NONEXISTENCE OF MULTI-BUBBLE SOLUTIONS FOR A HIGHER ORDER MEAN FIELD EQUATION ON CONVEX DOMAINS

FUTOSHI TAKAHASHI

Abstract. In this note, we show that there does not exist any blowing-up solution sequence with multiple blow up points to a $2p$-th order mean field equation

$$\begin{cases} (-\Delta)^p u = \rho \frac{V(x)e^u}{\int_\Omega V(x)e^u dx} & \text{in } \Omega \subset \mathbb{R}^{2p}, \\ (-\Delta)^j u = 0 \quad \text{on } \partial\Omega, & (j = 0, 1, \cdots p-1) \end{cases}$$

for $p \in \mathbb{N}$, if a bounded smooth domain Ω is convex and the function V satisfies some conditions.

Keywords: blowing-up solution, higher-order mean field equation, Green's function.

2010 Mathematics Subject Classifications: 35J35, 35J40.

1. Introduction

Recently, many authors have been interested in the study of non-linear elliptic partial differential equations involving the higher-order differential operator, because of its connection to the conformal geometry. One of the most important conformally invariant differential operators on a four-dimensional Riemannian manifold (M, g) is a Paneitz operator, defined as

$$P_g = \Delta_g^2 - \delta_g \left(\frac{2}{3} S_g - 2Ric_g \right) d$$

where Δ_g denotes the Laplace-Beltrami operator with respect to g, δ_g the co-differential, d the exterior differential, S_g and Ric_g denote the scalar and Ricci curvature of the metric g. By this symbol, the equation of prescribing Q-curvature on (M, g) is described as

$$P_g u + 2Q_g = 2\bar{Q}_{g_u} e^{4u}$$

where Q_g is the Q-curvature of the original metric g, $\bar{Q}_{g_u}$ is the Q-curvature of the new metric $g_u = e^{4u}g$. If (M, g) is $\mathbb{R}^4$ with its standard euclidean metric, the Paneitz operator P_g is nothing but $\Delta^2 = \Delta\Delta$ where $\Delta = \sum_{i=1}^4 \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in $\mathbb{R}^4$, and the equation of prescribing Q-curvature becomes of the form

$$\Delta^2 u = \rho \frac{V(x)e^{4u}}{\int_\Omega V(x)e^{4u}dx}.$$

See for example, [7], [10], [8] and the references therein.

In this paper, we consider a generalization of it, namely, we concern the following $2p$-th order mean field equation ($p \in \mathbb{N}$)

$$\begin{cases} (-\Delta)^p u = \rho \frac{V(x)e^u}{\int_\Omega V(x)e^u dx} \quad \text{in } \Omega \subset \mathbb{R}^{2p}, \\ (-\Delta)^j u = 0 \quad \text{on } \partial\Omega, \quad (j = 0, 1, \cdots p-1), \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^{2p}$, ρ is a positive parameter and $V \in C^{2,\beta}(\Omega)$ is a positive function. Let us define the variational functional $I_\rho : X \to \mathbb{R}$,

$$I_\rho(u) = \frac{1}{2}\int_\Omega |(-\Delta)^{\frac{p}{2}} u|^2 dx - \rho \log \int_\Omega V(x)e^u dx$$

where

$$X = H^p(\Omega) \cap \{u \mid (-\Delta)^j u \in H_0^1(\Omega),\ j = 0, 1, \cdots \left[\frac{p-1}{2}\right]\},$$

and we admit the notation that

$$(-\Delta)^{\frac{p}{2}} u = \begin{cases} \nabla(-\Delta)^{k-1}u, \quad (p = 2k-1), \\ (-\Delta)^k u, \quad (p = 2k), \end{cases}$$

for $k \in \mathbb{N}$. Then (1.1) is the Euler-Lagrange equation of I_ρ.

