On the number of maximum points of least energy solutions to a two-dimensional Hénon equation with large exponent

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Abstract. In this note, we prove that least energy solutions of the two-dimensional Hénon equation
\[-\Delta u = |x|^{2\alpha} u^p \quad x \in \Omega, \quad u > 0 \quad x \in \Omega, \quad u = 0 \quad x \in \partial \Omega,\]
where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^2\) with \(0 \in \Omega\), \(\alpha \geq 0\) is a constant and \(p > 1\), have only one global maximum point when \(\alpha > e - 1\) and the nonlinear exponent \(p\) is sufficiently large. This answers positively to a recent conjecture by C. Zhao (preprint, 2011).

Keywords: Hénon equation, global maximum point, large exponent.


1. Introduction.

In this note we consider the problem
\[
\begin{cases}
-\Delta u = |x|^{2\alpha} u^p & x \in \Omega, \\
u > 0 & x \in \Omega, \\
u = 0 & x \in \partial \Omega,
\end{cases}
\]
where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^2\) with \(0 \in \Omega\), \(\alpha \geq 0\) is a constant and \(p > 1\). Since the Sobolev embedding \(H^1_0(\Omega) \hookrightarrow L^{p+1}(\Omega)\) is compact for any \(p > 1\), we can obtain at least one solution of (1.1) by a standard variational method. In fact, let us consider the constrained minimization problem
\[
S_p = \inf \left\{ \int_\Omega |\nabla v|^2 dx \mid v \in H^1_0(\Omega), \int_\Omega |x|^{2\alpha}|v|^{p+1} dx = 1 \right\}.
\]
Standard variational method implies that $S_p$ is achieved by a positive function $v_p \in H^1_0(\Omega)$ and then $u_p = S_p^{1/(p-1)} v_p$ solves (1.1). We call $u_p$ a least energy solution to the problem (1.1).

When $\alpha = 0$, several studies on the asymptotic behavior of least energy solutions $u_p$ as $p \to \infty$ have been done in [4], [5], [3] and [1]. Recently, Chunyi Zhao [9] extended the study to the case when $\alpha > 0$, and obtained some results. First he showed that for any $\alpha > 0$, there exists $\delta > 0$ such that the least energy solution $u_p$ satisfies $1 - \delta \leq \|u_p\|_{\infty} \leq \sqrt{e} + \delta$ for $p$ large enough. To state his results further, we introduce some notations: Let $x_p$ be a global maximum point of $u_p$ and define $\varepsilon_p > 0$ by the relation

$$\varepsilon_p^2 |x_p|^{2\alpha_p} \|u_p\|_{\infty}^{p-1} = 1.$$

Also define the function $\tilde{u}_p : \Omega_p = \frac{\Omega - x_p}{\varepsilon_p} \to \mathbb{R}$ such that

$$\tilde{u}_p(y) = \frac{p}{\|u_p\|_{\infty}} \left\{ u_p(\varepsilon_p y + x_p) - u_p(x_p) \right\}.$$

By using these symbols, the main result of C. Zhao reads as follows:

**Theorem 1** (Chunyi Zhao [9]) Assume $\alpha > e - 1$. Then $\varepsilon_p \to 0$ and $\Omega_p \to \mathbb{R}^2$ as $p \to \infty$. Also for any sequence $p_n \to \infty$ as $n \to \infty$, there exists a subsequence (again denoted by the same symbol) such that

$$\tilde{u}_{p_n}(y) \to U(y) := -2 \log \left( 1 + \frac{|y|^2}{8} \right) \text{ in } C^2_{\text{loc}}(\mathbb{R}^2), \quad |x_{p_n}|^{2+2\alpha_p} \|u_{p_n}\|_{\infty}^{p_n-1} \to \infty, \quad \text{dist}(x_{p_n}, \partial \Omega)^2 \|u_{p_n}\|_{\infty}^{p_n-1} \to \infty \quad (1.4)$$

as $n \to \infty$. Moreover, the least energy solution $u_p$ has at most two global maximum points in $\Omega$ for large $p$.

After obtaining these results, Zhao conjectured that $u_p$ has only one global maximum point when $p$ large in Theorem 1.

Main purpose of this note is to answer the conjecture affirmatively.

