Classification of complex projective towers up to dimension 8 and cohomological rigidity

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Abstract. A complex projective tower or simply a $\mathbb{C}P$-tower is an iterated complex projective fibrations starting from a point. In this paper we classify all 6-dimensional $\mathbb{C}P$-towers up to diffeomorphism, and as a consequence, we show that all such manifolds are cohomologically rigid, i.e., they are completely determined up to diffeomorphism by their cohomology rings. We also show that cohomological rigidity is not valid for 8-dimensional $\mathbb{C}P$-towers by classifying all $\mathbb{C}P^1$-fibrations over $\mathbb{C}P^3$ up to diffeomorphism. As a corollary we show that such $\mathbb{C}P$-towers are diffeomorphic if they are homotopy equivalent.

Contents

1. Introduction

A *complex projective tower* (or simply a $\mathbb{C}P$-*tower*) of height m is a sequence of complex projective fibrations

$$C_m \xrightarrow{\pi_m} C_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} C_1 \xrightarrow{\pi_1} C_0 = \{\text{a point}\} \tag{1.1}$$

where $C_i = P(\xi_{i-1})$ is the projectivization of a complex vector bundle ξ_{i-1} over C_{i-1}. It is also called an *m-stage $\mathbb{C}P$-tower*. We call each C_i the *ith stage* of the tower. Hence a $\mathbb{C}P$-tower is an iterated complex projective bundles starting from a point.

The $\mathbb{C}P$-towers contain many interesting classes of manifolds. For example, if each complex vector bundle ξ_i is a Whitney sum of complex line bundles, such $\mathbb{C}P$-tower is a *generalized Bott tower*, introduced in [**CMS10**]. If each ξ_i is a sum of two complex line bundles, then it is a *Bott tower*, introduced in [**BoSa**] (also see [**GrKa**]). In particular, Hirzebruch surfaces are nothing but 2-stage Bott towers. Moreover, flag manifolds of type A, i.e., $U(n+1)/T^{n+1} \cong \mathcal{F}\ell(\mathbb{C}^{n+1})$, and type C, i.e., $Sp(n)/T^n$ have n-stage $\mathbb{C}P$-tower structures, see Example 2.3 and 2.4, and the Milnor surface $H_{ij} \subset \mathbb{C}P^i \times \mathbb{C}P^j$ has a structure of 2-stage $\mathbb{C}P$-tower, see Examle 2.6.

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It is well known that there are only two diffeomorphism types of Hirzebruch surfaces, namely, $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, and their cohomology rings are not isomorphic. Hence, Hirzebruch surfaces are classified up to diffeomorphism by their cohomology rings. One might ask whether the same is true for Bott towers or generalized Bott towers. Namely, the cohomological rigidity question for (generalized) Bott towers asks whether the diffeomorphism classes of (generalized) Bott towers are determined by their cohomology rings. There are some partial affirmative answers to the question in [**CMS10, CPS, MaPa**], and we refer the reader to [**CMS11**] for the summary of the most recent developments about the question. In particular, the class of m-stage Bott towers for $m \leq 4$ ([**Ch**] and [**CMS10**]) and the class of 2-stage generalized Bott towers [**CMS10**] are cohomologically rigid, i.e., their diffeomorphism types are determined by their cohomology rings.

Since the (generalized) Bott tower is a special kind of $\mathbb{C}P$-towers, one might ask the cohomological rigidity question for $\mathbb{C}P$-towers. On the other hand, if one note that the cohomology ring of a projective bundle $P(\xi)$ is determined by the cohomology ring of the base space of $P(\xi)$ and the Chern classes of the complex bundle ξ (see (2.1)), then the expectation for the affirmative answer to the question can not be high, because complex vector bundles are not classified by their Chern classes in general. Therefore, it might be interesting to determine whether cohomological rigidity indeed fails to hold for $\mathbb{C}P$-towers, and if so, exactly in what dimension, does it fail? In this paper, we answer these questions by complete classification of $\mathbb{C}P$-towers up to dimension 6, and some special 2-stage $\mathbb{C}P$-towers of dimension 8.

We now describe our classification results. Note that the only 2-dimensional $\mathbb{C}P$-tower is $\mathbb{C}P^1$. Any 4-dimensional $\mathbb{C}P$-tower is either $\mathbb{C}P^2$ or a 2-stage $\mathbb{C}P$-tower which is in fact nothing but a Hirzebruch surface. So they are either $H_0 := \mathbb{C}P^1 \times \mathbb{C}P^1$ or $H_1 := \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. For 6-dimensional $\mathbb{C}P$-towers, we have to consider one-stage $\mathbb{C}P$-tower which is $\mathbb{C}P^3$, two-stage $\mathbb{C}P$-towers, and three-stage $\mathbb{C}P$-towers separately. For 2-stage 6-dimensional $\mathbb{C}P$-towers, there are two cases; the cases when the first stages are $C_1 = \mathbb{C}P^1$ and $C_1 = \mathbb{C}P^2$. When $C_1 = \mathbb{C}P^1$, then $C_2 = P(\xi)$ where ξ is a sum of three line bundles. Therefore, C_2 must be a 2-stage generalized Bott tower, which is completely determined in [**CMS10**]. In fact, there are only three diffeomorphism types $P(\gamma_1^k \oplus \epsilon \oplus \epsilon) \to \mathbb{C}P^1$ for $k = 0,\ 1,\ 2$, where γ_1 is the tautological line bundle over $\mathbb{C}P^1$.

For 2-stage 6-dimensional $\mathbb{C}P$-towers with $C_1 = \mathbb{C}P^2$, the second stage $C_2 = P(\xi)$, where ξ is a rank 2-complex vector bundle over $\mathbb{C}P^2$, which is determined by its Chern classes $c_1 \in H^2(\mathbb{C}P^2) \simeq \mathbb{Z}$ and $c_2 \in H^4(\mathbb{C}P^2) \simeq \mathbb{Z}$. It is proved that the diffeomorphism types of such $\mathbb{C}P$-towers are $P(\eta_{(0,\alpha)}) \to \mathbb{C}P^2$ and $P(\eta_{(1,\alpha)}) \to \mathbb{C}P^2$ for $\alpha \in H^4(\mathbb{C}P^2) \simeq \mathbb{Z}$, where $\eta_{(s,\alpha)}$ is a $\mathbb{C}$-vector bundle over $\mathbb{C}P^2$ whose Chern classes are $(c_1, c_2) = (s, \alpha)$.

For 3-stage $\mathbb{C}P$-towers $C_3 \to C_2 \to C_1$, there are two cases, i.e., when $C_2 = H_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ and $C_2 = H_1 = \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. Then $C_3 = P(\xi)$ where ξ is a complex 2-dimensional vector bundle over C_2. Again, it is proved in Lemma 4.1 that ξ is classified by its Chern classes c_1 and c_2. Let $\eta_{(s,r,\alpha)}$ (resp. $\xi_{(s,r,\alpha)}$) be the complex 2-dimensional bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$ (resp. $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$) whose first Chern class $c_1(\eta_{(s,r,\alpha)}) = (s,r) \in H^2(\mathbb{C}P^1 \times \mathbb{C}P^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$ (resp. $c_1(\xi_{(s,r,\alpha)}) = (s,r) \in H^2(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})$) and the second Chern class $c_2(\eta_{(s,r,\alpha)}) = \alpha \in H^4(\mathbb{C}P^1 \times \mathbb{C}P^1) \simeq \mathbb{Z}$ (resp. $c_2(\xi_{(s,r,\alpha)}) = \alpha \in H^4(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})$). Then, it is proved that all diffeomorphism types of 3-stage $\mathbb{C}P$-towers are $P(\zeta_{(s,r,\alpha)}) \to H_0$ and $P(\xi_{(s,r,\alpha)}) \to H_1$ for $\alpha \in \mathbb{Z}$ and $(s,r) = (0,0)$, $(1,0)$ or $(1,1)$.

We thus have the following classification result of 6-dimensional $\mathbb{C}P$-towers.

THEOREM 1.1. *Any 6-dimensional $\mathbb{C}P$-tower is diffeomorphic to one of the following distinct manifolds:*

- $\mathbb{C}P^3$;
- $P(\gamma_1^k \oplus \epsilon \oplus \epsilon) \to \mathbb{C}P^1$ *for* $k = 0, 1, 2$;
- $P(\eta_{(0,\alpha)}) \to \mathbb{C}P^2$ *for* $\alpha \in \mathbb{Z} \setminus \{0\}$;
- $P(\eta_{(1,\alpha)}) \to \mathbb{C}P^2$ *for* $\alpha \in \mathbb{Z}$;
- $P(\zeta_{(0,0,\alpha)}) \to H_0$ *for* $\alpha \in \mathbb{Z}_{\geq 0}$;
- $P(\zeta_{(1,0,\alpha)}) \to H_0$ *for* $\alpha \in \mathbb{Z}_{\geq 0}$;
- $P(\zeta_{(1,1,\alpha)}) \to H_0$ *for* $\alpha \in \mathbb{N}$;

- *$P(\xi_{(0,0,\alpha)}) \to H_1$ for $\alpha \in \mathbb{N}$;*
- *$P(\xi_{(1,0,\alpha)}) \to H_1$ for $\alpha \in \mathbb{Z}_{\geq 0}$;*
- *$P(\xi_{(1,1,\alpha)}) \to H_1$ for $\alpha \in \mathbb{Z}$,*

where $H_0 := \mathbb{C}P^1 \times \mathbb{C}P^1$, $H_1 := \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, and the symbols $\mathbb{N}$, $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}$ represent natural numbers, non-negative integers and integers, respectively.