In the following, let $\alpha_0(p)$ denote the best constant for the Adams version Trudinger-Moser inequality [1]: there exists $C(\Omega) < +\infty$ such that for any $\alpha \le \alpha_0(p)$ and $u \in C_0^\infty(\Omega)$ with

$$\|(-\Delta)^{\frac{p}{2}} u\|_{L^2(\Omega)} \le 1,$$

there holds

$$\int_\Omega e^{\alpha u^2} dx \le C(\Omega).$$

The same holds for $u \in X$ by standard density argument. It is known that $\alpha_0(1) = 4\pi, \alpha_0(2) = 32\pi^2$, and generally, $\alpha_0(p) = \frac{2p}{\sigma_{2p}}(2\pi)^{2p} = 2^{2p}\pi^p p!$, where $\sigma_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ denotes the volume of the unit sphere in

$\mathbb{R}^N$. Also $G = G(x, y)$ will denote the Green function of $(-\Delta)^p$ under the Navier boundary condition:

$$\begin{cases} (-\Delta)^p G(\cdot, y) = \delta_y & \text{in } \Omega \subset \mathbb{R}^{2p}, \\ G(\cdot, y) = (-\Delta)^j G(\cdot, y) = 0 & \text{on } \partial\Omega, \quad (j = 1, \cdots p-1). \end{cases}$$

We decompose G as $G(x, y) = \Gamma(x, y) - H(x, y)$, where $\Gamma(x, y)$ is the fundamental solution of $(-\Delta)^p$ on $\mathbb{R}^{2p}$, defined as

$$\Gamma(x, y) = C_p \log \frac{1}{|x-y|}, \quad C_p = \frac{1}{\{2^{p-1}(p-1)!\}^2 \sigma_{2p}},$$

and $H = H(x, y) \in C^\infty(\Omega \times \Omega)$ is called the regular part of the Green function. Finally, let $R(y) = H(y, y)$ denote the Robin function of the Green function of $(-\Delta)^p$ with the Navier boundary condition.

On the asymptotic behavior of blowing-up solutions to (1.1), C-S. Lin and J-C. Wei proved, among others, the following result; see [13], [11], [12].

Proposition 1.1. *Assume $V \in C^{2,\beta}(\Omega)$, $\inf_\Omega V > 0$. Let u_{ρ_n} be a solution sequence to (1.1) with $\rho = \rho_n > 0$ such that $\|u_{\rho_n}\|_{L^\infty(\Omega)} \to \infty$ while $\rho_n = O(1)$ as $n \to \infty$. Then there exists a subsequence (again denoted by ρ_n) and m-points set $\mathcal{S} = \{a_1, \cdots, a_m\} \subset \Omega$ (***blow up set***) such that*

$$\rho_n \to 2\alpha_0(p) m, \quad \textbf{(mass quantization)}$$

$$u_{\rho_n} \to 2\alpha_0(p) \sum_{j=1}^m G(\cdot, a_j) \quad \text{in } C^{2p}_{loc}(\overline{\Omega} \setminus \mathcal{S}),$$

$$\rho_n \frac{V(x) e^{u_{\rho_n}}}{\int_\Omega V(x) e^{u_{\rho_n}} dx} \rightharpoonup 2\alpha_0(p) \sum_{i=1}^m \delta_{a_i}$$

in the sense of measures. Finally, each blow up point $a_i \in \mathcal{S}$ must satisfy

$$\frac{1}{2} \nabla R(a_i) - \sum_{j=1, j\neq i}^m \nabla_x G(a_i, a_j) - \frac{1}{2\alpha_0(p)} \nabla \log V(a_i) = \vec{0}, \tag{1.2}$$

for $i = 1, \cdots, m$. **(Characterization of blow up points)**

The main difficult point in the proof is to show that the blow up set $\mathcal{S}$ consists of interior points of Ω. In [11], [12], the authors used the local version of the method of moving planes to overcome the difficulty. After showing that $\mathcal{S} \subset \Omega$, the rest of claims follow by the argument in [13].

As for the actual existence of multi-bubble solutions to (1.1), which exhibits the asymptotic behavior described in Proposition 1.1 with $m \geq 2$, some affirmative results are known by recent papers [2] [6] when $p = 2$.

Proposition 1.2. *Let $p = 2$ and $m \geq 2$ be an integer. Set $\Omega^m = \Omega \times \cdots \times \Omega$ (m times) and $\Delta = \{(\xi_1, \cdots, \xi_m) \in \Omega^m | \xi_i = \xi_j \text{ for some } i \neq j\}$. Define the Hamiltonian function*

$$\mathcal{F}(\xi_1, \cdots, \xi_m) = \sum_{i=1}^{m} \left(R(\xi_i) - \frac{1}{32\pi^2} \log V(\xi_i) \right) - \sum_{\substack{i \neq j \\ 1 \leq i,j \leq m}} G(\xi_i, \xi_j)$$