**Theorem 2** Under the assumption of Theorem 1, the number of global maximum points of least energy solution $u_p$ is exactly 1 for $p$ large enough.
For the proof, we will use the Morse index characterization of least energy solutions and an argument of [6]. Relations between the number of blowing-up points and the Morse indices of blowing-up solutions to a two-dimensional Liouville equation have been studied in [7], [8].

2. Proof of Theorem 2.

As in §1, let \( v_p \) denote a solution of (1.2), which may be chosen positive. Then \( v_p \) solves the equation

\[
-\Delta v_p = S_p |x|^{2\alpha} v_p^p \quad x \in \Omega, \quad v_p > 0 \quad x \in \Omega, \quad v_p = 0 \quad x \in \partial \Omega.
\]

Let \( u_p = S_p^{\frac{1}{p-1}} v_p \) be a least energy solution to (1.1). First, we recall the well-known fact, which says that the Morse index of \( u_p \) is less than or equal to 1 for any \( p > 1 \).

Lemma 3 Let \( L_p = -\Delta - p|x|^{2\alpha} u_p^{p-1}(x) : H^1_0(\Omega) \to H^{-1}(\Omega) \) denote the linearized operator around \( u_p \). Then the second eigenvalue of \( L_p \), denoted by \( \lambda_2(L_p, \Omega) \), is nonnegative.

Proof. When \( \alpha = 0 \), a proof of this lemma is shown, for example, in [2]. Proof in the case of \( \alpha > 0 \) is similar. Here we recall it for the sake of completeness.

Let \( \tilde{L}_p = -\Delta - pS_p x^2 u_p^{p-1} : H^1_0(\Omega) \to H^{-1}(\Omega) \) denote the linearized operator around \( v_p \). Since \( S_p u_p^{p-1} = u_p^{p-1} \), it is enough to show that the second eigenvalue of \( \tilde{L}_p \), denoted by \( \lambda_2(\tilde{L}_p, \Omega) \), is nonnegative. For this purpose, let us define

\[
f(t) = \frac{\int_\Omega |\nabla (v_p + t\varphi)|^2 dx}{(\int_\Omega |x|^{2\alpha}|v_p + t\varphi|^p + 1 dx)^{\frac{p}{p+1}}}
\]

for any \( \varphi \in H^1_0(\Omega) \). By the minimality of \( v_p \), we have \( f'(0) = 0 \) and \( f''(0) \geq 0 \). Calculation using \( \int_\Omega |x|^{2\alpha} v_p^{p+1} dx = 1 \) and \( \int_\Omega |\nabla v_p|^2 dx = S_p \) shows that

\[
f''(0) = 2 \left\{ \int_\Omega |\nabla \varphi|^2 dx - pS_p \int_\Omega |x|^{2\alpha} v_p^{p-1} \varphi^2 dx + (p - 1)S_p \left( \int_\Omega |x|^{2\alpha} v_p^p \varphi dx \right)^2 \right\}
\]

\[
= 2(\tilde{L}_p \varphi, \varphi)_{L^2(\Omega)} + 2(p - 1)S_p \left( \int_\Omega |x|^{2\alpha} v_p^p \varphi dx \right)^2.
\]
Combining this to a variational characterization of \( \lambda_2(\mathcal{T}_p, \Omega) \), we have

\[
\lambda_2(\mathcal{T}_p, \Omega) = \sup_{L \subset H^1_0(\Omega), \text{codim} L = 1} \inf_{\varphi \in L} \frac{\langle \mathcal{T}_p \varphi, \varphi \rangle_{L^2(\Omega)}}{\|\varphi\|_{L^2(\Omega)}} \geq \inf_{\varphi \in H^1_0(\Omega), \varphi \perp |x|^{2\alpha} v_p^2} \frac{1}{2} \|\varphi\|_{L^2(\Omega)}^2 = 0.
\]

By using this fact, we prove Theorem 2 by a contradiction argument.