Since the cohomology rings of the manifolds in Theorem 1.1 are not mutually isomorphic, we have the following corollary on cohomological rigidity of $\mathbb{C}P$-towers.

Corollary 1.2. Let M_1 and M_2 be two $\mathbb{C}P$-towers of dimension less than or equal to 6. Then, M_1 and M_2 are diffeomorphic if and only if their cohomology rings $H^*(M_1)$ and $H^*(M_2)$ are isomorphic.

This corollary is a generalization of the cohomological rigidity theorem for Bott manifolds up to dimension less than or equal to 6 proved in [**CMS10**].

For Bott manifolds of dimension 8 cohomological rigidity theorem is also proved to be true by Choi in [**Ch**]. However, it is not the case for $\mathbb{C}P$-towers. Namely, we classify all 8-dimensional 2 stage $\mathbb{C}P$-towers $C_2 \to C_1$ when $C_1 = \mathbb{C}P^3$. In this case $C_2 = P(\xi)$ where ξ is a complex 2-dimensional vector bundle over $\mathbb{C}P^3$. By the result of Atiyah and Rees [**AtRe**], any complex 2-dimensional vector bundle ξ over $\mathbb{C}P^3$ is determined by its first and the second Chern classes c_1 and c_2 and an invariant $\alpha \in \mathbb{Z}_2$ which is 0 when c_1 is odd. Let $\eta_{(\alpha,c_1,c_2)}$ be the complex 2-dimensional vector bundle with the given invariants α, c_1 and c_2. Then we have the following classification theorem of $P(\eta_{(\alpha,c_1,c_2)})$.

Theorem 1.3. *Let M be the projectivization of a 2-dimensional complex vector bundle over $\mathbb{C}P^3$. Then, M is diffeomorphic to one of the following distinct manifolds:*

- *$M_0(u) = P(\eta_{(0,0,u)})$;*
- *$M_1(u) = P(\eta_{(1,0,u)})$;*
- *$N(u) = P(\eta_{(0,1,u)})$,*

for $u \in H^4(\mathbb{C}P^3) \simeq \mathbb{Z}$.

By the Borel-Hirzebruch formula (2.1), we have $H^*(M_0(u)) \simeq H^*(M_1(u))$, while $M_0(u)$ is not diffeomorphic to $M_1(u)$. This proves that 8-dimensional $\mathbb{C}P$-towers are not cohomologically rigid.

On the other hand, we prove that $\pi_6(M_0(u)) \not\simeq \pi_6(M_1(u))$ in Proposition 5.8. Therefore, we have the following homotopical rigidity result.

Corollary 1.4. Let M_1 and M_2 be the projectivizations of two complex 2-dimensional vector bundles over $\mathbb{C}P^3$. Then M_1 and M_2 are homotopic if and only if they are diffeomorphic

Moreover, we prove $Sp(2)/T^2 \cong M_1(1)$ in Appendix. Therefore, Theorem 1.3 also says that there is a $\mathbb{C}P$-tower which has the same cohomology ring with flag manifolds of type C, i.e., $Sp(n)/T^n$, but it is not the flag manifold of types C.

The organization of this paper is as follows. In Section 2, we prepare some basics and some examples. In Section 3, we classify 6-dimensional $\mathbb{C}P$-towers with height 2 up to diffeomorphism. In Section 4, we classify 6-dimensional $\mathbb{C}P$-towers with height 3. Theorem 1.1 is proved as a consequence of the classification. In Section 5, we classify the projectivizations of 2-dimensional complex vector bundles over $\mathbb{C}P^3$, and Theorem 1.3 is proved. In Appendix, we prove $Sp(2)/T^2 \cong M_1(1)$.

2. Some preliminaries

In this section, we prepare some basic facts which will be used in later sections. Let ξ be an n-dimensional complex vector bundle over a topological space X, and let $P(\xi)$ denote its projectivization. Then the *Borel-Hirzebruch formula* in [**BoHi**] says

$$H^*(P(\xi);\mathbb{Z}) \simeq H^*(X;\mathbb{Z})[x]/\langle x^{n+1} + \sum_{i=1}^{n} (-1)^i c_i(\pi^*\xi) x^{n+1-i} \rangle \tag{2.1}$$

where $\pi^*\xi$ is the pull-back of ξ along $\pi : P(\xi) \to X$ and $c_i(\pi^*\xi)$ is the ith Chern class of $\pi^*\xi$. Here x can be viewed as the first Chern class of the canonical line bundle over $P(\xi)$, i.e., the complex 1-dimensional sub-bundle γ_ξ in $\pi^*\xi \to P(\xi)$ such that the restriction $\gamma_\xi|_{\pi^{-1}(a)}$ is the canonical line bundle over $\pi^{-1}(a) \cong \mathbb{C}P^{n-1}$ for all $a \in X$. Therefore $\deg x = 2$. Since it is well-known that the induced homomorphism $\pi^* : H^*(X;\mathbb{Z}) \to H^*(P(\xi);\mathbb{Z})$ is injective, we often confuse $c_i(\pi^*\xi)$ with $c_i(\xi)$.

We apply the formula (2.1) to an m-stage $\mathbb{C}P$-tower

$$C_m \xrightarrow{\pi_m} C_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} C_1 \xrightarrow{\pi_1} C_0 = \{\text{a point}\}$$

with $C_i = P(\xi_{i-1})$, to get the following isomorphisms.

$$\begin{aligned} H^*(C_m;\mathbb{Z}) &\simeq H^*(C_{m-1};\mathbb{Z})[x_m]/\langle x_m^{n_m+1} + \sum_{i=1}^{n_m}(-1)^i c_i(\xi_{m-1})x_m^{n_m+1-i}\rangle \\ &\simeq H^*(C_{m-2};\mathbb{Z})[x_{m-1},x_m]/\langle x_k^{n_k+1} + \sum_{i=1}^{n_k}(-1)^i c_i(\xi_k)x_k^{n_k+1-i} \mid k = m-1,\ m\rangle \\ &\ \ \vdots \\ &\simeq \mathbb{Z}[x_1,\dots,x_m]/\langle x_k^{n_k+1} + \sum_{i=1}^{n_k}(-1)^i c_i(\xi_k)x_k^{n_k+1-i} \mid k = 1,\cdots,m\rangle. \end{aligned} \tag{2.2}$$

In order to prove the main theorem, we often use the following lemmas.

Lemma 2.1. *Let γ be any line bundle over M, and let $P(\xi)$ be the projectivization of a complex vector bundle ξ over M. Then, $P(\xi)$ is diffeomorphic to $P(\xi \otimes \gamma)$.*

Proof. By the definition of the projectivization of a complex vector bundle, the statement follows immediately. □

Lemma 2.2. *Let γ be a complex line bundle, and let ξ be a 2-dimensional complex vector bundle over a manifold M. Then the Chern classes of the tensor product $\xi \otimes \gamma$ are as follows.*

$$c_1(\xi\otimes\gamma) = c_1(\xi) + 2c_1(\gamma);$$
$$c_2(\xi\otimes\gamma) = c_1(\gamma)^2 + c_1(\gamma)c_1(\xi) + c_2(\xi).$$

Proof. Let us consider the following pull-back diagram:

$$\begin{tikzcd} \pi^*\xi\otimes\pi^*\gamma \arrow[r] \arrow[d] & \xi\otimes\gamma \arrow[d] \\ P(\xi\otimes\gamma) \arrow[r, "\pi"] & M \end{tikzcd}$$

Let $\varphi : P(\xi\otimes\gamma) \to P(\xi)$ be the diffeomorphism from Lemma 2.1, and let $\pi_\xi : P(\xi) \to M$ be the projection of the fibration. Then we can see easily that $\pi = \pi_\xi \circ \varphi$. Taking the canonical line bundle γ_ξ in $\pi_\xi^*\xi$, we may regard $\pi_\xi^*\xi \equiv \gamma_\xi \oplus \gamma_\xi^\perp$, where $\gamma_\xi^\perp$ is the normal (line) bundle of γ_ξ in $\pi_\xi^*\xi$. By using the decomposition $\pi = \pi_\xi\circ\varphi$, we have the following equation:

$$\begin{aligned} \pi^*c(\xi\otimes\gamma) &= c(\varphi^*\gamma_\xi\otimes\pi^*(\gamma))c(\varphi^*\gamma_\xi^\perp\otimes\pi^*(\gamma)) \\ &= (1+\varphi^*c_1(\gamma_\xi)+\pi^*c_1(\gamma))(1+\varphi^*c_1(\gamma_\xi^\perp)+\pi^*c_1(\gamma)). \end{aligned}$$

Because $\pi^*c_1(\xi) = \varphi^*c_1(\gamma_\xi)+\varphi^*c_1(\gamma_\xi^\perp)$ and $\pi^*c_2(\xi) = \varphi^*c_1(\gamma_\xi)\varphi^*c_1(\gamma_\xi^\perp)$, we have

$$\begin{aligned} \pi^*c_1(\xi\otimes\gamma) &= \pi^*c_1(\xi) + 2\pi^*c_1(\gamma); \\ \pi^*c_2(\xi\otimes\gamma) &= \pi^*c_2(\xi) + \pi^*c_1(\xi)\pi^*c_1(\gamma) + \pi^*c_1(\gamma)^2. \end{aligned}$$

As is well-known, $\pi^* : H^*(M) \to H^*(P(\xi\otimes\gamma))$ is injective. Hence we have the formula in the lemma. □

We now give two examples of $\mathbb{C}P$-towers.