on $\Omega^m \setminus \Delta$. If $\mathcal{F}$ has a nondegenerate critical point (Baraket-Dammak-Ouni-Pacard [2], $V \equiv 1$ case), or, a "minimax value in an appropriate subset" (Clapp-Munõz-Musso [6]), that is, if $(a_1, \cdots, a_m) \in \Omega^m \setminus \Delta$ satisfies

$$\frac{1}{2} \nabla R(a_i) - \sum_{j=1, j \neq i}^{m} \nabla_x G(a_i, a_j) - \frac{1}{64\pi^2} \nabla \log V(a_i) = \vec{0}$$

for $i = 1, 2, \cdots, m$ and some additional conditions, then there exists a solution sequence $\{u_\rho\}$ which blows up exactly on $\mathcal{S} = \{a_1, \cdots, a_m\}$.

For the precise meaning that $\mathcal{F}$ has a "minimax value in an appropriate subset", we refer to [6]. By this proposition, we know that if Ω has the cohomology group $H^d(\Omega) \neq 0$ for some $d \in \mathbb{N}$, or, if Ω is an m-dumbbell shaped domain (roughly, a simply-connected domain made by m balls those connected to each other by thin tubes), then there exist m-points blowing up solutions for any $m \geq 2$ [6].

In this paper, on the contrary, we prove the nonexistence of multi-bubble solutions to (1.1) on *convex* domains, under an additional assumption on the coefficient function V.

Theorem 1.3. *Assume $\Omega \subset \mathbb{R}^{2p}$ be a bounded convex domain. Let $\{u_{\rho_n}\}$ be a solution sequence to (1.1) satisfying $\|u_{\rho_n}\|_{L^\infty(\Omega)}$ is not bounded while $\rho_n > 0$ is bounded as $n \to \infty$. Assume $\inf_\Omega V > 0$ and $R - \frac{1}{\alpha_0(p)} \log V$ is a strictly convex function on Ω. Then there exists $a \in \Omega$ such that, for the full sequence, we have*

$$\rho_n \to 2\alpha_0(p),$$
$$u_{\rho_n} \to 2\alpha_0(p) G(\cdot, a) \quad \text{in } C^{2p}_{loc}(\overline{\Omega} \setminus \{a\}),$$
$$\rho_n \frac{V(x) e^{u_{\rho_n}}}{\int_\Omega V(x) e^{u_{\rho_n}} dx} \rightharpoonup 2\alpha_0(p) \delta_a \quad \text{in the sense of measures}$$

as $n \to \infty$.

In this theorem, we can claim also that $a \in \Omega$ is the unique minimum point of the strictly convex function $R - \frac{1}{\alpha_0(p)} \log V$.

We remark here that, for the 2nd order case, the Robin function of $-\Delta$ with the Dirichlet boundary condition on a bounded convex domain Ω in $\mathbb{R}^N$ is strictly convex on Ω. This fact was first proved by Caffarelli and Friedman [4] when $N = 2$, and later extended to $N \geq 3$ by Cardaliaguet and Tahraoui [5]. By using this fact, Grossi and Takahashi [9] proved that blowing-up solutions with multiple blow up points do not exist on convex domains for various semilinear problems with blowing-up or concentration phenomena. It is open whether the same convexity holds true or not for the Robin function of $(-\Delta)^p$ under the Navier boundary condition when $p \geq 2$. Thus at this stage, we cannot drop the assumption on V and we do not know whether the same result as Theorem 1.3 is true when V is a constant.

This paper is organized as follows. In §2, we prove a lemma which is crucial to our argument. In this lemma, we do not need the assumption of the convexity of Ω. In §3, we prove Theorem 1.3 by using the key lemma in §2 and the characterization of blow up points (1.2).

2. New Pohozaev identity for the Green function.

In this section, we prove an integral identity for the Green function of $(-\Delta)^p$ with the Navier boundary condition, which is a key for the proof of Theorem 1.3. Corresponding identity when $p = 1$ was former proved in [9].

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2p)$ be a smooth bounded domain. For any $P \in \mathbb{R}^N$ and $a, b \in \Omega, a \neq b$, it holds*

$$\sum_{k=1}^{p} \int_{\partial\Omega} (x - P) \cdot \nu(x) \left(\frac{\partial(-\Delta)^{p-k} G_a}{\partial \nu_x} \right) \left(\frac{\partial(-\Delta)^{k-1} G_b}{\partial \nu_x} \right) ds_x$$
$$= (2p - N)G(a, b) + (P - a) \cdot \nabla_x G(a, b) + (P - b) \cdot \nabla_x G(b, a),$$

where $G_a(x) = G(x, a), G_b(x) = G(x, b)$ and $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$.

