Greeks formulas for asset price model with some Lévy processes *

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1 Preliminaries

Let $T>0$ be fixed throughout the paper, and $(\Omega,\mathscr{F},\mathbb{P})$ our underlying probability space. In particular, we assume that throughout the paper that $\mathbb{P}$ is a risk-neutral probability measure that is one of infinitely many of its kind. We denote by $\mathbb{E}_{\mathbb{P}}[\,\cdot\,]$ the expectation taken under the measure $\mathbb{P}$. Let $\mathbb{D}\big([0,T]\,;\mathbb{R}\big)$ be the set of right-continuous functions on $[0,T]$ with left hand limits.

Let $X=\big\{X_t\,;\,t\in[0,T]\big\}$ be a gamma process, that is, the one-sided pure-jump Lévy process with the Lévy measure given by

$$\nu_X(\mathrm{d}z)=\frac{a}{z}\,\exp(-bz)\,\mathbb{I}_{(0,+\infty)}(z)\,\mathrm{d}z, \tag{1}$$

where a and b are positive constants. For each time $0<t\leq T$, the characteristic function of its marginal X_t is

$$\mathbb{E}_{\mathbb{P}}\big[\exp\big(ixX_t\big)\big]=\left(1-\frac{ix}{b}\right)^{-at},\quad x\in\mathbb{R}, \tag{2}$$

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and the marginal density at time t is given in closed form:

$$p_t^X(x) = \frac{b^{at}}{\Gamma(at)}\, x^{at-1} \exp(-bx), \quad x \in [0,+\infty). \tag{3}$$

Let G and M be positive constants, and $0 < \alpha_p, \alpha_n < 2$ with $\alpha_p \neq 1$ and $\alpha_n \neq 1$. Let $Y = \left\{Y_t\,; t \in [0,T]\right\}$ be a tempered stable process, that is, a Lévy process without Gaussian component independent of the process X, which can be divided into two independent Lévy processes $Y^{(p)} = \left\{Y_t^{(p)}\,; t \in [0,T]\right\}$ and $Y^{(n)} = \left\{Y_t^{(n)}\,; t \in [0,T]\right\}$:

$$\left\{Y_t = Y_t^{(p)} + Y_t^{(n)}\,; t \in [0,T]\right\} \tag{4}$$

with the characteristic functions

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(iyY_t^{(p)}\right)\right] = \exp\left[t\,C_p\,\Gamma(-\alpha_p)\,\left\{(M-iy)^{\alpha_p} - M^{\alpha_p} + iy\,\alpha_p\,M^{-1+\alpha_p}\,\mathbb{I}_{(1,2)}(\alpha_p)\right\}\right], \tag{5}$$

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(iyY_t^{(n)}\right)\right] = \exp\left[t\,C_n\,\Gamma(-\alpha_n)\,\left\{(G+iy)^{\alpha_n} - G^{\alpha_n} - iy\,\alpha_n\,G^{-1+\alpha_n}\,\mathbb{I}_{(1,2)}(\alpha_n)\right\}\right], \tag{6}$$

where C_p, C_n are non-negative constants, and with the Lévy measures given by

$$\nu_Y^{(p)}(\mathrm{d}z) = C_p\,z^{-1-\alpha_p}\,\exp(-Mz)\,\mathbb{I}_{(0,+\infty)}(z)\,\mathrm{d}z, \tag{7}$$

$$\nu_Y^{(n)}(\mathrm{d}z) = C_n\,|z|^{-1-\alpha_n}\,\exp(-G|z|)\,\mathbb{I}_{(-\infty,0)}(z)\,\mathrm{d}z, \tag{8}$$

respectively. The Lévy processes $Y^{(p)}$ and $Y^{(n)}$ consist of jumps only in a single direction. This accounts for the downward and upward moves of the market, respectively. Moreover, they are subordinators in the positive and negative directions when $0 < \alpha_p, \alpha_n < 1$, while they are centered with jumps when $1 < \alpha_p, \alpha_n < 2$.

