1&e^{\sqrt{-1}\hat{t}_0\frac{3}{2}}\xi_3\end{pmatrix}\\
=&\begin{pmatrix}0&0&0\\0&e^{-\sqrt{-1}3\hat{t}_0}\xi_2&e^{-\sqrt{-1}\frac{3}{2}\hat{t}_0}x_1\\0&e^{-\sqrt{-1}\frac{3}{2}\hat{t}_0}\bar{x}_1&\xi_3\end{pmatrix}.
\end{aligned}$$

In particular,

$$\rho_\Lambda(\exp(\hat{t}_0\sqrt{-1}R(e_1-2e_2+e_3)))\begin{pmatrix}0&0&0\\0&0&0\\0&0&\xi_3\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\xi_3\end{pmatrix}$$

for each $\hat{t}_0 \in \mathbf{R}$. Hence,

$$(V_\Lambda)_{K_0} \cong \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} \mid \xi_3 \in \mathbf{C} \right\}.$$

But as a generator of $\mathbf{Z}_4$ of $K_{[\mathfrak{a}]}$, the action of

$$\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3) \in K_{[\mathfrak{a}]}$$

is given by

$$\begin{aligned} &(\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3)) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} \\ =&(\alpha_{23}(\pi)) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\xi_3 \end{pmatrix}. \end{aligned}$$

Therefore $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$ and $\Lambda = 6y_0 + y_1 \notin D(K, K_{[\mathfrak{a}]})$. Similarly, $\Lambda = -6y_0 + y_1 \notin D(K, K_{[\mathfrak{a}]})$.

11.16.4. *On* $(p_0, p_1, p_2, p_3, p_4, p_5) = (6, 1, 1, 1, 0, 0)$.

$$\begin{aligned} &\Lambda = 6y_0 + y_1 + y_2 + y_3 \in D(K, K_0), \\ &\Lambda' = 6y_0 + y_1 + y_2 + y_3 \in D(K_2, K_0), \\ &\kappa\Lambda' = -2\hat{y}_1 + \hat{y}_2 + \hat{y}_3 \in D(K_2, K_0), \\ &\Lambda'' = -2\hat{y}_1 \in D(K_1, K_0), \\ &\Lambda''' = 0 \in D(K_0) \end{aligned}$$

$C_L = 30$.

In this case,

$$\begin{aligned} \rho'_{\Lambda'} =&\mathrm{Id} \boxtimes \mu_{-2} \boxtimes \mathrm{Ad}^{\mathbf{C}}_{Spin(8)} = \mathrm{Id} \boxtimes \mu_{-2} \boxtimes \mathrm{Ad}^{\mathbf{C}}_{SO(8)} \\ &\in \mathcal{D}(K_2) = \mathcal{D}((S^1 \times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4). \end{aligned}$$

Here $W_{\Lambda'} = \mathfrak{o}(8)^{\mathbf{C}} = \mathfrak{o}(2)^{\mathbf{C}} \oplus \mathfrak{o}(6)^{\mathbf{C}} \oplus M(2,6;\mathbf{R})^{\mathbf{C}}$, and the subgroups S^1 of $K_2 = (S^1 \times (Spin(2) \cdot Spin(8))/\mathbf{Z}_4$ acts trivially on $\mathfrak{o}(8)^{\mathbf{C}}$. The subgroup $Spin(6)$ of $K_0 = S^1 \cdot Spin(6) \subset K_2$ acts trivially on $\mathfrak{o}(2)^{\mathbf{C}}$, hence

$$(W_{\Lambda'})_{K_0} = \mathfrak{o}(2)^{\mathbf{C}},$$

that is, $\Lambda \in D(K, K_0)$.

Notice that

$$\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3) \in K_{[\mathfrak{a}]} \subset K_2,$$

which corresponds to a generator of $\mathbf{Z}_4$. $\alpha_{23}(\pi) \in Spin(2)$ and $(\alpha_1, \alpha_2, \alpha_3)$ commute each other. The action of $\alpha_{23}(\pi) \in Spin(2)$ on $H_3(\mathbb{K}^{\mathbf{C}})$ is given by

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 & \sqrt{-1}x_3 & -\sqrt{-1}\bar{x}_2 \\ \sqrt{-1}\bar{x}_3 & -\xi_2 & x_1 \\ -\sqrt{-1}x_2 & \bar{x}_1 & -\xi_3 \end{pmatrix}.$$