EXAMPLE 2.3. The flag manifold $\mathcal{F}l(\mathbb{C}^{n+1}) = \{\{0\} \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}\}$, called type A, is well-known to be diffeomorphic to the homogeneous space $U(n+1)/T^{n+1}(\cong SU(n+1)/T^n)$. We will show that the flag manifold $U(n+1)/T^{n+1}$ is a $\mathbb{C}P$-tower with height n. Recall that if M is a smooth manifold with free K action and H is a subgroup of K, then we have a diffeomorphism $M/H \cong M \times_K (K/H)$. Also recall that $\mathbb{C}P^n \cong U(n+1)/(T^1 \times U(n))$. By using these facts, it is easy to check that there is the following $\mathbb{C}P$-tower structure of height n in $U(n+1)/T^{n+1}$:

$$\begin{array}{c}
U(n+1) \times_{(T^1 \times U(n))} (U(n) \times_{(T^1 \times U(n-1))} (U(n-1) \times_{(T^1 \times U(n-2))} \cdots (U(3) \times_{(T^1 \times U(2))} \mathbb{C}P^1) \cdots) \\
\downarrow \\
\vdots \\
\downarrow \\
U(n+1) \times_{(T^1 \times U(n))} (U(n) \times_{(T^1 \times U(n-1))} \mathbb{C}P^{n-2}) \\
\downarrow \\
U(n+1) \times_{(T^1 \times U(n))} \mathbb{C}P^{n-1} \\
\downarrow \\
\mathbb{C}P^n,
\end{array}$$

where the $U(k)$ action on $\mathbb{C}P^{k-1}$ in each stage is induced from the usual $U(k)$ action on $\mathbb{C}^k$.

EXAMPLE 2.4. The flag manifold of type C is defined by the homogeneous space $Sp(n)/T^n$. We claim that $Sp(n)/T^n$ is a $\mathbb{C}P$-tower with height n. It is well known that $Sp(n)/(T^1 \times Sp(n-1)) \cong S^{4n-1}/T^1 \cong \mathbb{C}P^{2n-1}$, because $Sp(n)/Sp(n-1) \cong S^{4n-1}$. By using this fact and the method similar to that demonstrated in Example 2.3, it is easy to check that there is the following $\mathbb{C}P$-tower structure of height n in $Sp(n)/T^n$:

$$\begin{array}{c}
Sp(n) \times_{(T^1 \times Sp(n-1))} (Sp(n-1) \times_{(T^1 \times Sp(n-2))} \cdots (Sp(2) \times_{(T^1 \times Sp(1))} \mathbb{C}P^1) \cdots) \\
\downarrow \\
\vdots \\
\downarrow \\
Sp(n) \times_{(T^1 \times Sp(n-1))} (Sp(n-1) \times_{(T^1 \times Sp(n-2))} \mathbb{C}P^{2n-5}) \\
\downarrow \\
Sp(n) \times_{(T^1 \times Sp(n-1))} \mathbb{C}P^{2n-3} \\
\downarrow \\
\mathbb{C}P^{2n-1},
\end{array}$$

where the $Sp(k)$-action on $\mathbb{C}P^{2k-1}$ in each stage is induced from the $Sp(k)$-action on $\mathbb{C}^{2k}(\simeq \mathbb{H}^k)$ induced by the following representation to $U(2k)$:

$$A + Bj \longrightarrow \begin{pmatrix} A & -B \\ \overline{B} & \overline{A} \end{pmatrix}.$$

Here $A,\ B \in M(k;\mathbb{C})$ satisfy $A\overline{A} + B\overline{B} = I_k$ and $BA - AB = O$.

REMARK 2.5. By computing the generators of flag manifolds of other types (B_n $(n \geq 3)$, D_n $(n \geq 4)$, G_2, F_4, E_6, E_7, E_8), they do not admit the structure of $\mathbb{C}P$-towers, see [**Bo**] (or [**FIM**] for classical types).

EXAMPLE 2.6. The *Milnor hypersurface* $H_{i,j} \subset \mathbb{C}P^i \times \mathbb{C}P^j$, $1 \leq i \leq j$ is defined by the following equation (see [**BuPa**, Example 5.39]):

$$H_{i,j} = \{[z_0 : \cdots : z_i] \times [w_0 : \cdots : w_j] \in \mathbb{C}P^i \times \mathbb{C}P^j \mid \sum_{q=0}^{i} z_q w_q = 0\}.$$

We can show easily that the natural projection onto the first coordinate of $H_{i,j}$ gives the structure of a $\mathbb{C}P^{j-1}$-bundle over $\mathbb{C}P^i$. Moreover, by the proof in [**BuPa**, Theorem 5.39], this bundle may be regarded as the projectivization of $\gamma^\perp \subset \epsilon^{j+1}$, where ϵ^{j+1} is the trivial $\mathbb{C}^{j+1}$-bundle over $\mathbb{C}P^i$ and $\gamma^\perp$ is the normal bundle of the canonical line bundle γ over $\mathbb{C}P^i$ in ϵ^{j+1}. Therefore, the Milnor hypersurface admits the structure of a $\mathbb{C}P$-tower with height 2.

Remark 2.7. As is well-known, both of the flag manifold $U(n+1)/T^{n+1}$ (and $Sp(n)/T^n$) with $n \geq 2$ and the Milnor hypersurface $H_{i,j}$ with $i \geq 2$ do not admit the structure of a *toric manifold* (see e.g. [**BuPa**]). On the other hand, $U(2)/T^2 \cong Sp(1)/T^1 \cong \mathbb{C}P^1$ and $H_{1,j} \to \mathbb{C}P^1$ are toric manifolds.

3. 6-dimensional $\mathbb{C}P$-towers of height 2

Let M be a 6-dimensional $\mathbb{C}P$-tower. Then, the height of M is at most 3. If its height is one, then M is diffeomorphic to $\mathbb{C}P^3$. Therefore, it is enough to analyze the case when the height is 2 and 3. In this section, we focus on the classification of 6-dimensional $\mathbb{C}P$-towers of height 2.

To state the main theorem of this section, we first set up some notation. Let $\mathcal{M}_2^6$ be the set of all 6-dimensional $\mathbb{C}P$-towers of height 2, up to diffeomorphisms. Let γ_i denote the tautological line bundle over $\mathbb{C}P^i$, and let x denote the generator $-c_1(\gamma_2) \in H^2(\mathbb{C}P^2)$. Let $\eta_{(s,\alpha)}$ as the complex 2-dimensional vector bundle over $\mathbb{C}P^2$ whose total Chern class is $1 + sx + \alpha x^2$ for $s,\ \alpha \in \mathbb{Z}$, let $P(\eta_{(s,\alpha)})$ be its projectivization. We now state the main theorem of this section.

Theorem 3.1. *The set $\mathcal{M}_2^6$ consists of the following distinct $\mathbb{C}P$-towers.*

$$\begin{aligned}
&P(\gamma_1 \oplus \epsilon \oplus \epsilon) \longrightarrow \mathbb{C}P^1;\\
&P(\gamma_1^2 \oplus \epsilon \oplus \epsilon) \longrightarrow \mathbb{C}P^1,\ \textit{where } \gamma_1^2 \equiv \gamma_1 \otimes \gamma_1;\\
&P(\eta_{(0,\alpha)}) \longrightarrow \mathbb{C}P^2 \quad \textit{for} \quad \alpha \in \mathbb{Z};\\
&P(\eta_{(1,\beta)}) \longrightarrow \mathbb{C}P^2 \quad \textit{for} \quad \beta \in \mathbb{Z}.
\end{aligned}$$

Proof. Take $M \in \mathcal{M}_2^6$. Then the first stage C_1 of M is either $\mathbb{C}P^1$ or $\mathbb{C}P^2$. We treat these two cases separately below.

CASE I: $C_1 = \mathbb{C}P^1$. Note that any complex vector bundles over $\mathbb{C}P^1$ decomposes into a Whitney sum of line bundles. Therefore a $\mathbb{C}P$-tower $M \in \mathcal{M}_2^6$ with $C_1 = \mathbb{C}P^1$ is a 2-stage generalized Bott tower, and such Bott towers are completely classified in [**CMS10**]. (See also [**CPS**].) Due to the cited result, we have the following proposition.

Proposition 3.2. Let $M \in \mathcal{M}_2^6$ be a generalized Bot manifold with $C_1 = \mathbb{C}P^1$. Then M is diffeomorphic to one of the following three distinct manifolds:

$$\begin{aligned}
&P(\gamma_1^0 \oplus \epsilon \oplus \epsilon) \cong \mathbb{C}P^1 \times \mathbb{C}P^2,\ \text{where } \gamma_1^0 \equiv \epsilon;\\
&P(\gamma_1 \oplus \epsilon \oplus \epsilon);\\
&P(\gamma_1^2 \oplus \epsilon \oplus \epsilon).
\end{aligned}$$

CASE II: $C_1 = \mathbb{C}P^2$. Because $\dim M = 6$ and $C_1 = \mathbb{C}P^2$, the bundle $E_1 \to C_1$ is a compex 2-dimensional vector bundle. Such vector bundles are determined by their Chern classes c_1 and c_2 (see [**Sh, Sw**]). Hence, by Lemmas 2.1 and 2.2, we may denote E_1 by $\eta_{(s,\alpha)}$ such that $c_1(\eta_{(s,\alpha)}) = sx$ for $s = 0, 1$ and $c_2(\eta_{(s,\alpha)}) = \alpha x^2 \in H^4(\mathbb{C}P^2)$ for $\alpha \in \mathbb{Z}$. In Case II, we have the following classification result.