Let $B = \left\{B_t\,; t \in [0,T]\right\}$ and $W = \left\{W_t\,; t \in [0,T]\right\}$ be independent one-dimensional standard Brownian motion, which are also independent of the processes X and Y. Let $S_0 > 0$. Consider asset price dynamics models $S = \left\{S_t\,; t \in [0,T]\right\}$ given by

$$S_t = S_0\,\exp\left[\theta X_t + \kappa Y_t + \tau B_t + \sigma W_{X_t} + (c+\mu)\,t\right], \tag{9}$$

where $\theta \in \mathbb{R}$, $\tau \geq 0$, $\sigma \geq 0$, $0 \leq \kappa \leq M$, $c = c(\theta,\tau,\sigma)$ with the each partial derivatives $\partial_\theta c$, $\partial_\tau c$, $\partial_\sigma c$ being well defined, and

$$\mu = -C_p\,\Gamma(-\alpha_p)\,\left\{(M-\kappa)^{\alpha_p} - M^{\alpha_p} + \kappa\,\alpha_p\,M^{-1+\alpha_p}\,\mathbb{I}_{(1,2)}(\alpha_p)\right\}$$

$$- C_n \Gamma(-\alpha_n) \left\{ (G+\kappa)^{\alpha_n} - G^{\alpha_n} - \kappa \alpha_n G^{-1+\alpha_n} \mathbb{I}_{(1,2)}(\alpha_n) \right\}.$$

We shall remark that

$$\mathbb{E}_{\mathbb{P}} [\exp(\kappa Y_t)] = \exp(-\mu t).$$

Our model considered here includes a lot of well-known models of practical interest, such as the Black-Scholes model, the variance gamma model, the finite moment log stable model, and the CGMY model. In this paper, we shall pay attention to the Greeks formula, in particular, the Delta formula, via the Malliavin calculus for jump processes by making full unse of the Girsanov transform. Our asset price models can be of a pure-jump type, and also of infinite activity type. This is based upon joint works ([6, 7]) with Rei-ichiro Kawai (University of Leicester, UK).

2 Key lemmas

Here, we shall introduce some lemmas which play crucial roles in our argument. Recall $0 < \alpha_p, \alpha_n < 2$ with $\alpha_p, \alpha_n \neq 1$, and $C_p, C_n \geq 0$. Let $\mathbb{P}_0$ be a probability measure, under which the process $L^{(p)} = \left\{ L_t^{(p)} ; t \in [0,T] \right\}$ is an α_p-stable process without Gaussian component with characteristic functions

$$\mathbb{E}_{\mathbb{P}_0} \left[\exp\left(i y L_t^{(p)} \right) \right] = \exp\left[t C_p \Gamma(-\alpha_p) \left(\cos \frac{\pi \alpha_p}{2} \right) |y|^{\alpha_p} \left(1 - i \tan \frac{\pi \alpha_p}{2} \operatorname{sgn}(y) \right) \right], \qquad (10)$$

and the process $L^{(n)} = \left\{ L_t^{(n)} ; t \in [0,T] \right\}$ an α_n-stable process without Gaussian component with characteristic functions

$$\mathbb{E}_{\mathbb{P}_0} \left[\exp\left(i y L_t^{(n)} \right) \right] = \exp\left[t C_n \Gamma(-\alpha_n) \left(\cos \frac{\pi \alpha_n}{2} \right) |y|^{\alpha_n} \left(1 + i \tan \frac{\pi \alpha_n}{2} \operatorname{sgn}(y) \right) \right], \qquad (11)$$

which are independent each other. Their Lévy measures are given by

$$\nu_{L^{(p)}}(\mathrm{d}z) = C_p z^{-1-\alpha_p} \mathbb{I}_{(0,+\infty)}(z) \, \mathrm{d}z, \qquad (12)$$

$$\nu_{L^{(n)}}(\mathrm{d}z) = C_n |z|^{-1-\alpha_n} \mathbb{I}_{(-\infty,0)}(z) \, \mathrm{d}z, \qquad (13)$$

respectively. When $0 < \alpha_p, \alpha_n < 1$, the processes $L^{(p)}$ and $L^{(n)}$ are subordinators in the positive and negative directions, respectively. On the other hand, when $1 < \alpha_p, \alpha_n < 2$, they are centered with jumps only in a single direction. Define the process $L = \{L_t\,; t \in [0,T]\}$ by

$$L_t = L_t^{(p)} + L_t^{(n)}, \quad t \in [0,T]. \tag{14}$$

Remark that the process L is a stable process only if $\alpha_p = \alpha_n$. Then, it holds that