In particular, $\alpha_{23}(\pi)$ transforms u_2 to $-\sqrt{-1}u_2$ and $\mathbf{e}u_2$ to $-\sqrt{-1}\mathbf{e}u_2$, which says that $\alpha_{23}(\pi)$ acts on $\mathfrak{o}(2) \cong \mathbf{R}1 + \mathbf{Re}$ as the matrix multiplication by $\begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$. Thus $\mu_{-2}(\alpha_{23}(\pi))$ acts on $\mathfrak{o}(2) \cong \mathbf{R}1 + \mathbf{Re}$ is just the matrix multiplication by $-\mathrm{Id}$. On the other hand, α_2 of $(\alpha_1, \alpha_2, \alpha_3)$ acts on $\mathbf{R}1 + \mathbf{R}e$ as $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Thus the $Spin(2)$-factor of $(\alpha_1, \alpha_2, \alpha_3)$ in $Spin(2) \cdot Spin(6)$ corresponds to

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in O(2).$$

Hence the adjoint action of $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in O(2)$ on $\mathfrak{o}(2)^{\mathbf{C}}$ is $-\mathrm{Id}$. Therefore, $V_{K_{[\mathfrak{a}]}} = \mathfrak{o}(2)^{\mathbf{C}}$, i.e., $\Lambda = 6y_0 + y_1 + y_2 + y_3 \in D(K, K_{[\mathfrak{a}]}) = \mathfrak{o}(2)^{\mathbf{C}}$. Thus $\Lambda = 6y_0 + y_1 + y_2 + y_3 \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1. Similarly, $\Lambda = -6y_0 + y_1 + y_2 + y_3 \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1.

11.16.5. *On* $(p_0, p_1, p_2, p_3, p_4, p_5) = (0, 1, 1, 1, 1, 0)$.

$$\begin{aligned}
&\Lambda = y_1 + y_2 + y_3 + y_4 \in D(K, K_0), \\
&\Lambda' = y_2 + y_3 + y_4 + y_5 \in D(K_2, K_0), \\
&\kappa\Lambda' = 2\hat{y}_2 \in D(K_2, K_0), \\
&\Lambda''_1 = 0 \in D(K_1, K_0), \\
&\Lambda''_2 = 2\hat{y}_2 \in D(K_1, K_0), \\
&\Lambda''_3 = -2\hat{y}_2 \in D(K_1, K_0), \\
&\Lambda''' = 0 \in D(K_0)
\end{aligned}$$

$C_L = 30$.

$$V_\Lambda \supset W_{\Lambda'} \supset U_{\Lambda''}$$

$$V_\Lambda \supset (V_\Lambda)_{K_0} = U_{\Lambda''_1(0,0,0,0,0,0)} \oplus U_{\Lambda''_2(0,0,2,0,0,0)} \oplus U_{\Lambda''_3(0,0,-2,0,0,0)},$$

$$W_{\Lambda'} \cong S_0^2(\mathbf{C}^8) \cong S_0^2(\mathbb{K}^8).$$

Let $\{1, e_1, \cdots, e_7\}$ denote the standard basis of the Cayley algebra $\mathbb{K}$ and denote $e := e_4$. Then

$$3(1\cdot 1+e\cdot e)-(e_1\cdot e_1+e_2\cdot e_2+e_3\cdot e_3+e_5\cdot e_5+e_6\cdot e_6+e_7\cdot e_7) \in S_0^2(\mathbb{K}^{\mathbf{C}}).$$

For any $A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2)$, $A(1,e) = (1,e)\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$.

$$\begin{aligned}
A(1\cdot 1) =&(\cos t1 + \sin te)\cdot(\cos t1+\sin te)\\
=&\cos^2 t(1\cdot 1) + \sin^2 t(e\cdot e) + 2\sin t\cos t(1\cdot e),\\
A(e\cdot e) =&(-\sin t1 + \cos te)\cdot(-\sin t1+\cos te)\\
=&\sin^2 t(1\cdot 1) + \cos^2 t(e\cdot e) - 2\sin t\cos t(1\cdot e),\\
A(1\cdot e) =&(\cos t1 + \sin te)\cdot(-\sin t1+\cos te)\\
=&-\sin t\cos t(1\cdot 1) + \sin t\cos t(e\cdot e) + (\cos^2 t-\sin^2 t)(1\cdot e)\\
=&-\frac{1}{2}\sin 2t(1\cdot 1) + \frac{1}{2}\sin 2t(e\cdot e) + (\cos^2 t-\sin^2 t)(1\cdot e)\\
=&-\frac{1}{2}\sin 2t(1\cdot 1 - e\cdot e) + \cos 2t(1\cdot e).
\end{aligned}$$