Proposition 3.3. The following are equivalent for $s_1,\ s_2 \in \{0,\ 1\}$ and $\alpha_1,\ \alpha_2 \in \mathbb{Z}$.

(1) $(s_1, \alpha_1) = (s_2, \alpha_2)$.
(2) Two manifolds $P(\eta_{(s_1,\alpha_1)})$ and $P(\eta_{(s_2,\alpha_2)})$ are diffeomorphic.
(3) Two cohomology rings $H^*(P(\eta_{(s_1,\alpha_1)}))$ and $H^*(P(\eta_{(s_2,\alpha_2)}))$ isomorphic.

Theorem 3.1 follows from Proposition 3.2 and 3.3. □

It remains to prove Proposition 3.3.

Proof of Proposition 3.3. (1) ⇒ (2) and (2) ⇒ (3) are obvious. We now prove (3) ⇒ (1). We prove this by proving the three claims: (1) $H^*(P(\eta_{(0,\alpha)})) \not\simeq H^*(P(\eta_{(1,\beta)}))$ for every $\alpha,\ \beta \in \mathbb{Z}$, (2) if $H^*(P(\eta_{(0,\alpha_1)})) \simeq H^*(P(\eta_{(0,\alpha_2)}))$ then $\alpha_1 = \alpha_2$, and (3) if $H^*(P(\eta_{(1,\beta_1)})) \simeq H^*(P(\eta_{(1,\beta_2)}))$ then $\beta_1 = \beta_2$.

Claim 1: $H^*(P(\eta_{(0,\alpha)})) \not\simeq H^*(P(\eta_{(1,\beta)}))$ for every $\alpha,\ \beta \in \mathbb{Z}$. By using the Borel-Hirzebruch formula (2.1), we have the following isomorphisms:

$$\begin{aligned} H^*(P(\eta_{(0,\alpha)})) &\simeq \mathbb{Z}[X,Y]/\langle X^3, Y^2+\alpha X^2\rangle; \\ H^*(P(\eta_{(1,\beta)})) &\simeq \mathbb{Z}[x,y]/\langle x^3, y^2+xy+\beta x^2\rangle, \end{aligned}$$

where $\deg X = \deg Y = \deg x = \deg y = 2$. We write the $\mathbb{Z}$-module structures of $H^*(P(\eta_{(0,\alpha)}))$ and $H^*(P(\eta_{(1,\beta)}))$ by indicating their generators as follows:

$$\begin{aligned} &\mathbb{Z} \oplus \mathbb{Z}X \oplus \mathbb{Z}Y \oplus \mathbb{Z}X^2 \oplus \mathbb{Z}XY \oplus \mathbb{Z}X^2Y; \\ &\mathbb{Z} \oplus \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}x^2 \oplus \mathbb{Z}xy \oplus \mathbb{Z}x^2y. \end{aligned}$$

If there exits a graded ring isomorphism $f : H^*(P(\eta_{(0,\alpha)})) \to H^*(P(\eta_{(1,\beta)}))$, then we may put $f(X) = ax+by$ and $f(Y) = cx+dy$ for some $a,b,c,d \in \mathbb{Z}$ such that

$$ad - bc = \pm 1. \tag{3.1}$$

Because f preserves the ring structure, we have

$$\begin{aligned} f(X^3) &= (ax+by)^3 \\ &= (3a^2b - 3ab^2 + b^3 - \beta b^3)x^2y = 0; \\ f(Y^2+\alpha X^2) &= (cx+dy)^2 + \alpha(ax+by)^2 \\ &= (c^2+\alpha a^2 - \beta d^2 - \alpha\beta b^2)x^2 + (2cd + 2\alpha ab - d^2 - \alpha b^2)xy = 0. \end{aligned}$$

This implies the following equations:

$$b(3a^2 - 3ab + b^2 - \beta b^2) = 0; \tag{3.2}$$

$$c^2 + \alpha a^2 - \beta d^2 - \alpha\beta b^2 = 0; \tag{3.3}$$

$$2cd + 2\alpha ab - d^2 - \alpha b^2 = 0. \tag{3.4}$$

If $b = 0$, then $2c = d = \pm 1$ by (3.1) and (3.4). But this contradicts to the fact that c is an integer (i.e., $c \in \mathbb{Z}$). Hence $b \neq 0$, and by (3.2) we have $3a^2 - 3ab + b^2 - \beta b^2 = 0$. We also have the following commutative diagram of free $\mathbb{Z}$-modules.

$$\begin{tikzcd} \mathbb{Z}X \oplus \mathbb{Z}Y \arrow[r, "\cdot X"] \arrow[d, "f"] & \mathbb{Z}X^2 \oplus \mathbb{Z}XY \arrow[d, "f"] \\ \mathbb{Z}x \oplus \mathbb{Z}y \arrow[r, "\cdot f(X)"] & \mathbb{Z}x^2 \oplus \mathbb{Z}xy \end{tikzcd},$$

where the horizontal maps are induced from the multiplication by X and $f(X)$, respectively. Let us represent the linear map $\cdot f(X) = \cdot(ax+by) : \mathbb{Z}x \oplus \mathbb{Z}y \to \mathbb{Z}x^2 \oplus \mathbb{Z}xy$ by the matrix

$$A = \begin{pmatrix} a & -\beta b \\ b & a-b \end{pmatrix}$$

with respect to the generators. Note that $\cdot X : \mathbb{Z}X \oplus \mathbb{Z}Y \to \mathbb{Z}X^2 \oplus \mathbb{Z}XY$ is an isomorphism. Therefore $\cdot f(X)$ is also an isomorphism, and hence

$$\det A = a^2 - ab + \beta b^2 = \pm 1. \tag{3.5}$$

Because $b \neq 0$, it follows from (3.2) and (3.5) that we have $b = \pm 1$, $\beta = 1$ and $a = 0$ or b. If $a = b$, then $c = d$ or $c = -d$ by (3.3). However, it is easy to check that both of these cases give contradictions to (3.1) and $c, d \in \mathbb{Z}$. Hence, $a = 0$. In this case, $\alpha = c^2 - d^2$ by (3.3) and $\alpha = 2cd - d^2$ by (3.4). Therefore we have $c = 0$ or $2d$. However, both of these cases give contradictions to (3.1) and $c, d \in \mathbb{Z}$. This establishes that there is no ring isomorphism between $H^*(P(\eta_{(0,\alpha)}))$ and $H^*(P(\eta_{(1,\beta)}))$.

Claim 2: If $H^*(P(\eta_{(0,\alpha_1)})) \simeq H^*(P(\eta_{(0,\alpha_2)}))$, then $\alpha_1 = \alpha_2$. By (2.1), we have the isomorphisms

$$\begin{aligned} H^*(P(\eta_{(0,\alpha_1)})) &\simeq \mathbb{Z}[X,Y]/\langle X^3, Y^2+\alpha_1 X^2\rangle, \text{ and} \\ H^*(P(\eta_{(0,\alpha_2)})) &\simeq \mathbb{Z}[x,y]/\langle x^3, y^2+\alpha_2 x^2\rangle. \end{aligned}$$

Assume that there exists an isomorphism $f : H^*(P(\eta_{(0,\alpha_1)})) \to H^*(P(\eta_{(0,\alpha_2)}))$ for some $\alpha_1,\ \alpha_2 \in \mathbb{Z}$, and let $f(X) = ax+by$ and $f(Y) = cx+dy$, so that $ad-bc = \pm 1$. Because $f(X^3) = (ax+by)^3 = 0$, we have that

$$b(3a^2 - b^2\alpha_2) = 0.$$

Suppose $b \neq 0$. Then $3a^2 - b^2\alpha_2 = 0$. Because the map

$$f : H^6(P(\eta_{(0,\alpha_1)})) = \mathbb{Z}X^2Y \longrightarrow \mathbb{Z}x^2y = H^6(P(\eta_{(0,\alpha_2)})),$$

is an isomorphism, we have

$$f(X^2Y) = (ax+by)^2(cx+dy) = \pm x^2y. \tag{3.6}$$

Using (3.6) and the ring structures, we have that

$$a^2d + 2abc - b^2d\alpha_2 = \pm 1.$$

Because $3a^2 - b^2\alpha_2 = 0$, we have $-2a^2d + 2abc = -2a(ad-bc) = \pm 1$. However, this gives a contradiction to $a \in \mathbb{Z}$, because $ad - bc = \pm 1$. Hence, $b = 0$ and $ad = \pm 1$; in particular, we have $a,\ d = \pm 1$. Then, we have the following equations:

$$\begin{aligned} f(Y^2 + \alpha_1 X^2) &= (cx+dy)^2 + \alpha_1(ax+by)^2 \\ &= (c^2 - \alpha_2 + \alpha_1)x^2 + 2cdxy = 0 \end{aligned}$$

Therefore, we have that $c = 0$ and $\alpha_1 = \alpha_2$. This proves the claim.