Lemma 1 (cf. [10] Theorem 33.2) *The two probability measures $\mathbb{P}$ and $\mathbb{P}_0$ are mutually absolute continuous through the Radon-Nikodým derivative*

$$\left.\frac{d\mathbb{P}}{d\mathbb{P}_0}\right|_{\mathscr{F}_T} = \frac{\exp\left(G L_T^{(n)} - M L_T^{(p)}\right)}{\mathbb{E}_{\mathbb{P}_0}\left[\exp\left(G L_T^{(n)} - M L_T^{(p)}\right)\right]} = \exp\left(G L_T^{(n)} - M L_T^{(p)} - \gamma_1 T\right), \quad \mathbb{P}_0\text{-}a.s., \tag{15}$$

where $(\mathscr{F}_t\,; t \in [0,T])$ is the natural filtration generated by the processes $L^{(p)}$, $L^{(n)}$, and

$$\gamma_1 = C_n \Gamma(-\alpha_n) G^{\alpha_n} + C_p \Gamma(-\alpha_p) M^{\alpha_p}.$$

Moreover, it holds that

$$\mathbb{P}_0\left(\{L_t + \gamma_2 t\,; t \in [0,T]\} \in A\right) = \mathbb{P}\left(\{Y_t\,; t \in [0,T]\} \in A\right), \tag{16}$$

for $A \in \mathscr{B}(\mathbb{D}([0,T]\,;\mathbb{R}))$, where

$$\gamma_2 = C_n \Gamma(1-\alpha_n) G^{-1+\alpha_n} \mathbb{I}_{(1,2)}(\alpha_n) - C_p \Gamma(1-\alpha_p) M^{-1+\alpha_p} \mathbb{I}_{(1,2)}(\alpha_p).$$

Let E_p and E_n be independent standard exponential random variables, and define

$$U_p = -C_p \Gamma(-\alpha_p) \cos\left(\frac{\pi \alpha_p}{2}\right) \frac{\sin\left(\alpha_p (V_p + \eta_p)\right)}{\left\{\cos(\alpha_p \eta_p) \cos V_p\right\}^{1/\alpha_p}} \left\{\cos\left(\alpha_p \beta_p V_p - \alpha_p \eta_p\right)\right\}^{\beta_p}, \tag{17}$$

$$U_n = -C_n \Gamma(-\alpha_n) \cos\left(\frac{\pi \alpha_n}{2}\right) \frac{\sin\left(\alpha_n (V_n + \eta_n)\right)}{\left\{\cos(\alpha_n \eta_n) \cos V_n\right\}^{1/\alpha_n}} \left\{\cos\left(\alpha_n \beta_n V_n - \alpha_n \eta_n\right)\right\}^{\beta_n}, \tag{18}$$

where $\eta_p = \arctan\left(-\tan\frac{\pi \alpha_p}{2}\right)/\alpha_p$, $\eta_n = \arctan\left(-\tan\frac{\pi \alpha_n}{2}\right)/\alpha_n$, $\beta_p = (1-\alpha_p)/\alpha_p$, $\beta_n = (1-\alpha_n)/\alpha_n$, and V_p and V_n are independent uniform random variables on $(-\pi/2, \pi/2)$, which are also independent of E_p and E_n. Denote by

$$\tilde{L}^{(p)} = \left\{\tilde{L}_t^{(p)} := t^{1/\alpha_p} U_p E_p^{-\beta_p}\,; t \in [0,T]\right\}, \tag{19}$$

$$\tilde{L}^{(n)} = \left\{ \tilde{L}_t^{(n)} := t^{1/\alpha_n} U_n E_n^{-\beta_n} \,; t \in [0,T] \right\}, \tag{20}$$

$$\tilde{L} = \left\{ \tilde{L}_t := \tilde{L}_t^{(p)} + \tilde{L}_t^{(n)} \,; t \in [0,T] \right\}. \tag{21}$$