Hence

$$A(1\cdot 1+e\cdot e) = 1\cdot 1+e\cdot e$$

and

$$\begin{aligned}
&A(3(1\cdot 1+e\cdot e)-(e_1\cdot e_1+e_2\cdot e_2+e_3\cdot e_3+e_5\cdot e_5+e_6\cdot e_6+e_7\cdot e_7))\\
=&3(1\cdot 1+e\cdot e)-(e_1\cdot e_1+e_2\cdot e_2+e_3\cdot e_3+e_5\cdot e_5+e_6\cdot e_6+e_7\cdot e_7).
\end{aligned}$$

$$3(1\cdot 1+e\cdot e)-(e_1\cdot e_1+e_2\cdot e_2+e_3\cdot e_3+e_5\cdot e_5+e_6\cdot e_6+e_7\cdot e_7) \in U_{\Lambda''(0,0,0,0,0,0)}$$

On the other hand,

$$\begin{aligned}
1\cdot 1 - e\cdot e + 2\sqrt{-1}(1\cdot e) \in S_0^2(\mathbb{K}^{\mathbf{C}}),\\
1\cdot 1 - e\cdot e - 2\sqrt{-1}(1\cdot e) \in S_0^2(\mathbb{K}^{\mathbf{C}}),
\end{aligned}$$

and

$$\begin{aligned}
&A(1\cdot 1 - e\cdot e + 2\sqrt{-1}1\cdot e)\\
=&(\cos 2t - \sqrt{-1}\sin 2t)(1\cdot 1 - e\cdot e + 2\sqrt{-1}1\cdot e)\\
=&e^{-\sqrt{-1}2t}(1\cdot 1 - e\cdot e + 2\sqrt{-1}1\cdot e),\\
&A(1\cdot 1 - e\cdot e - 2\sqrt{-1}1\cdot e)\\
=&e^{\sqrt{-1}2t}(1\cdot 1 - e\cdot e - 2\sqrt{-1}1\cdot e).
\end{aligned}$$

Hence,

$$1 \cdot 1 - e \cdot e + 2\sqrt{-1}1 \cdot e \in U_{\Lambda''(0,0,-2,0,0,0)},$$
$$1 \cdot 1 - e \cdot e - 2\sqrt{-1}1 \cdot e \in U_{\Lambda''(0,0,2,0,0,0)}.$$

Therefore

$$\begin{aligned}
&(V_\Lambda)_{K_0}\\
&= \mathbf{C}(1 \cdot 1 - e \cdot e + 2\sqrt{-1}1 \cdot e)\\
&\quad \oplus \mathbf{C}(1 \cdot 1 - e \cdot e - 2\sqrt{-1}1 \cdot e)\\
&\quad \oplus \mathbf{C}(3(1 \cdot 1 + e \cdot e) - (e_1 \cdot e_1 + e_2 \cdot e_2 + e_3 \cdot e_3 + e_5 \cdot e_5 + e_6 \cdot e_6 + e_7 \cdot e_7)).
\end{aligned}$$

Since

$$(\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3))(2\sqrt{-1}(1 \cdot e)) = 2(\sqrt{-1}e \cdot (-1)) = -2\sqrt{-1}(1 \cdot e),$$
$$(\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3))(1 \cdot 1 - e \cdot e) = 1 \cdot 1 - e \cdot e,$$
$$(\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3))(1 \cdot 1 + e \cdot e) = -(1 \cdot 1 + e \cdot e),$$

we obtain

$$(V_\Lambda)_{K_{[\mathfrak{a}]}} = \mathbf{C}(1 \cdot 1 - e \cdot e).$$

and

$$\Lambda = y_1 + y_2 + y_3 + y_4 \in D(K, K_{[\mathfrak{a}]})$$

with the multiplicity 1.