Proof. We follow the argument used in [9], which originates from [3]. In order to introduce the idea clearly, first we show a formal computation. Let us denote $G_a(x) = G(x, a), G_b(x) = G(x, b)$ and define $w(x) = (x - P) \cdot \nabla G_a(x)$. Since $\Delta^j ((x - P) \cdot \nabla) = 2j\Delta^j +$

$((x-P)\cdot\nabla\Delta^j)$ for $j\in\{0\}\cup\mathbb{N}$, we have

$$\begin{cases}(-\Delta)^p w(x) &= (x-P)\cdot\nabla\delta_a(x)+2p\delta_a(x),\\ (-\Delta)^p G_b(x) &= \delta_b(x),\end{cases}$$

where δ_a, δ_b are the Dirac delta functions supported on a, b respectively. Multiplying $G_b(x)$, $w(x)$ respectively to the above equations, and subtracting, we obtain

$$\int_\Omega ((-\Delta)^p w(x))\, G_b(x) - ((-\Delta)^p G_b(x))\, w(x)dx$$
$$= \int_\Omega \left\{(x-P)\cdot\nabla\delta_a(x)G_b(x) + 2p\delta_a(x)G_b(x) - \delta_b(x)w(x)\right\} dx. \quad (2.1)$$

By an iterated use of Green's second formula, we see

$$\begin{aligned}\text{LHS of (2.1)} &= (-1)^p\sum_{k=1}^p\int_{\partial\Omega}\left(\frac{\partial\Delta^{p-k}w}{\partial\nu}\Delta^{k-1}G_b - \frac{\partial\Delta^{k-1}G_b}{\partial\nu}\Delta^{p-k}w\right)ds_x\\ &= (-1)^{p+1}\sum_{k=1}^p\int_{\partial\Omega}\left((x-P)\cdot\nabla\Delta^{p-k}G_a\right)\left(\frac{\partial\Delta^{k-1}G_b}{\partial\nu}\right)ds_x\\ &= \sum_{k=1}^p\int_{\partial\Omega}(x-P)\cdot\nu(x)\left(\frac{\partial(-\Delta)^{p-k}G_a}{\partial\nu_x}\right)\left(\frac{\partial(-\Delta)^{k-1}G_b}{\partial\nu_x}\right)ds_x,\end{aligned}$$

here we have used $\Delta^{k-1}G_b=0$ and $\Delta^{p-k}w=(x-P)\cdot\nabla\Delta^{p-k}G_a$ on $\partial\Omega$.

On the other hand,

$$\begin{aligned}\text{RHS of (2.1)} &= 2pG_b(a) - w(b) + \int_\Omega (x-P)\cdot\nabla\delta_a(x)G_b(x)dx\\ &= 2pG_b(a) - w(b) + \sum_{i=1}^N\int_\Omega (x_i-P_i)\frac{\partial\delta_a}{\partial x_i}G_b(x)dx\\ &= 2pG_b(a) - w(b) - \sum_{i=1}^N\int_\Omega \frac{\partial}{\partial x_i}\{(x_i-P_i)G_b(x)\}\delta_a(x)dx\\ &= 2pG_b(a) - w(b) - \sum_{i=1}^N\frac{\partial}{\partial x_i}\left\{(x_i-P_i)G_b(x)\right\}\Big|_{x=a}\\ &= (2p-N)G(a,b) + (P-a)\cdot\nabla_x G(a,b) + (P-b)\cdot\nabla_x G(b,a).\end{aligned}$$

Thus we obtain the conclusion.

To make this argument rigorously, we use standard approximations. Define $\delta_{a,\rho}(x)=\frac{1}{|B_\rho|}\chi_{B_\rho(a)}(x)$ where $\chi_{B_\rho(a)}$ is the characteristic function of the ball $B_\rho(a)$ with radius $\rho>0$ and center $a\in\Omega$. Denote $\delta^\varepsilon_{a,\rho}(x)=$