Claim 3: If $H^*(P(\eta_{(1,\beta_1)})) \simeq H^*(P(\eta_{(1,\beta_2)}))$, then $\beta_1 = \beta_2$. By (2.1), we have the isomorphisms

$$\begin{aligned} H^*(P(\eta_{(1,\beta_1)})) &\simeq \mathbb{Z}[X,Y]/\langle X^3, Y^2 + XY + \beta_1 X^2\rangle, \text{ and} \\ H^*(P(\eta_{(1,\beta_2)})) &\simeq \mathbb{Z}[x,y]/\langle x^3, y^2 + xy + \beta_2 x^2\rangle. \end{aligned}$$

Assume that there exists an isomorphism $f : H^*(P(\eta_{(1,\beta_1)})) \to H^*(P(\eta_{(1,\beta_2)}))$ for some $\beta_1,\ \beta_2 \in \mathbb{Z}$, and let $f(X) = ax + by$ and $f(Y) = cx + dy$, so that $ad - bc = \pm 1$. Because of the relations $f(X^3) = (ax+by)^3 = 0$ and $f(Y^2+XY+\beta_1X^2) = (cx+dy)^2+(ax+by)(cx+dy)+\beta_1(ax+by)^2 = 0$, we have that

$$b(3a^2 - 3ab + b^2 - b^2\beta_2) = 0; \tag{3.7}$$

$$c^2 - d^2\beta_2 + ac - bd\beta_2 + a^2\beta_1 - b^2\beta_1\beta_2 = 0; \tag{3.8}$$

$$2cd - d^2 + ad + bc - bd + 2\beta_1 ab - \beta_1 b^2 = 0. \tag{3.9}$$

We first assume $b = 0$. From the equation $ad - bc = \pm 1$, we have $a,\ d = \pm 1$. Now plug $b = 0$ and $d = \pm 1$ into (3.9) to get the equation

$$2c + a = d = \pm 1.$$

Together with $a = \pm 1$, this equation implies that either $c = 0$ and $a = d$, or $c \neq 0$ and $c = -a = d$. Now plug these into (3.8) to obtain $\beta_1 = \beta_2$ in either cases, which proves the claim when $b = 0$.

We now assume $b \neq 0$. Then from (3.7), we have $3a^2 - 3ab + b^2 - b^2\beta_2 = 0$. By using the same argument as the one used to get (3.5), we have

$$a^2 - ab + \beta_2 b^2 = \epsilon, \tag{3.10}$$

where $\epsilon = \pm 1$. Substitute (3.10) into the equation $3a^2 - 3ab + b^2 - b^2\beta_2 = 0$. Then, we obtain the equation

$$b^2(4\beta_2 - 1) = 3\epsilon.$$

Therefore, $b = \pm 1$ and $\beta_2 = \epsilon = 1$. Hence, together with (3.10), we have that $a = 0$ or $a = b$.

If $a = 0$, then $c = \pm 1$ by the equation $ad - bc = \pm 1$. Substitute these equations into (3.8) and (3.9). Then, we have the equations

$$\beta_1 = 1 - d^2 - bd = 2cd - d^2 + bc - bd.$$

Therefore, we have that $(2d+b)c = 1$. Moreover, because $c = \pm 1$ and $b = \pm 1$, we have $(b,d) = (c,0)$ or $(-c,c)$. Hence, $\beta_1 = 1 = \beta_2$.

If $a = b = \pm 1$, then $d - c = \pm 1$ by the equation $ad - bc = \pm 1$. Put $a = b = \pm 1$ in (3.9) to obtain the equation

$$\beta_1 = d^2 - 2cd - bc. \tag{3.11}$$

Moreover, by substituting $a = b = \pm 1$ and $\beta_2 = 1$ into (3.8), we have

$$(c - d)(a + c + d) = 0.$$

This together with $d - c = \pm 1$ implies that $c + d = -a = \pm 1$. It follows that either $d = 0$ and $c = -a = -b$, or $d = -a = -b$ and $c = 0$. By (3.11), we have $\beta_1 = 1 = \beta_2$. This proves the claim, and hence the proof of the proposition is complete. □

We can show easily that $P(\eta_{(s,\alpha)})$ is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^2$ if and only if $(s, \alpha) = (0, 0)$ by comparing their cohomology rings. Therefore, by Propositions 3.2 and 3.3, we have Theorem 3.1. Moreover, by Theorem 3.1, we have the following corollary.

Corollary 3.4. Let $\mathcal{M}^6_{\leq 2}$ be the class of all 6-dimensional $\mathbb{C}P$-towers of height at most 2, up to diffeomorphism. Then two $\mathbb{C}P$-towers M and M' in $\mathcal{M}^6_{\leq 2}$ are diffeomorphic if and only if their cohomology rings $H^*(M)$ and $H^*(M')$ are isomorphic. In other words, the class $\mathcal{M}^6_{\leq 2}$ is cohomologically rigid.

4. 3-stage 6-dimensional $\mathbb{C}P$-towers

In this section, we focus on 6-dimensional $\mathbb{C}P$-towers of height 3. The 3-stage 6-dimensional $\mathbb{C}P$-towers are of the form

$$P(\xi) \xrightarrow{\mathbb{C}P^1} H_k \xrightarrow{\mathbb{C}P^1} \mathbb{C}P^1.$$

Here, ξ is a complex 2-dimensional vector bundle over H_k, and H_k is the *Hirzebruch surface* $P(\gamma_1^k \oplus \epsilon)$ where ϵ is the trivial complex line bundle and γ_1^k is the k-th tensor power of the tautological line bundle γ_1 over $\mathbb{C}P^1$. As is well known, H_k is diffeomrophic to H_0 if k is even, and to H_1 if k is odd (see [**Hi, MaSu**]).

Lemma 4.1. *Let* $\mathrm{Vect}^2_{\mathbb{C}}(H_k)$ *be the set of complex* 2*-dimensional vector bundles over* H_k *up to isomorphisms. Then the correspondence*

$$\begin{array}{ccc} \mathrm{Vect}^2_{\mathbb{C}}(H_k) & \xrightarrow{c} & H^2(H_k) \oplus H^4(H_k) \\ \Psi & & \Psi \\ \xi & \longmapsto & c_1(\xi) \oplus c_2(\xi) \end{array}$$

is bijective.

Proof. Since $\dim_{\mathbb{R}} H_k = 4$, any two bundles η_1 and $\eta_2 \in \mathrm{Vect}^2_{\mathbb{C}}(H_k)$ are isomorphic if and only if they are stably isomorphic, i.e., $\eta_1 \oplus \epsilon^\ell \equiv \eta_2 \oplus \epsilon^\ell$ for some trivial complex ℓ-dimensional bundle ϵ^ℓ, see [**Hu**, 1.5 Theorem in Chapter 9]. Therefore η_1 and η_2 represent the same element in $\widetilde{K}(H_k)$, the stable K-ring of H_k, if and only if $\eta_1 \equiv \eta_2$. Therefore the map $\mathrm{Vect}^2_{\mathbb{C}}(H_k) \to \widetilde{K}(H_k)$ defined by $\xi \mapsto [\xi]$ is bijective. Hence, it is enough to prove that the induced map

$$c' : \widetilde{K}(H_k) \to H^2(H_k) \oplus H^4(H_k), \qquad [\xi] \mapsto (c_1(\xi), c_2(\xi))$$

is bijective.

Let $s : \mathbb{C}P^1 \to H_k = P(\gamma_1^k \oplus \epsilon^1)$ be the section defined by $s([p]) = [p, [0 : 1]]$, and let $i : \mathbb{C}P^1 \to H_k$ be an inclusion to a fiber in the fibrarion $H_k \to \mathbb{C}P^1$. Then $s(\mathbb{C}P^1) \cup i(\mathbb{C}P^1) \cong \mathbb{C}P^1 \vee \mathbb{C}P^1$, and we have the following inclusion and collapsing sequence

$$\mathbb{C}P^1 \vee \mathbb{C}P^1 \longrightarrow H_k \longrightarrow H_k/(\mathbb{C}P^1 \vee \mathbb{C}P^1).$$

Since H_k admits a CW-structure with one 0-cell, two 2-cells, and one 4-cell (e.g. see [**DaJa**]), $H_k/(\mathbb{C}P^1 \vee \mathbb{C}P^1)$ may be regarded as the collapsing of two 2-cells to the one 0-cell. Therefore, the space $H_k/(\mathbb{C}P^1 \vee \mathbb{C}P^1)$ is homeomorphic to S^4. Hence, we have the following exact sequence of reduced K groups (see [**Hu**, 2.1 Proposition in Chapter 10]):

$$\widetilde{K}(S^4) \to \widetilde{K}(H_k) \to \widetilde{K}(\mathbb{C}P^1 \vee \mathbb{C}P^1).$$

As is well known, we have the following isomorphisms

$$\widetilde{K}(S^4) \simeq \widetilde{K}(S^2) \simeq \widetilde{K}(\mathbb{C}P^1) \simeq \mathbb{Z}, \text{ and} \tag{4.1}$$

$$\widetilde{K}(\mathbb{C}P^1 \vee \mathbb{C}P^1) \simeq \widetilde{K}(\mathbb{C}P^1) \oplus \widetilde{K}(\mathbb{C}P^1) \simeq \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2. \tag{4.2}$$