Therefore,

$$\begin{aligned}
&\dim_{\mathbf{C}} V_{(6,1,1,1,0,0)} + \dim_{\mathbf{C}} V_{(-6,1,1,1,0,0)} + \dim_{\mathbf{C}} V_{(0,1,1,1,0,0)}\\
&= 120 + 120 + 210 = 450\\
&= \dim SO(32) - \dim U(1) \cdot Spin(10) = n_{kl}(\mathcal{G}),
\end{aligned}$$

$\mathcal{G}(N^{30}) \subset Q_{30}(\mathbf{C})$ is strictly Hamiltonian stable.

Theorem.

$$K/K_{[\mathfrak{a}]} = (U(1) \cdot Spin(10))/(\mathbf{Z}_4 \cdot S^1 \cdot Spin(6)) \subset Q_{30}(\mathbf{C})$$

is Hamiltonian stable and Hamiltonian rigid, and hence it is strictly Hamiltonian stable.

12. Appendix

12.1. **The principle of triality for $SO(8)$.**

$$\mathbb{K} \cong \mathbf{R}^8,$$
$$SO(\mathbb{K}) \cong SO(8).$$

Theorem 12.1 (The Principle of Triality for $SO(8)$)**.** *For any $\alpha_3 \in SO(8)$, there exist $\alpha_1, \alpha_2 \in SO(8)$ such that*

$$(\alpha_1 x)(\alpha_2 y) = \alpha_3(xy)$$

for each $x, y \in \mathbb{K}$. Such α_1, α_2 for α_3 are only α_1, α_2 and $-\alpha_1, -\alpha_2$.

Lemma 12.1. *Suppose that $\alpha_1, \alpha_2, \alpha_3 \in SO(8)$ satisfy*

$$(\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})}$$

for each $x, y \in \mathbb{K}$. Then

$$(\alpha_2 x)(\alpha_3 y) = \overline{\alpha_1(\overline{xy})},$$
$$(\alpha_3 x)(\alpha_1 y) = \overline{\alpha_2(\overline{xy})}$$

for each $x, y \in \mathbb{K}$.

Lemma 12.2. *Suppose that $\alpha_1, \alpha_2, \alpha_3 \in O(8)$ satisfy*

$$(\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})}$$

for each $x, y \in \mathbb{K}$. Then $\alpha_1, \alpha_2, \alpha_3 \in SO(8)$.

Set

$\mathfrak{d}_4 := \{D \in \mathrm{Hom}_{\mathbf{R}}(\mathbb{K}) \mid (Dx, y) + (x, Dy) = 0 \text{ for each } x, y \in \mathbb{K}\}$.

$\kappa, \pi, \nu : \mathfrak{d}_4 \to \mathfrak{d}_4$ are defined by

$$(\kappa D)x := \overline{D\overline{x}} \quad \text{for each } x \in \mathbb{K},$$
$$\pi(G_{ij}) := F_{ij},\ i, j = 0, 1, 2, 3, 4, 5, 6, 7,$$
$$\nu := \pi \circ \kappa = \pi\kappa.$$

Let $\mathrm{Aut}(\mathfrak{d}_4)$ be the automorphism group of Lie algebra $\mathfrak{d}_4$. Then $\kappa, \pi, \nu \in \mathrm{Aut}(\mathfrak{d}_4)$ and they satisfy

$$\kappa^2 = 1, \pi^2 = 1, \nu^3 = 1, \nu = \pi\kappa.$$

Proposition 12.1. *Let $\mathfrak{S}_3$ be a subgroup of $\mathrm{Aut}(\mathfrak{d}_4)$ generated by κ and π and S_3 the symmetry group of degree 3. Then there is a group isomorphism*

$$\mathfrak{S}_3 \cong S_3.$$

Theorem 12.2 (The Principle of Triality for $\mathfrak{d}_4$)**.** *For any $D_1 \in \mathfrak{d}_4$, there exist $D_2, D_3 \in \mathfrak{d}_4$ such that*

$$(D_1 x)y + x(D_2 y) = D_3(xy)$$

for each $x, y \in \mathbb{K}$. Such D_2, D_3 for D_1 is unique and $D_2 = \nu D_1$ and $D_3 = \pi D_1$.

Lemma 12.3. *Suppose that $D_1, D_2, D_3 \in \mathfrak{d}_4$ satisfy*

$$(D_1 x)y + x(D_2 y) = D_3(xy)$$

for each $x, y \in \mathbb{K}$. Then

$$(D_2 x)y + x(D_3 y) = D_1(xy),$$
$$(D_3 x)y + x(D_1 y) = D_2(xy)$$

for each $x, y \in \mathbb{K}$.