$j_\varepsilon * \delta_{a,\rho}(x)$ where $j(x) \geq 0, \mathrm{supp} j \subset B_1(0), \int_{\mathbb{R}^N} j(x)dx = 1$ and $j_\varepsilon(x) = \varepsilon^{-N} j(\frac{x-a}{\varepsilon})$. For a point $a \in \Omega$ and for $\rho > 0$ and $\varepsilon > 0$ sufficiently small such that $B_{\rho+\varepsilon}(a) \subset \Omega$, $\delta^\varepsilon_{a,\rho}$ is well-defined and a smooth function on Ω. Let $u^\varepsilon_{a,\rho}$ denote the unique solution of the problem

$$\begin{cases} (-\Delta)^p u^\varepsilon_{a,\rho} = \delta^\varepsilon_{a,\rho} & \text{in } \Omega, \\ (-\Delta)^j u^\varepsilon_{a,\rho} = 0 & \text{on } \partial\Omega, \ (j = 0, 1, \cdots, p-1). \end{cases}$$

Define $\delta^\varepsilon_{b,\rho}, u^\varepsilon_{b,\rho}$ in the same way. Since $\delta^\varepsilon_{a,\rho} \to \delta_{a,\rho}$ as $\varepsilon \to 0$ in $L^q(\Omega)$ for any $1 \leq q < \infty$, $u^\varepsilon_{a,\rho} \to u_{a,\rho}$ in $W^{2p,q}(\Omega)$ as $\varepsilon \to 0$, where $u_{a,\rho}$ is the unique solution of

$$\begin{cases} (-\Delta)^p u_{a,\rho} = \delta_{a,\rho} & \text{in } \Omega, \\ (-\Delta)^j u_{a,\rho} = 0 & \text{on } \partial\Omega, \ (j = 0, 1, \cdots, p-1). \end{cases}$$

Since $\delta_{a,\rho} \to \delta_a$ as $\rho \to 0$, we have

$$\lim_{\rho \to 0} \lim_{\varepsilon \to 0} u^\varepsilon_{a,\rho} = G(\cdot, a)$$

in $C^k_{loc}(\overline{\Omega} \setminus \{a\})$ for any $k \in \mathbb{N}$, and the same holds for $u^\varepsilon_{b,\rho}$.

Define $w(x) = (x - P) \cdot \nabla u^\varepsilon_{a,\rho}(x)$. Simple calculation shows that w satisfies

$$(-\Delta)^p w = (x - P) \cdot \nabla_x \delta^\varepsilon_{a,\rho} + 2p\delta^\varepsilon_{a,\rho}. \tag{2.2}$$

Multiply $u^\varepsilon_{b,\rho}$ to (2.2), w to the equation $-\Delta u^\varepsilon_{b,\rho} = \delta^\varepsilon_{b,\rho}$, subtracting, and integrating on Ω, we have

$$\begin{aligned} &\int_\Omega \left((-\Delta)^p u^\varepsilon_{b,\rho}\right) w - ((-\Delta)^p w) u^\varepsilon_{b,\rho} dx \\ &= \int_\Omega \left[2p\delta^\varepsilon_{a,\rho}(x) u^\varepsilon_{b,\rho}(x) + (x - P) \cdot \nabla_x \delta^\varepsilon_{a,\rho}(x) u^\varepsilon_{b,\rho}(x) - \delta^\varepsilon_{b,\rho}(x) w(x)\right] dx. \end{aligned} \tag{2.3}$$

The LHS of (2.3) is

$$
\begin{aligned}
&(-1)^p \sum_{k=1}^{p} \int_{\partial\Omega} \left(\frac{\partial \Delta^{p-k} w}{\partial \nu} \Delta^{k-1} u^{\varepsilon}_{b,\rho} - \frac{\partial \Delta^{k-1} u^{\varepsilon}_{b,\rho}}{\partial \nu} \Delta^{p-k} w \right) ds_x \\
&= (-1)^{p+1} \sum_{k=1}^{p} \int_{\partial\Omega} \left((x-P) \cdot \nabla \Delta^{p-k} u^{\varepsilon}_{a,\rho} \right) \left(\frac{\partial \Delta^{k-1} u^{\varepsilon}_{b,\rho}}{\partial \nu} \right) ds_x \\
&= \sum_{k=1}^{p} \int_{\partial\Omega} (x-P) \cdot \nu(x) \left(\frac{\partial (-\Delta)^{p-k} u^{\varepsilon}_{a,\rho}}{\partial \nu_x} \right) \left(\frac{\partial (-\Delta)^{k-1} u^{\varepsilon}_{b,\rho}}{\partial \nu_x} \right) ds_x \\
&\to \sum_{k=1}^{p} \int_{\partial\Omega} (x-P) \cdot \nu(x) \left(\frac{\partial (-\Delta)^{p-k} G_a}{\partial \nu_x} \right) \left(\frac{\partial (-\Delta)^{k-1} G_b}{\partial \nu_x} \right) ds_x
\end{aligned}
$$

as $\varepsilon \to 0$ and then $\rho \to 0$.