These isomorphisms are induced by taking the Chern classes of vector bundles. Let $c' = (c_1', c_2') : \widetilde{K}(H_k) \to H^2(H_k) \oplus H^4(H_k) \simeq \mathbb{Z}^2 \oplus \mathbb{Z}$, where $c_1'([\xi]) = c_1(\xi)$ and $c_2'([\xi]) = c_2(\xi)$. Then $c_1' : \widetilde{K}(H_k) \to H^2(H_k)$ is surjective because for any $\alpha \in H^2(H_k) \simeq \mathbb{Z}^2$ can be realized as the first Chern class $c_1(\gamma)$ of a complex line bundle γ over H_k. Indeed, for a given $\alpha_1 x + \alpha_2 y \in \mathbb{Z}x \oplus \mathbb{Z}y = H^2(H_k)$, the line bundle $\gamma = \pi^*(\gamma_1^{\alpha_1}) \otimes \gamma_{H_k}^{\alpha_2}$ has the first Chern class $\alpha_1 x + \alpha_2 y$, where $\pi : H_k \to \mathbb{C}P^1$ is the projection, γ_{H_k} is the canonical line bundle over $H_k = P(\gamma_1^k \oplus \epsilon^1)$ induced from the vector bundle $\pi^*(\gamma_1^k \oplus \epsilon^1)$, and x, y are generators induced by $c_1(\pi^*\gamma_1)$, $c_1(\gamma_{H_k})$ respectively. We also claim that $c_2' : \widetilde{K}(H_k) \to H^4(H_k)$ is surjective. By the fundamental results of fibre bundle, we can construct all complex 2-dimensional vector bundles over $H_k/(\mathbb{C}P^1 \vee \mathbb{C}P^1) \cong S^4$ by using the continuous map $S^4 \to BU(2)$ up to homotopy. Because $\pi_4(BU(2)) \simeq \mathbb{Z}$, for a given $\beta \in H^4(H_k/(\mathbb{C}P^1 \vee \mathbb{C}P^1))$ we can construct the complex 2-dimensional vector bundle η' such that $c(\eta') = 1 + \beta$. Now the collapsing map $\rho : H_k \to H_k/(\mathbb{C}P^1 \vee \mathbb{C}P^1)$ induces the isomorphism $H^4(H_k/(\mathbb{C}P^1 \vee \mathbb{C}P^1)) \simeq H^4(H_k) \simeq \mathbb{Z}$; therefore, its pull-back $\eta = \rho^*\eta'$ over H_k satisfies $c(\eta) = 1 + \beta$. This implies that c_2' is surjective. Because $\gamma \oplus \eta$ is a complex 3-dimensional vector bundle and $\dim_{\mathbb{R}} H_k = 4$, the bundle $\gamma \oplus \eta$ is in the stable range. Therefore, there is the complex 2-dimensional vector bundle ξ such that $\xi \oplus \epsilon^1 \equiv \gamma \oplus \eta$, where ϵ^1 is the trivial line bundle over H_k, and $c(\xi) = c(\gamma \oplus \eta) = 1 + c_1(\gamma) + c_2(\eta)$. Therefore, the map $c' : \widetilde{K}(H_k) \to H^2(H_k) \oplus H^4(H_k)$ is surjective. Now consider the following diagram.

$$\begin{array}{ccccccccc} & & \widetilde{K}(S^4) & \longrightarrow & \widetilde{K}(H_k) & \longrightarrow & \widetilde{K}(\mathbb{C}P^1 \vee \mathbb{C}P^1) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 0 \end{array}$$

Here the vertical maps from the left are the isomorphism in (4.1), the map $c' : \widetilde{K}(H_k) \to H^2(H_k) \oplus H^4(H_k)$ and the isomorphism in (4.2), and the horizontal sequences are exact. One can see easily that the diagram is commutative. From the commutativity of the diagram and the surjectivity of the map c', we can see that $\widetilde{K}(S^4) \to \widetilde{K}(H_k) \to \widetilde{K}(\mathbb{C}P^1 \vee \mathbb{C}P^1)$ is a short exact sequence, and the map c' is bijective. Consequently, there exists the bijective map $\mathrm{Vect}^2_{\mathbb{C}}(H_k) \to H^2(H_k) \oplus H^4(H_k)$ defined by $\xi \mapsto c_1(\xi) \oplus c_2(\xi)$. This establishes the lemma. $\square$

By Lemma 4.1, any complex 2-dimensional vector bundles over H_0 and H_1 can be written by

$$\eta_{(s,r,\alpha)} \to H_0, \text{ and } \quad \xi_{(s,r,\beta)} \to H_1$$

where

$$c_1(\eta_{(s,r,\alpha)}) = (s, r) \in H^2(H_0) \simeq \mathbb{Z} \oplus \mathbb{Z}, \quad c_2(\eta_{(s,r,\alpha)}) = \alpha \in H^4(H_0) \simeq \mathbb{Z};$$
$$c_1(\xi_{(s,r,\beta)}) = (s, r) \in H^2(H_1) \simeq \mathbb{Z} \oplus \mathbb{Z}, \quad c_2(\xi_{(s,r,\beta)}) = \beta \in H^4(H_1) \simeq \mathbb{Z}.$$

Moreover, by taking tensor product with an appropriate line bundle if necessary, we may assume $(s, r) \in \{0, 1\}^2$, see Lemma 2.2. Let $\mathcal{M}^6_3$ be the set of all 6-dimensional $\mathbb{C}P$-towers of height 3, up to diffeomorphism. The main theorem of this section is the following.

Theorem 4.2. *The set $\mathcal{M}^6_3$ consists of the following distinct manifolds:*

$P(\eta_{(0,0,\alpha)})$ *for* $\alpha \in \mathbb{Z}_{\geq 0}$;
$P(\eta_{(1,0,\alpha)})$ *for* $\alpha \in \mathbb{Z}_{\geq 0}$;
$P(\eta_{(1,1,\alpha)})$ *for* $\alpha \in \mathbb{N}$;
$P(\xi_{(0,0,\beta)})$ *for* $\beta \in \mathbb{N}$;
$P(\xi_{(1,0,\beta)})$ *for* $\beta \in \mathbb{Z}_{\geq 0}$;
$P(\xi_{(0,1,\beta)})$ *for* $\beta \in \mathbb{Z}$.

Moreover, we have the diffeomorphisms $P(\eta_{(1,0,\alpha)}) \cong P(\eta_{(0,1,\alpha)})$, $P(\eta_{(0,0,1)}) \cong P(\xi_{(0,0,0)})$, *and* $P(\xi_{(0,1,\beta)}) \cong P(\xi_{(1,1,-\beta)})$.

To prove Theorem 4.2, we first observe the following. For $H_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$, there is a self-diffeomorphism on H_0 defined by exchanging the first and second terms, i.e., $(p,q) \mapsto (q,p)$ for $(p,q) \in H_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$. This diffeomorphism induces a bundle isomorphism between $\eta_{(s,r,\alpha)}$ and $\eta_{(r,s,\alpha)}$. Therefore, we may assume $(s,r) = (0,0),\ (1,0)$ or $(1,1)$ in the case of $\eta_{(s,r,\alpha)}$.

We also need the following lemma.

Lemma 4.3. *If the cohomology ring* $H^*(P(\eta_{(s,r,\alpha)}))$ *is isomorphic to* $H^*(P(\xi_{(s',r',\beta)}))$, *then* $(s,r,\alpha) = (1,0,0)$ *and* $(s',r',\beta) = (0,0,0)$. *Furthermore,* $P(\eta_{(1,0,0)})$ *is diffeomorphic to* $P(\xi_{(0,0,0)})$.

Proof. By the Borel Hirzebruch formula (2.1), we have the isormophisms

$$\begin{aligned} H^*(P(\eta_{(s,r,\alpha)})) &\simeq \mathbb{Z}[X,Y,Z]/\langle X^2,\ Y^2,\ Z^2 + sZX + rZY + \alpha XY\rangle, \text{ and} \\ H^*(P(\xi_{(s',r',\beta)})) &\simeq \mathbb{Z}[x,y,z]/\langle x^2,\ y^2 + xy,\ z^2 + s'zx + r'zy + \beta xy\rangle, \end{aligned}$$

where $(s,r) = (0,0),\ (1,0)$ or $(1,1)$ in $\eta_{(s,r,\alpha)}$, and $(s',r') = (0,0),\ (1,0),\ (0,1)$ or $(1,1)$ in $\xi_{(s',r',\alpha)}$. For each (s,r,α) and (s',r',β), we express the $\mathbb{Z}$-module structures of the above cohomology rings using their generators as follows:

$$\mathbb{Z} \oplus \mathbb{Z}X \oplus \mathbb{Z}Y \oplus \mathbb{Z}Z \oplus \mathbb{Z}XY \oplus \mathbb{Z}YZ \oplus \mathbb{Z}ZX \oplus \mathbb{Z}XYZ;$$
$$\mathbb{Z} \oplus \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \oplus \mathbb{Z}xy \oplus \mathbb{Z}yz \oplus \mathbb{Z}zx \oplus \mathbb{Z}xyz.$$

Assume there exists an isomorphism $f : H^*(P(\eta_{(s,r,\alpha)})) \to H^*(P(\xi_{(s',r',\beta)}))$. Let $f(X) = a_1x + b_1y + c_1z$, $f(Y) = a_2x + b_2y + c_2z$ and $f(Z) = a_3x + b_3y + c_3z$, and let A_f denote the corresponding 3×3 matrix of f. Because f is a graded ring isomorphism, it satisfies the following relations:

$$f(X)^2 = (a_1x + b_1y + c_1z)^2 = (2a_1b_1 - b_1^2 - \beta c_1^2)xy + (2a_1c_1 - s'c_1^2)xz + (2b_1c_1 - r'c_1^2)yz = 0;$$
$$f(Y)^2 = (a_2x + b_2y + c_2z)^2 = (2a_2b_2 - b_2^2 - \beta c_2^2)xy + (2a_2c_2 - s'c_2^2)xz + (2b_2c_2 - r'c_2^2)yz = 0$$

in $H^*(P(\xi_{(s',r',\beta)}))$. Therefore, we have

$$\begin{aligned} &2a_ib_i - b_i^2 - \beta c_i^2 = 0; \\ &2a_ic_i - s'c_i^2 = 0; \\ &2b_ic_i - r'c_i^2 = 0, \end{aligned}$$

for $i = 1, 2$.