12.1.1. *Spinor group $Spin(7)$.*

$$\mathbb{K} \cong \mathbf{R}^8,$$
$$\mathrm{Im}(\mathbb{K}) \cong \mathbf{R}^7.$$

$$\begin{aligned}\tilde{B}_3 :=\{\tilde{\alpha} \in SO(\mathbb{K}) \mid & \text{ There exists } \alpha \in SO(\mathrm{Im}(\mathbb{K})) \text{ such that} \\ & (\alpha x)(\tilde{\alpha} y) = \tilde{\alpha}(xy) \text{ for each } x, y \in \mathbb{K}\}.\end{aligned}$$

$\tilde{B}_3 \cong Spin(7)$

$$p : Spin(7) \cong \tilde{B}_3 \ni \tilde{\alpha} \longmapsto \alpha \in SO(\mathrm{Im}(\mathbb{K})) \cong SO(7).$$

12.1.2. *Spinor group $Spin(8)$.*

$$\begin{aligned}\tilde{D}_4 :=\{(\alpha_1, \alpha_2, \alpha_3) \in SO(\mathbb{K}) \times SO(\mathbb{K}) \times SO(\mathbb{K}) \mid & \\ (\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})} & \text{ for each } x, y \in \mathbb{K}\}\end{aligned}$$

$$p : \tilde{D}_4 \ni (\alpha_1, \alpha_2, \alpha_3) \longmapsto \alpha_1 \in SO(\mathbb{K}) \cong SO(8)$$

is a universal covering map. Then $\tilde{D}_4 \cong Spin(8)$.

Lie group isomorphisms $\kappa, \pi, \nu : Spin(8) \to Spin(8)$ defined by

$$\kappa(\alpha_1, \alpha_2, \alpha_3) := (\kappa\alpha_1, \kappa\alpha_3, \kappa\alpha_2),$$
$$\pi(\alpha_1, \alpha_2, \alpha_3) := (\kappa\alpha_3, \kappa\alpha_2, \kappa\alpha_1),$$
$$\pi(\alpha_1, \alpha_2, \alpha_3) := (\alpha_2, \alpha_3, \alpha_1),$$

where $\kappa : SO(8) \to SO(8)$ is defined by $\kappa\alpha(x) := \overline{\alpha(\overline{x})}$ for each $x \in \mathbb{K}$.

$$Spin(7) \cong \tilde{B}_3 \ni \tilde{\alpha} \longmapsto (\alpha, \tilde{\alpha}, \kappa\tilde{\alpha}) \in \tilde{D}_4$$

is an injective Lie group homomorphism.

$$Spin(7) \cong \{\alpha \in Spin(8) \mid \kappa\alpha = \alpha\}.$$

12.1.3. *Subgroup $Spin(8)$ of $F_4 \subset E_6$.*

$$D_4 := \{a \in F_4 \mid ae_i = e_i \ (i = 1, 2, 3)\}$$

$$\varphi : \tilde{D}_4(\cong Spin(8)) \ni (\alpha_1, \alpha_2, \alpha_3) \longmapsto \varphi(\alpha_1, \alpha_2, \alpha_3) \in D_4 \subset F_4 \subset E_6$$

defined by

$$\varphi(\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} := \begin{pmatrix} \xi_1 & \alpha_3 x_3 & \overline{\alpha_2 x_2} \\ \overline{\alpha_3 x_3} & \xi_2 & \alpha_1 x_1 \\ \alpha_2 x_2 & \overline{\alpha_1 x_1} & \xi_3 \end{pmatrix}.$$

φ is a Lie group isomorphism.

$$p_i : \tilde{D}_4(\cong Spin(8)) \ni (\alpha_1, \alpha_2, \alpha_3) \longmapsto \alpha_i \in SO(\mathbb{K}) \cong SO(8) \quad (i = 1, 2, 3)$$

12.1.4. *Subgroup $Spin(6) \cong SU(4)$ of $Spin(8)$.*

$$\begin{aligned} Spin(8) &\longrightarrow SO(8) \\ U(1) \times SU(4) &\longrightarrow U(4) \\ Spin(7) &\longrightarrow SO(7) \\ SU(4) \cong Spin(6) &\longrightarrow SO(6) \end{aligned}$$

$$\begin{aligned} \mathfrak{a} \cong & \{ \begin{pmatrix} 0 & 0 & m_2 - \sqrt{-1}a_2 e \\ 0 & 0 & 0 \\ m_2 + \sqrt{-1}a_2 e & 0 & 0 \end{pmatrix} \mid m_2, a_2 \in \mathbf{R}\} \\ \subset & (\mathfrak{J}^{\mathbf{C}})_{\tau\gamma, -\sigma} \subset (\mathfrak{J}^{\mathbf{C}})_{-\sigma} \ (\cong \mathfrak{p}) \\ \subset & (\mathfrak{J}^{\mathbf{C}}) \end{aligned}$$