The RHS of (2.3) is

$$
2p \int_{\Omega} \delta^{\varepsilon}_{a,\rho}(x) u^{\varepsilon}_{b,\rho}(x) dx + \int_{\Omega} \sum_{i=1}^{N} (x_i - P_i) \left(\frac{\partial \delta^{\varepsilon}_{a,\rho}}{\partial x_i}(x) \right) u^{\varepsilon}_{b,\rho}(x) dx - \int_{\Omega} \delta^{\varepsilon}_{b,\rho}(x) w(x) dx.
$$

Now, integrating by parts, we have

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\Omega} (x_i - P_i) \left(\frac{\partial \delta^{\varepsilon}_{a,\rho}(x)}{\partial x_i} \right) u^{\varepsilon}_{b,\rho}(x) dx \\
&= -\sum_{i=1}^{N} \int_{\Omega} \frac{\partial}{\partial x_i} \left\{ (x_i - P_i) u^{\varepsilon}_{b,\rho}(x) \right\} \delta^{\varepsilon}_{a,\rho}(x) dx \\
&= -N \int_{\Omega} \delta^{\varepsilon}_{a,\rho}(x) u^{\varepsilon}_{b,\rho}(x) dx - \int_{\Omega} (x-P) \cdot \nabla u^{\varepsilon}_{b,\rho}(x) \delta^{\varepsilon}_{a,\rho}(x) dx,
\end{aligned}
$$

thus

$$
\begin{aligned}
&\text{RHS of (2.3)} = (2p-N) \int_{\Omega} \delta^{\varepsilon}_{a,\rho}(x) u^{\varepsilon}_{b,\rho}(x) dx \\
&- \int_{\Omega} (x-P) \cdot \nabla u^{\varepsilon}_{b,\rho}(x) \delta^{\varepsilon}_{a,\rho}(x) dx - \int_{\Omega} (x-P) \cdot \nabla u^{\varepsilon}_{a,\rho}(x) \delta^{\varepsilon}_{b,\rho}(x) dx \\
&\to (2p-N) G(a,b) \\
&- \int_{\Omega} (x-P) \cdot \nabla_x G(x,b) \delta_a(x) dx - \int_{\Omega} (x-P) \cdot \nabla_x G(x,a)(x) \delta_b(x) dx \\
&= (2p-N) G(a,b) + (P-a) \cdot \nabla_x G(a,b) + (P-b) \cdot \nabla_x G(b,a)
\end{aligned}
$$

as $\varepsilon \to 0$ and then $\rho \to 0$. This proves Lemma 2.1. □

3. Proof of Theorem 1.3.

In this section, we prove Theorem 1.3 along the same line in [9].

Step 1.

We argue by contradiction and assume that there exists a m-points set $\mathcal{S} = \{a_1, \cdots, a_m\} \subset \Omega$ $(m \geq 2)$ satisfying (1.2). Set $K(x) = \frac{1}{2}R(x) - \frac{1}{2\alpha_0(p)} \log V(x)$.

$P \in \Omega$ is chosen later. Multiplying $P - a_i$ to (1.2) and summing up, we have

$$\sum_{i=1}^{m}(P - a_i) \cdot \nabla K(a_i) = \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} (P - a_i) \cdot \nabla_x G(a_i, a_j) \tag{3.1}$$

$$= \sum_{1\leq j<k\leq m} \left\{(P - a_j) \cdot \nabla_x G(a_j, a_k) + (P - a_k) \cdot \nabla_x G(a_k, a_j)\right\}.$$

Step 2.