Then, we see that, for each $t > 0$, the random variables $\tilde{L}_t^{(p)}$ and $L_t^{(p)}$, or, $\tilde{L}_t^{(n)}$ and $L_t^{(n)}$ have the same law, respectively (cf. [4]). Let $\Theta = (\Theta_1, \Theta_2)$ be a standard normal random vector in $\mathbb{R}^2$, which is independent of the processes X and Y. Define the new process $\tilde{S} = \left\{ \tilde{S}_t \,; t \in [0,T] \right\}$ by

$$\tilde{S}_t := S_0 \exp\left[\theta X_t + \kappa \left(\tilde{L}_t + \gamma_2 t \right) + \tau \sqrt{t}\, \Theta_1 + \sigma \sqrt{X_t}\, \Theta_2 + (c + \mu)\, t \right]. \tag{22}$$

Remark 1 The probability law of S_t is not equivalent to the one of $\tilde{S}_t$, because of Lemma 1. $\square$

Let $\delta = (\delta_1, \delta_2) \in [0,+\infty) \times [0,+\infty)$, $\xi \in [0,+\infty)$ with $\lambda \xi < b$, and $(\zeta_1, \zeta_2) \in [0,+\infty) \times [0,+\infty)$ with $\lambda \zeta_i < 1$ $(i = 1,2)$. For each $t \in [0,T]$, denote by $\mathscr{G}_t$ the minimal σ-field generated by $\sigma\left[\{X_s \,; s \in [0,t]\}\right]$, $\sigma[\Theta]$, and $\sigma[(E_p, E_n)]$. Then, define the new probability measure $\mathbb{Q}_\lambda$ equivalent to $\mathbb{P}_0$ over $\mathscr{G}_T$, via the Radon-Nikodým derivative

$$\begin{aligned}
\left. \frac{\mathrm{d}\mathbb{Q}_\lambda}{\mathrm{d}\mathbb{P}_0} \right|_{\mathscr{G}_T} &:= \frac{\exp(\lambda \xi X_T)}{\mathbb{E}_{\mathbb{P}_0}[\exp(\lambda \xi X_T)]} \frac{\exp(\lambda \langle \delta, \Theta \rangle)}{\mathbb{E}_{\mathbb{P}_0}[\exp(\lambda \langle \delta, \Theta \rangle)]} \frac{\exp(\lambda (\zeta_1 E_p + \zeta_2 E_n))}{\mathbb{E}_{\mathbb{P}_0}[\exp(\lambda (\zeta_1 E_p + \zeta_2 E_n))]} \\
&= \left(1 - \frac{\lambda \xi}{b} \right)^{aT} (1 - \lambda \zeta_1)(1 - \lambda \zeta_2) \\
&\qquad \times \exp\left(\lambda \xi X_T + \lambda \langle \delta, \Theta \rangle - \frac{\lambda^2 |\delta|^2}{2} + \lambda (\zeta_p E_p + \zeta_n E_n) \right), \quad \mathbb{P}_0\text{-a.s.}.
\end{aligned} \tag{23}$$

Lemma 2 (the Esscher transform, cf. [6, 7]) *For $x \in [0,+\infty)$, $(z_p, z_n) \in [0,+\infty) \times [0,+\infty)$, $(\theta_1, \theta_2) \in \mathbb{R}^2$, and $t \geq 0$, we have*

$$\begin{aligned}
&\mathbb{Q}_\lambda (X_t \leq x,\ \Theta_1 \leq \theta_1,\ \Theta_2 \leq \theta_2,\ E_p \leq z_p,\ E_n \leq z_n) \\
&= \mathbb{P}_0\left(\frac{b X_t}{b - \lambda \xi} \leq x \right) \mathbb{P}_0(\Theta_1 + \lambda \delta_1 \leq \theta_1)\, \mathbb{P}_0(\Theta_2 + \lambda \delta_2 \leq \theta_2) \\
&\qquad \times \mathbb{P}_0\left(\frac{E_p}{1 - \lambda \zeta_1} \leq z_p \right) \mathbb{P}_0\left(\frac{E_n}{1 - \lambda \zeta_2} \leq z_n \right).
\end{aligned} \tag{24}$$