Assume $c_1 = 0$. Then, by using the first equation above and $\det A_f = \pm 1$, we have either $b_1 = 0$ and $a_1 = \epsilon_1$, or $b_1 = 2a_1 = 2\epsilon_1$, where $\epsilon_1 = \pm 1$. If $c_2 = 0$, then it is easy to check that this gives a contradiction to $\det A_f = \pm 1$. Hence, $c_2 \neq 0$. By using the second and the third equations above, we have $s'c_2 = 2a_2$ and $r'c_2 = 2b_2$. Hence it can be seen easily from $\det A_f = \pm 1$ that only $(s',r') = (0,0)$ is possible, and in this case $(a_2,b_2,c_2) = (0,0,\epsilon_2)$ and $\beta = 0$, where $\epsilon_2 = \pm 1$. Hence, we have that $(s',r',\beta) = (0,0,0)$.

If $(a_1,b_1,c_1) = (\epsilon_1,0,0)$, then $b_3 = \epsilon_3$ because $\det A_f = \pm 1$. Therefore, it follows from $f(Z)^2 = -sf(X)f(Z) - rf(Y)f(Z) - \alpha f(X)f(Y)$ that

$$\begin{aligned} &2a_3\epsilon_3 - 1 = -s\epsilon_1\epsilon_3; \\ &2a_3c_3 = -s\epsilon_1c_3 - r\epsilon_2a_3 - \alpha\epsilon_1\epsilon_2; \\ &2\epsilon_3c_3 = -r\epsilon_2\epsilon_3. \end{aligned}$$

Using the third equation above, we have $r = c_3 = 0$. Therefore, by the second equation, we also have $\alpha = 0$. Moreover, from the first equation $s = 1$. Hence, $(s,r,\alpha) = (1,0,0)$.

If $(a_1,b_1,c_1) = (\epsilon_1,2\epsilon_1,0)$, then $b_3 - 2a_3 = \epsilon_3$ because $\det A_f = \pm 1$. Therefore, it follows from $f(Z)^2 = -sf(X)f(Z) - rf(Y)f(Z) - \alpha f(X)f(Y)$ that

$$\begin{aligned} &2a_3b_3 - b_3^2 = s\epsilon_1b_3 - 2s\epsilon_1a_3; \\ &2a_3c_3 = -s\epsilon_1c_3 - r\epsilon_2a_3 - \alpha\epsilon_1\epsilon_2; \\ &2b_3c_3 = -r\epsilon_2b_3 - 2s\epsilon_1c_3 - 2\alpha\epsilon_1\epsilon_2. \end{aligned}$$

Using the first equation and $b_3 - 2a_3 = \epsilon_3$, we have $b_3 = -s\epsilon_1$. Therefore, by using the third equation, we have $sr = -2\alpha$. This implies that $\alpha = 0$ and $sr = 0$. If $s = 0$, then $b_3 = -s\epsilon_1 = 0$; however, $b_3 - 2a_3 = -2a_3 = \epsilon_3$ and this gives a contradiction. Therefore $(s, r, \alpha) = (1, 0, 0)$. This establishes the first statement of the lemma when $c_1 = 0$ case.

In the case when $c_1 \neq 0$ and $c_2 = 0$, by a similar argument to the above case, we have the same result. When $c_1 \neq 0$ and $c_2 \neq 0$, by some routine computation, we can see that this case gives a contradiction. This establishes the first statement of the lemma.

Because $\eta_{(1,0,0)} \equiv \gamma_x \oplus \epsilon$, where γ_x is the tautological line bundle along the first factor of $\mathbb{C}P^1 \times \mathbb{C}P^1$, we can easily check that $P(\eta_{(1,0,0)}) \cong (S^3 \times \mathbb{C}P^1) \times_{T^1} P(\mathbb{C}_1 \oplus \mathbb{C})$, where T^1 acts on S^3 as diagonal multiplications in its coordinates and trivially on $\mathbb{C}P^1$ and $\mathbb{C}_1$ is a complex 1-dimensional T^1 representation such that $t \cdot z = tz$ for $t \in T^1$ and $z \in \mathbb{C}_1$. On the other hand, because $\xi_{(0,0,0)}$ is the trivial bundle over H_1 (by Lemma 4.1), we have that$P(\xi_{(0,0,0)}) = S^3 \times_{T^1} P(\mathbb{C}_1 \oplus \mathbb{C}) \times \mathbb{C}P^1$. Therefore, we have that $P(\eta_{(1,0,0)}) \cong P(\xi_{(0,0,0)})$. This establishes the second statement. □

In order to prove Theorem 4.2, we may divide the proof into the following two cases.

CASE I: $P(\eta_{(s,r,\alpha)})$ with the base space H_0. In this case $(s, r) = (0, 0), (1, 0)$ and $(1, 1)$.

CASE II: $P(\xi_{(s,r,\alpha)})$ with the base space H_1. In this case $(s, r) = (0, 0), (1, 0), (0, 1)$ and $(1, 1)$. Moreover if $(s, r) = (0, 0)$ then $\alpha \neq 0$.

The rest of the section in devoted to the proof of Theorem 4.2 by treating the two cases separately.

CASE I: $P(\eta_{(s,r,\alpha)})$ **with the base space** H_0**.** We prove the cohomological rigidity for $P(\eta_{(s,r,\alpha)})$. Namely, we prove the following proposition.

Proposition 4.4. The following statements are equivalent.

(1) Two manifolds $P(\eta_{(s_1,r_1,\alpha_1)})$ and $P(\eta_{(s_2,r_2,\alpha_2)})$ are diffeomorphic.
(2) Two cohomology rings $H^*(P(\eta_{(s_1,r_1,\alpha_1)}))$ and $H^*(P(\eta_{(s_2,r_2,\alpha_2)}))$ are isomorphic.
(3) $(s_1, r_1) = (s_2, r_2)$, and α_1 and α_2 are as follows:
   (a) if $(s_1, r_1) = (s_2, r_2) = (0, 0)$, then $\alpha_2 = \alpha_1$ or $-\alpha_1$;
   (b) if $(s_1, r_1) = (s_2, r_2) = (1, 0)$ (or $(0, 1)$), then $\alpha_2 = \alpha_1$ or $-\alpha_1$;
   (c) if $(s_1, r_1) = (s_2, r_2) = (1, 1)$, then $\alpha_2 = \alpha_1$ or $-\alpha_1 + 1$.

Proof. (1) ⇒ (2) is trivial.

We first prove (2) ⇒ (3). By (2.1), we have the following isomorphisms

$$\begin{aligned} H^*(P(\eta_{(s_1,r_1,\alpha_1)})) &\simeq \mathbb{Z}[X, Y, Z]/\langle X^2,\ Y^2,\ Z^2 + s_1 ZX + r_1 ZY + \alpha_1 XY\rangle, \text{ and} \\ H^*(P(\eta_{(s_2,r_2,\alpha_2)})) &\simeq \mathbb{Z}[x, y, z]/\langle x^2,\ y^2,\ z^2 + s_2 zx + r_2 zy + \alpha_2 xy\rangle. \end{aligned}$$

Assume there exists a graded ring isomorphism $f : H^*(P(\eta_{(s_1,r_1,\alpha_1)})) \simeq H^*(P(\eta_{(s_2,r_2,\alpha_2)}))$, and put the matrix representation of $f : H^2(P(\eta_{(s_1,r_1,\alpha_1)})) \simeq H^2(P(\eta_{(s_2,r_2,\alpha_2)}))$ with respect to the given module generators as

$$A_f = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix},$$

i.e., $f(X) = a_1x + b_1y + c_1z$, $f(Y) = a_2x + b_2y + c_2z$, $f(Z) = a_3x + b_3y + c_3z$. Note that $\det A_f = \pm 1$. Because $X^2 = Y^2 = 0$ and f is a ring isomorphism,

$$\begin{aligned} f(X)^2 &= (2a_1b_1 - \alpha_2 c_1^2)xy + (2a_1 - s_2c_1)c_1xz + (2b_1 - r_2c_1)c_1yz = 0; \\ f(Y)^2 &= (2a_2b_2 - \alpha_2 c_2^2)xy + (2a_2 - s_2c_2)c_2xz + (2b_2 - r_2c_2)c_2yz = 0 \end{aligned}$$

in $H^*(P(\eta_{(s_2,r_2,\alpha_2)}))$. Therefore, we have

$$2a_ib_i - \alpha_2 c_i^2 = 0, \tag{4.3}$$

$$(2a_i - s_2c_i)c_i = 0, \tag{4.4}$$

$$(2b_i - r_2c_i)c_i = 0, \tag{4.5}$$

for $i = 1,\ 2$. We divide the proof into the following three cases: **Case 1** $(s_2, r_2) = (1, 1)$; **Case 2** $(s_2, r_2) = (0, 0)$; **Case 3** $(s_2, r_2) = (1, 0)$.