By proposition 2.1, we obtain

$$(P - a_j) \cdot \nabla_x G(a_j, a_k) + (P - a_k) \cdot \nabla_x G(a_k, a_j)$$

$$= \sum_{l=1}^{p} \int_{\partial\Omega} (x - P) \cdot \nu(x) \left(\frac{\partial(-\Delta)^{p-l} G(x, a_j)}{\partial \nu_x}\right) \left(\frac{\partial(-\Delta)^{l-1} G(x, a_k)}{\partial \nu_x}\right) ds_x.$$

By the convexity of Ω, we have $(x - P) \cdot \nu(x) > 0$ on $\partial\Omega$. Also by Hopf lemma, we obtain $\frac{\partial(-\Delta)^{p-l}G(x,a_j)}{\partial \nu_x} < 0$, $\frac{\partial(-\Delta)^{l-1}G(x,a_k)}{\partial \nu_x} < 0$ for $x \in \partial\Omega$. Thus we see the right hand side of (3.1) is positive, and get

$$\sum_{i=1}^{m}(a_i - P) \cdot \nabla K(a_i) < 0. \tag{3.2}$$

Step 3.

By assumption, $K(x) = \frac{1}{2}R(x) - \frac{1}{2\alpha_0(p)} \log V(x)$ is strictly convex. Thus, all level sets of K is strictly star-shaped with respect to its unique minimum point $P \in \Omega$. Choose P as the minimum point. Then

$$(a - P) \cdot \nabla K(a) \geq 0, \quad \forall a \in \Omega \setminus \{P\}. \tag{3.3}$$

In particular,

$$\sum_{i=1}^{m}(a_i - P) \cdot \nabla K(a_i) \geq 0.$$

Now, (3.2) and (3.3) leads to an obvious contradiction. Thus we have $m = 1$ and the rest of proof is easily done by Proposition 1.1. □

Acknowledgement. This work was partly supported by JSPS Grant-in-Aid for Scientific Research (KAKENHI) (B), No. 23340038.

References

[1] D. R. Adams: *A sharp inequality of J. Moser for higher order derivatives,* Ann. of Math. **128**(2) (1988) 385-398.

[2] S. Baraket, M. Dammak, T. Ouni and F. Pacard: *Singular limits for a 4-dimensional semilinear elliptic problem with exponential nonlinearity,* Ann. Inst. H. Poincaré Anal. Non Linéaire, **24** (2007) 875–895.

[3] H. Brezis, and L.A. Peletier: *Asymptotics for elliptic equations involving critical growth,* Partial differential equations and calculus of variations, Vol.1 ,vol. 1 of Progress. Nonlinear Differential Equations Appl. Birkhüser Boston, Boston, MA, (1989) 149-192.

[4] L. A. Caffarelli, and A. Friedman: *Convexity of solutions of semilinear elliptic equations,* Duke Math. J. **52**(2) (1985) 431-456.

[5] P. Cardaliaguet, and R. Tahraoui: *On the strict concavity of the harmonic radius in dimension* $N \geq 3$, J. Math. Pures Appl. **81**(9) (2002) 223-240.

[6] M. Clapp, C. Muños, and M. Musso: *Singular limits for the bi-Laplacian operator with exponential nonlinearity in* $\mathbb{R}^4$, Ann. Inst. H. Poincaré Anal. Non Linéaire, **25** (2008) 1015–1041.

[7] Z. Djadli, E. Hebey and M. Ledoux: *Paneitz-type operators and applications,* Duke Math. J. **104** (2000) no.1, 129-169.

[8] P. Esposito and F. Robert: *Mountain pass critical points for Paneitz-Branson operators,* Calc. Var. Partial Differential Equations **15** (2002) no.4, 493-517.

[9] M. Grossi, and F. Takahashi: *Nonexistence of multi-bubble solutions to some elliptic equations on convex domains,* J. Funct. Anal. **259** (2010) 904-917.

[10] E. Hebey and F. Robert: *Coercivity and Struwe's compactness for Paneitz-type operators with constant coefficients,* Calc. Var. Partial Differential Equations **13** (2001) no.4, 497-517.

[11] C-S. Lin, and J-C. Wei: *Locating the peaks of solutions via the maximum principle II: a local version of the method of moving planes,* Comm. Pure and Appl. Math. **56** (2003) 784-809.

[12] C-S. Lin, and J-C. Wei: *Sharp estimates for bubbling solutions of a fourth order mean field equation,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6**, no.4 (2007) 599-630.

[13] J-C. Wei: *Asymptotic behavior of a nonlinear fourth order eigenvalue problem,* Comm. Partial Differential Equations, **21** (1996) no.9-10, 1451-1467.

Futoshi Takahashi, Department of Mathematics & OCAMI, Osaka City University, Sumiyoshi-ku, Osaka, 558-8585, Japan

E-mail address: futoshi@sci.osaka-cu.ac.jp