3 Results and sketch of proofs

Let $0<\varepsilon<1$, and $\Psi_\varepsilon \in C_b^\infty(\mathbb{R};\mathbb{R})$ such that

$$\Psi_\varepsilon(V)=\begin{cases}0 & (|V|\le\varepsilon)\\ 1 & (|V|\ge 2\varepsilon).\end{cases}$$

In order to avoid lengthy expression, let us prepare some auxiliary notations. Define

$$\begin{aligned}
F_T &:= \xi\left(\theta X_T+\frac{\sigma}{2}\sqrt{X_T}\,\Theta_2\right)+\delta_1 b\,\tau\sqrt{T}+\delta_2 b\,\sigma\sqrt{X_T},\\
H_T &:= \xi\,\frac{\sigma}{4}\sqrt{X_T}\,\Theta_2+\delta_1 b\,\tau\sqrt{T},\\
J_T &:= \zeta_1\left(1-E_p+\beta_p M\tilde{L}_T^{(p)}\right)+\zeta_2\left(1-E_n-\beta_n G\tilde{L}_T^{(n)}\right),\\
K_T &:= \zeta_1\beta_p\tilde{L}_T^{(p)}+\zeta_2\beta_n\tilde{L}_T^{(n)},\quad N_T:=\zeta_1^2\beta_p^2\tilde{L}_T^{(p)}+\zeta_2^2\beta_n^2\tilde{L}_T^{(n)},\\
\Xi_T &:= \frac{\partial}{\partial\lambda}\left(\ln\frac{\tilde{S}_T^{(\lambda)}}{S_0}\right)\Bigg|_{\lambda=0}\left(=\frac{F_T}{b}-\kappa K_T\right),\\
\Gamma_T &:= \Psi_\varepsilon(\Xi_T)\left\{\frac{\frac{\xi}{b}\left(bX_T-aT\right)+\langle\delta,\Theta\rangle-J_T}{\Xi_T}+\frac{\frac{\xi}{b^2}(F_T-H_T)+\kappa N_T}{\Xi_T^2}\right\}\\
&\qquad -\Psi'_\varepsilon(\Xi_T)\,\frac{\frac{\xi}{b^2}(F_T-H_T)+\kappa N_T}{\Xi_T}.
\end{aligned}$$

Theorem 1 (cf. [6, 7]) *For $\Phi\in C_b^1(\mathbb{R};\mathbb{R})$, it holds that*

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_0}\left[\frac{\exp\left(G\tilde{L}_T^{(n)}-M\tilde{L}_T^{(p)}\right)}{\mathbb{E}_{\mathbb{P}_0}\left[\exp\left(GL_T^{(n)}-ML_T^{(p)}\right)\right]}\,\Phi'\left(\tilde{S}_T\right)\tilde{S}_T\,\Psi_\varepsilon\left(\Xi_T\right)\right]\\
&=\mathbb{E}_{\mathbb{P}_0}\left[\frac{\exp\left(G\tilde{L}_T^{(n)}-M\tilde{L}_T^{(p)}\right)}{\mathbb{E}_{\mathbb{P}_0}\left[\exp\left(GL_T^{(n)}-ML_T^{(p)}\right)\right]}\,\Phi\left(\tilde{S}_T\right)\Gamma_T\right].
\end{aligned}\tag{25}$$

Proof. From Lemma 2, we have

$$\mathbb{E}_{\mathbb{P}_0}\left[\frac{\exp\left(G\tilde{L}_T^{(n,\lambda)}-M\tilde{L}_T^{(p,\lambda)}\right)}{\mathbb{E}_{\mathbb{P}_0}\left[\exp\left(GL_T^{(n)}-ML_T^{(p)}\right)\right]}\,\frac{\Phi\left(\tilde{S}_T^{(\lambda)}\right)}{\Xi_T^{(\lambda)}}\,\Psi_\varepsilon\left(\Xi_T^{(\lambda)}\right)\right]$$