**Proof of Theorem 2.** Assume the contrary that there exist two global maximum points \( x^{i}_{p_n}, x^{2}_{p_n} \) of \( u_{p_n}, x^{i}_{p_n} \in \Omega, \|u_{p_n}\|_{\infty} = u_{p_n}(x^{i}_{p_n}), (i = 1, 2) \) for some sequence \( p_n \to \infty \). Define \( \varepsilon^{i}_{p_n} > 0 \) by the relation

\[
\varepsilon^{i}_{p_n} = \sup_{\varphi \in H^1_0(\Omega)} \inf_{\varphi \perp |x|^{2\alpha} v_p^2} \frac{\langle \mathcal{T}_p \varphi, \varphi \rangle_{L^2(\Omega)}}{\|\varphi\|_{L^2(\Omega)}} \geq 0.
\]

and the scaled functions

\[
\tilde{u}^{i}_{p_n}(y) = \frac{p_n}{\|u_{p_n}\|_{\infty}} \left\{ u_{p_n}(\varepsilon^{i}_{p_n} y + x^{i}_{p_n}) - u_{p_n}(x^{i}_{p_n}) \right\}, \quad y \in \Omega^{i}_{p_n} := \frac{\Omega - x^{i}_{p_n}}{\varepsilon^{i}_{p_n}}
\]

for \( i = 1, 2 \). Now we assume \( \alpha > e - 1 \), so all results of Theorem 1 hold true for \( \tilde{u}^{i}_{p_n}, (i = 1, 2) \). In particular, there exists a subsequence (denoted by the same symbol again) such that (1.3) holds for both \( \tilde{u}^{i}_{p_n} \).

Next, we define elliptic operators

\[
\tilde{L}^{i}_{p_n} = -\Delta y - \frac{\varepsilon^{i}_{p_n} y + x^{i}_{p_n}}{|x^{i}_{p_n}|^{2\alpha}} \left( 1 + \frac{\tilde{u}^{i}_{p_n}(y)}{p_n} \right)^{p_n-1}, \quad (i = 1, 2)
\]

acting on \( H^1_0(\Omega^{i}_{p_n}) \). These operators are related to the operator \( L_{p_n} \) by the formula

\[
\varepsilon^{i}_{p_n} L_{p_n} x = \delta x^{i}_{p_n}, \quad y \in \Omega^{i}_{p_n},
\]

for \( i = 1, 2 \). Also, eigenvalues are related with each other by the formula

\[
\varepsilon^{i}_{p_n} \lambda_j(L_{p_n}, D) = \lambda_j(\tilde{L}^{i}_{p_n}, D^{i}_{p_n}), \quad D^{i}_{p_n} = \frac{D - \varepsilon^{i}_{p_n} x^{i}_{p_n}}{\varepsilon^{i}_{p_n}},
\]
where \( \lambda_j(L_{p_n}, D) \) will denote a \( j \)-th eigenvalue of the operator \( L_{p_n} \) acting on \( H^1_0(D) \) for a domain \( D \), etc.

Let \( B(a, R) = B_R(a) \) denote an open ball of center \( a \in \mathbb{R}^2 \) with radius \( R \). We prove the following:

**Lemma 4** There exist disjoint balls \( B^i \) (\( i = 1, 2 \)), each ball is of the form \( B(x_{p_n}^i, \epsilon_{p_n}^i, R) \) for some \( R > 0 \), such that \( \lambda_1(L_{p_n}, B^i) < 0 \) for \( i = 1, 2 \) when \( n \) sufficiently large.

**Proof.** For \( R > 0 \), we define

\[
    w_R(y) = 2 \log \frac{8 + R^2}{8 + |y|^2}.
\]

Since \( w_R = 0 \) on \( \partial B_R(0) \), we see \( w_R \in H^1_0(B_R(0)) \).

We will prove that \( (\tilde{L}_{p_n}^i w_R, w_R)_{L^2(B_R(0))} < 0 \) for \( R > 0 \) sufficiently large and \( B_R(0) \subset \Omega_{p_n}^i \). Indeed,

\[
    (\tilde{L}_{p_n}^i w_R, w_R)_{L^2(B_R(0))} = \int_{B_R(0)} |\nabla w_R|^2 dy
    - \int_{B_R(0)} \frac{\epsilon_{p_n}^i}{|x_{p_n}^i|} y + \frac{x_{p_n}^i}{|x_{p_n}^i|} \left( 1 + \frac{\tilde{u}_{p_n}^i(y)}{p_n} \right)^{p_n - 1} w_R^2(y) dy
    =: I_1 - I_2.
\]