Question 1. Suppose that $(\alpha_1, \alpha_2, \alpha_3) \in SO(\mathfrak{C}) \times SO(\mathfrak{C}) \times SO(\mathfrak{C})$ such that

$$(\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})}$$

for each $x, y \in \mathfrak{C}$. If $\alpha_2(1) = 1$ and $\alpha_2(\sqrt{-1}e) = \sqrt{-1}e$, i.e. $\alpha_2(e) = e$, then what can you say about α_1 and α_3 ?

$$(\alpha_1 x)(\alpha_2 y) = \kappa\alpha_3(xy)$$

By putting $y = 1$,

$$\begin{aligned}(\alpha_1 x)(\alpha_2 1) &= \kappa\alpha_3(x)\\ (\alpha_1 x)1 &= \kappa\alpha_3(x)\\ (\alpha_1 x) &= \kappa\alpha_3(x)\end{aligned}$$

Thus $\alpha_1 = \kappa\alpha_3$ and hence

$$(\alpha_1 x)(\alpha_2 y) = \alpha_1(xy)$$

for each $x, y \in \mathfrak{C}$. By putting $y = e$,

$$\begin{aligned}(\alpha_1 x)(\alpha_2 e) &= \alpha_1(xe)\\ (\alpha_1 x)e &= \alpha_1(xe)\\ (\alpha_1 x)e &= \alpha_1(xe).\end{aligned}$$

As $\mathbf{C} = \mathbf{R} + \mathbf{R}e$,

$$\mathfrak{C} \cong \mathbf{C}^4 \cong \mathbf{C}1 + \mathbf{C}e_1 + \mathbf{C}e_2 + \mathbf{C}e_3.$$

$$Spin(8) \ni (\alpha_1, \alpha_2, \alpha_3) \longmapsto \alpha_2 \in SO(\mathfrak{C}) \cong SO(8)$$

$$\begin{aligned}Spin(6) \cong &\{(\alpha_1, \alpha_2, \alpha_3) \in Spin(8) \mid \alpha_2(1) = 1, \alpha_2(e) = e\}\\ \ni &(\alpha_1, \alpha_2, \alpha_3) \longmapsto \alpha_2 \in SO(6)\end{aligned}$$

$$\begin{aligned}Spin(6) \cong &\{(\alpha_1, \alpha_2, \alpha_3) \in Spin(8) \mid \alpha_2(1) = 1, \alpha_2(e) = e\}\\ \ni &(\alpha_1, \alpha_2, \alpha_3) \longmapsto \alpha_1 \in U(4)\end{aligned}$$

is an injective Lie group homomorphism. In fact, if $(\alpha_1, \alpha_2, \alpha_3) \in Spin(6)$, then $(\alpha_1, -\alpha_2, -\alpha_3) \in Spin(8)$ but $(\alpha_1, -\alpha_2, -\alpha_3) \notin Spin(6)$. Because $(-\alpha_2)(1) = -1 \neq 1$ and $(-\alpha_2)(e) = -e \neq e$. Since $Spin(6)$ is simple,

$$\{\alpha_1 \mid (\alpha_1, \alpha_2, \alpha_3) \in Spin(6)\} \cong SU(4) \subset U(4).$$

Therefore

$$\begin{aligned}Spin(6) \cong &\{(\alpha_1, \alpha_2, \alpha_3) \in Spin(8) \mid \alpha_2(1) = 1, \alpha_2(e) = e\}\\ \ni &(\alpha_1, \alpha_2, \alpha_3) \longmapsto \alpha_1 \in SU(4)\end{aligned}$$

is a Lie group isomorphism.

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Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P.R. CHINA
E-mail address: `hma@math.tsinghua.edu.cn`

Osaka City University Advanced Mathematical Institute & Department of Mathematics, Osaka City University, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, JAPAN
E-mail address: `ohnita@sci.osaka-cu.ac.jp`