$$
= \mathbb{E}_{\mathbb{P}_0}\left[\frac{\exp\left(G\tilde{L}_T^{(n)} - M\tilde{L}_T^{(p)}\right)}{\mathbb{E}_{\mathbb{P}_0}\left[\exp\left(GL_T^{(n)} - ML_T^{(p)}\right)\right]} \frac{\Phi\left(\tilde{S}_T\right)}{\Xi_T} \Psi_\varepsilon\left(\Xi_T\right) \frac{\mathrm{d}\mathbb{Q}_\lambda}{\mathrm{d}\mathbb{P}_0}\Big|_{\mathscr{G}_T} \right],
$$

where

$$
\tilde{L}_t^{(p,\lambda)} = t^{1/\alpha_p} U_p \left(\frac{E_p}{1-\lambda\,\zeta_1}\right)^{-\beta_p}, \ \tilde{L}_t^{(n,\lambda)} = t^{1/\alpha_n} U_n \left(\frac{E_n}{1-\lambda\,\zeta_2}\right)^{-\beta_n},
$$

$$
\tilde{L}_t^{(\lambda)} = \tilde{L}_t^{(p,\lambda)} + \tilde{L}_t^{(n,\lambda)}, \quad \Theta_1^{(\lambda)} = \Theta_1 + \lambda\,\delta_1, \quad \Theta_2^{(\lambda)} = \Theta_2 + \lambda\,\delta_2, \quad X_t^{(\lambda)} = \frac{bX_t}{b-\lambda\,\xi},
$$

$$
\tilde{S}_t^{(\lambda)} = S_0 \exp\left[\theta\, X_t^{(\lambda)} + \kappa\left(\tilde{L}_t^{(\lambda)} + \gamma_2\, t\right) + \tau\sqrt{t}\,\Theta_1^{(\lambda)} + \sigma\sqrt{X_t^{(\lambda)}}\,\Theta_2^{(\lambda)} + (c+\mu)\,t\right],
$$

$$
F_t^{(\lambda)} = \xi\left(\theta\, X_t^{(\lambda)} + \frac{\sigma}{2}\sqrt{X_t^{(\lambda)}}\,\Theta_2^{(\lambda)}\right) + \delta_1\, b\,\tau\sqrt{t} + \delta_2\, b\,\sigma\sqrt{X_t^{(\lambda)}},
$$

$$
K_t^{(\lambda)} = \zeta_1\,\beta_p\,\tilde{L}_t^{(p,\lambda)} + \zeta_2\,\beta_n\,\tilde{L}_t^{(n,\lambda)}, \quad \Xi_t^{(\lambda)} = \frac{F_t^{(\lambda)}}{b} - \kappa\, K_t^{(\lambda)}.
$$

Take the derivative with respect to $\lambda = 0$. The integrability of the processes and the condition on the function Φ enable us to check the interchange of the derivative and the expectation. $\square$

Finally, we shall define the class of functions, in order to present the practical sensitivity formula. Denote by $C_{LG}(\mathbb{R};\mathbb{R})$ the set of continuous functions with the linear growth order, and define

$$
\mathfrak{F}(\mathbb{R};\mathbb{R}) = \left\{ \sum_{k=1}^{n} \alpha_k\, f_k\, \mathbb{I}_{A_k};\, \alpha_k \in \mathbb{R},\ f_k \in C_{LG}(\mathbb{R};\mathbb{R}),\ A_k \subset \mathbb{R}\text{: interval} \right\}.
$$

The standard density argument stated in [6] enables us to extend the class of functions Φ from $C_b^1(\mathbb{R};\mathbb{R})$ to $\mathfrak{F}(\mathbb{R};\mathbb{R})$, in which the existence of a smooth density for the processes plays a crucial role.