We see

\[
    I_1 = \int_{B_R(0)} \frac{16|y|^2}{(8 + |y|^2)^2} dy = 2\pi \int_0^R \frac{16r^2}{(8 + r^2)^2} dr = 32\pi \left[ \log R + o_R(1) \right],
\]

where \( o_R(1) \to 0 \) as \( R \to \infty \). As for \( I_2 \), (1.4) implies \( \frac{\epsilon_{p_n}^i}{|x_{p_n}^i|} \to 0 \) as \( n \to \infty \) even if \( x_{p_n}^i \to 0 \). Also \( \frac{x_{p_n}^i}{|x_{p_n}^i|} \to \exists y_0, |y_0| = 1 \) for a subsequence. Therefore by choosing a subsequence, we have

\[
    \frac{\epsilon_{p_n}^i}{|x_{p_n}^i|} y + \frac{x_{p_n}^i}{|x_{p_n}^i|} \left( 1 + \frac{\tilde{u}_{p_n}^i(y)}{p_n} \right)^{p_n - 1} \to e^{U(y)}
\]

\( \tilde{u}_{p_n}^i \to 0 \) as \( n \to \infty \).
in $C^2_{\text{loc}}(\mathbb{R}^2)$ by (1.3). Thus,

$$I_2 = \int_{B_R(0)} \left| \frac{x_{p_n}^i}{|x_{p_n}^i|} y + \frac{x_{p_n}^i}{|x_{p_n}^i|} \right|^{2\alpha} \left( 1 + \frac{\tilde{v}_{p_n}^i(y)}{p_n} \right)^{p_n-1} w_R^2(y) dy$$

$$= \int_{B_R(0)} \frac{1}{\left( 1 + \frac{|y|^2}{8} \right)^2} \left\{ \log \frac{8 + R^2}{8 + |y|^2} \right\}^2 dy + o_n(1)$$

$$= 2\pi \int_0^R \frac{r}{\left( 1 + \frac{r^2}{8} \right)^2} \left\{ \log(8 + R^2) - \log(8 + r^2) \right\}^2 dr + o_n(1)$$

$$= 2\pi \cdot 8^2 \left\{ \log(8 + R^2) \right\}^2 \left[ \frac{1}{16} + o_R(1) \right] + o_n(1)$$

$$= 32\pi (\log R)^2 [1 + o_R(1)] + o_n(1),$$

where we have used $\int_0^\infty \frac{r}{(8+r^2)^2} dr = \frac{1}{16}$. Hence we obtain

$$(\tilde{L}_{p_n}^i w_R, w_R)_{L^2(B_R(0))} = I_1 - I_2 = -32\pi (\log R)^2 [1 + o_R(1)] < 0$$

by taking $n$ sufficiently large first, and then $R > 0$ large such that $B_R(0) \subset \Omega_{p_n}^i$. This implies that the first eigenvalue of the operator $\tilde{L}_{p_n}^i$ on $B_R$ is negative: $\lambda_1(\tilde{L}_{p_n}^i, B_R) < 0$. By this and the scaling formula (2.4) proves the first half part of the Lemma.

Recall that, under the assumption considered here, the following estimate is proved in [9] Lemma 5.1:

$$\frac{|x_{p} - x_{p_0}|}{\max\{\varepsilon_{p}^1, \varepsilon_{p_0}^1\}} \to \infty$$

as $p \to \infty$. This implies that these two balls $B(x_{p_n}^i, \varepsilon_{p_n}^i R)$ are disjoint for $n$ sufficiently large.

By Lemma 4, we have

$$\lambda_1(L_{p_n}, B^i) < 0 \quad i = 1, 2$$

for $n$ sufficiently large. On the other hand, a well known estimate of $\lambda_2(L_{p_n}, \Omega)$ claims

$$\lambda_2(L_{p_n}, \Omega) \leq \sum_{i=1}^2 \lambda_1(L_{p_n}, B^i).$$

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See, for example, [7] Appendix. From (2.5) and (2.6), we have $\lambda_2(L_{p_n}, \Omega) < 0$ for $n$ large, which contradicts to Lemma 3.

Acknowledgments. Part of this work was done while the author visited Laboratoire de mathématiques, Université de Cergy-Pontoise, Cergy, France, in November 2011. This work was supported by JSPS Grant-in-Aid for Scientific Research (KAKENHI) (B), No. 23340038.

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