Corollary 1 (Sensitivity formula, cf. [6, 7]) *For $\Phi \in \mathfrak{F}(\mathbb{R};\mathbb{R})$, it holds that*

$$
\begin{aligned}
&\frac{\partial}{\partial S_0}\mathbb{E}_{\mathbb{P}_0}\left[\frac{\exp\left(G\tilde{L}_T^{(n)} - M\tilde{L}_T^{(p)}\right)}{\mathbb{E}_{\mathbb{P}_0}\left[\exp\left(GL_T^{(n)} - ML_T^{(p)}\right)\right]} \Phi\left(\tilde{S}_T\right)\Psi_\varepsilon\left(\Xi_T\right) \right] \\
&= \frac{1}{S_0}\,\mathbb{E}_{\mathbb{P}_0}\left[\frac{\exp\left(G\tilde{L}_T^{(n)} - M\tilde{L}_T^{(p)}\right)}{\mathbb{E}_{\mathbb{P}_0}\left[\exp\left(GL_T^{(n)} - ML_T^{(p)}\right)\right]} \Phi\left(\tilde{S}_T\right)\Gamma_T \right].
\end{aligned} \tag{26}
$$

Remark 2 The class $\mathfrak{F}(\mathbb{R};\mathbb{R})$ is not as general as that of measurable functions Φ such that $\mathbb{E}_{\mathbb{P}}\left[|\Phi(S_T)|^2\right] < +\infty$. But, the class $\mathfrak{F}(\mathbb{R};\mathbb{R})$ is rich enough in the sense that the European payoffs of interest, either continuous or discontinuous. $\square$

Remark 3 If $1/\Xi_T \in \mathbb{L}^p(\Omega)$ for any $p > 1$, we can get rid of the effect $\Psi_\varepsilon(\Xi_T)$ from the formulas (22) and (23). Then, we can get the Delta formula of the form:

$$\begin{aligned}\frac{\partial}{\partial S_0}\mathbb{E}_{\mathbb{P}}\left[\Phi(S_T)\right] &= \frac{\partial}{\partial S_0}\mathbb{E}_{\mathbb{P}_0}\left[\frac{\exp\left(G\tilde{L}_T^{(n)} - M\tilde{L}_T^{(p)}\right)}{\mathbb{E}_{\mathbb{P}_0}\left[\exp\left(GL_T^{(n)} - ML_T^{(p)}\right)\right]}\,\Phi\big(\tilde{S}_T\big)\right] \\ &= \frac{1}{S_0}\,\mathbb{E}_{\mathbb{P}_0}\left[\frac{\exp\left(G\tilde{L}_T^{(n)} - M\tilde{L}_T^{(p)}\right)}{\mathbb{E}_{\mathbb{P}_0}\left[\exp\left(GL_T^{(n)} - ML_T^{(p)}\right)\right]}\,\Phi\big(\tilde{S}_T\big)\,\tilde{\Gamma}_T\right],\end{aligned} \tag{27}$$

where

$$\tilde{\Gamma}_T := \frac{\frac{\xi}{b}\left(bX_T - aT\right) + \langle\delta,\Theta\rangle - J_T}{\Xi_T} + \frac{\frac{\xi}{b^2}(F_T - H_T) + \kappa N_T}{\Xi_T^2}.$$

As for sufficient conditions on the higher order integrability of $1/\Xi_T$, see [6, 7]. $\square$

Remark 4 Our process $\left\{S_t\,;\,t\in[0,T]\right\}$ includes the well-known asset price dynamics models:

- *the Black-Scholes model*: $\theta = \kappa = \sigma = 0$ and $c = -\tau^2/2$.

- *the variance gamma model* (cf. [8]) : $\theta = \kappa = \tau = 0$ and $\sigma > 0$. On the other hand, choose the parameters $a = b =: \kappa^{-1}$ for the gamma process $X = \left\{X_t\,;\,t\in[0,T]\right\}$ in the model $\left\{\theta X_t + \sigma W_{X_t}\,;\,t\in[0,T]\right\}$. Then, the process $\left\{\theta X_t + \sigma W_{X_t}\,;\,t\in[0,T]\right\}$ is also the variance gamma process.

- *the CGMY model* (cf. [2]): $\theta = \tau = \sigma = 0$, $C_p = C_n$, $\alpha_p = \alpha_n$, and $\kappa > 0$. In particular, the model in case of $\alpha_p = \alpha_n = 1/2$ is known as *the inverse Gaussian model*.

- *the finite moment log stable model* (cf. [3, 7]): $\theta = \tau = \sigma = 0$, $\kappa > 0$, $1 < \alpha_n < 2$, and $C_p = 0$, by taking the limit as $G \to 0$ and $M \to 0$. $\square$

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