Case 1: $(s_2, r_2) = (1, 1)$**.** We first claim that $c_1 = c_2 = 0$ and $c_3 = \epsilon_3 = \pm 1$. If $c_i \neq 0$, for $i = 1$ or 2, then $2a_i = c_i$ by (4.4), $2b_i = c_i$ by (4.5) and $2a_ib_i = \alpha_2 c_i^2$ by (4.3). These equations imply that

$$4a_ib_i = c_i^2 = 2\alpha_2 c_i^2.$$

Because $c_i \neq 0$, we have that $1 = 2\alpha_2$. This gives a contradiction. Therefore, we have

$$c_1 = c_2 = 0.$$

This together with $\det A_f = \pm 1$ imply that

$$c_3 = \epsilon_3 = \pm 1.$$

Because $Z^2 = -s_1XZ - r_1YZ - \alpha_1XY$ in $H^*(P(\eta_{(s_1,r_1,\alpha_1)}))$, the ring isomorphism f induces the following equations

$$2a_3b_3 - \alpha_2\epsilon_3^2 = -s_1(a_1b_3 + a_3b_1) - r_1(a_2b_3 + a_3b_2) - \alpha_1(a_1b_2 + a_2b_1), \tag{4.6}$$

$$(2a_3 - \epsilon_3)\epsilon_3 = (-s_1a_1 - r_1a_2)\epsilon_3, \tag{4.7}$$

$$(2b_3 - \epsilon_3)\epsilon_3 = (-s_1b_1 - r_1b_2)\epsilon_3. \tag{4.8}$$

Using (4.3) and $c_1 = c_2 = 0$, we have $a_ib_i = 0$ for $i = 1,\ 2$. Moreover, from $\det A_f = \pm 1$, there are two possibilities, i.e., either $(a_1, b_2) = (0, 0)$ and $(a_2, b_1) = (\epsilon_1, \epsilon_2)$, or $(a_1, b_2) = (\epsilon_1, \epsilon_2)$ and $(a_2, b_1) = (0, 0)$ where $\epsilon_i = \pm 1$ for $i = 1, 2$.

If $(a_1, b_2) = (0, 0)$ and $(a_2, b_1) = (\epsilon_1, \epsilon_2)$, then it follows from (4.7) and (4.8) that

$$2a_3 = \epsilon_3 - r_1\epsilon_1;$$
$$2b_3 = \epsilon_3 - s_1\epsilon_2.$$

It is easy to check that if $s_1 = 0$ or $r_1 = 0$ then we have a contradiction to one of the equations above. Therefore, $(s_1, r_1) = (s_2, r_2) = (1, 1)$. We also have that if $\epsilon_3 = \epsilon_1$ (resp. $\epsilon_3 = \epsilon_2$) then $a_3 = 0$ (resp. $b_3 = 0$) and if $\epsilon_3 \neq \epsilon_1$ (resp. $\epsilon_3 \neq \epsilon_2$) then $a_3 = \epsilon_3$ (resp. $b_3 = \epsilon_3$). Thus, by the equation (4.6), we have that $\alpha_2 = \alpha_1$ or $\alpha_2 = -\alpha_1 + 1$.

If $(a_1, b_2) = (\epsilon_1, \epsilon_2)$ and $(a_2, b_1) = (0, 0)$, then similarly we have that $(s_1, r_1) = (s_2, r_2) = (1, 1)$ and $\alpha_2 = \alpha_1$ or $\alpha_2 = -\alpha_1 + 1$. This establishes $(3) - (c)$.

Case 2: $(s_2, r_2) = (0, 0)$**.** If $(s_1, r_1) = (1, 1)$ in this case, by the same argument as in Case 1 with (s_2, r_2) replaced by (s_1, r_1), we can see that $(s_2, r_2) = (1, 1)$ which contradicts to the hypothesis. Therefore $(s_1, r_1) = (0, 0)$ or $(1, 0)$, and hence, $Z^2 = -s_1XZ - \alpha_1XY$ in $H^*(P(\eta_{(s_1,r_1,\alpha_1)}))$. Therefore, the ring isomorphism f implies the following equations:

$$2a_3b_3 - \alpha_2c_3^2 = -s_1(a_1b_3 + a_3b_1) - \alpha_1(a_1b_2 + a_2b_1) + s_1c_1c_3\alpha_2 + \alpha_1c_1c_2\alpha_2; \tag{4.9}$$

$$2a_3c_3 = -s_1(a_1c_3 + a_3c_1) - \alpha_1(a_1c_2 + a_2c_1); \tag{4.10}$$

$$2b_3c_3 = -s_1(b_1c_3 + b_3c_1) - \alpha_1(b_1c_2 + b_2c_1). \tag{4.11}$$

Because of (4.4) and (4.5), we also have that $a_ic_i = b_ic_i = 0$. Then by (4.3), there are two cases to consider for $i = 1, 2$: **(2-i)** the case when $c_i \neq 0$, and hence, $a_i = b_i = \alpha_2 = 0$; **(2-ii)** the case when $c_i = 0$, and hence $a_ib_i = 0$.

(2-i) If $c_1 \neq 0$, and hence, $a_1 = b_1 = \alpha_2 = 0$, then $c_1 = \epsilon_3 = \pm 1$ because $\det A_f = \pm 1$. Furthermore, if $c_2 \neq 0$, then $a_2 = b_2 = 0$, which gives a contradiction to $\det A_f = \pm 1$. Therefore, $c_2 = 0$ and $a_2b_2 = 0$. Moreover $a_3b_3 = 0$ by (4.9). Since $\det A_f = \pm 1$, there are two possibilities for (a_2, a_3) and (b_2, b_3), i.e., either $(a_2, a_3) = (0, \epsilon_1)$ and $(b_2, b_3) = (\epsilon_2, 0)$, or $(a_2, a_3) = (\epsilon_1, 0)$ and $(b_2, b_3) = (0, \epsilon_2)$. If $a_2 = b_3 = 0$, then, by using (4.10) and (4.11), we have that $2c_3 = -s_1\epsilon_3$ and $\alpha_1 = \alpha_2 = 0$. Therefore, because $s_1 = 0$ or 1, we also have $c_3 = 0$ and $s_1 = s_2 = 0$. If $a_3 = b_2 = 0$, then we similarly have that $\alpha_1 = \alpha_2 = 0$ and $s_1 = s_2 = 0$.

(2-ii) If $c_1 = 0$, then $a_1b_1 = 0$. If $c_2 \neq 0$, then the proof is almost the same with the case when $c_1 \neq 0$; and we have that $\alpha_1 = \alpha_2 = 0$ and $s_1 = s_2 = 0$ as the conclusion. Therefore, we may put $c_2 = 0$ and $a_2b_2 = 0$. Because of $\det A_f = \pm 1$, we have that $c_3 = \epsilon_3 = \pm 1$ and there are the two possibilities, i.e., either $(a_1, a_2) = (0, \epsilon_1)$ and $(b_1, b_2) = (\epsilon_2, 0)$, or $(a_1, a_2) = (\epsilon_1, 0)$ and

$(b_1, b_2) = (0, \epsilon_2)$. If $a_1 = b_2 = 0$ (resp. $a_2 = b_1 = 0$), then it follows from (4.11) (resp. (4.10)) that $2b_3 = -s_1 b_1$ (resp. $2a_3 = -s_1 a_1$). Therefore, $s_1 = s_2 = 0$ and $b_3 = 0$ (resp. $a_3 = 0$). Moreover, by (4.9), we have that $\alpha_2 = \epsilon_1\epsilon_2\alpha_1$. This establishes $(3) - (a)$.

Case 3: $(s_2, r_2) = (1, 0)$**.** In this case, by the same arguments as above, we may assume $(s_1, r_1) = (1, 0)$, i.e., $Z^2 = -XZ - \alpha_1 XY$ in $H^*(P(\eta_{(s_1,r_1,\alpha_1)}))$. It is sufficient to show that $\alpha_2 = \alpha_1$ or $-\alpha_1$. Now, the ring isomorphism f implies the following equations:

$$2a_3b_3 - \alpha_2 c_3^2 = -(a_1b_3 + a_3b_1) - \alpha_1(a_1b_2 + a_2b_1) + c_1c_3\alpha_2 + \alpha_1c_1c_2\alpha_2; \tag{4.12}$$

$$2a_3c_3 - c_3^2 = -(a_1c_3 + a_3c_1) - \alpha_1(a_1c_2 + a_2c_1) + c_1c_3 + c_1c_2\alpha_1; \tag{4.13}$$

$$2b_3c_3 = -(b_1c_3 + b_3c_1) - \alpha_1(b_1c_2 + b_2c_1). \tag{4.14}$$

Because of (4.4) and (4.5), we also have $(2a_i - c_i)c_i = 0$ and $b_ic_i = 0$. By (4.3), if $c_i \neq 0$ then $b_i = \alpha_2 = 0$ and $c_i = 2a_i$, and if $c_i = 0$, then $a_ib_i = 0$.

(3-i) If $c_1 \neq 0$, then $b_1 = \alpha_2 = 0$, $c_1 = 2a_1$. Since $\det A_f = \pm 1$, we may put $a_1 = \epsilon_1 = \pm 1$. In this case, if $c_2 \neq 0$ then $b_2 = 0$ and $c_2 = 2a_2$, which contradicts to $\det A_f = \pm 1$. Therefore, $c_2 = 0$ and $a_2b_2 = 0$. It follows from (4.12) and (4.14) that

$$2a_3b_3 = -\epsilon_1(b_3 + \alpha_1 b_2) = b_3c_3.$$

Therefore, there are two cases to consider: the case when $b_3 = 0$, and hence $\alpha_1 b_2 = 0$; the case when $b_3 \neq 0$, and hence $c_3 = 2a_3$. If $b_3 \neq 0$ and $c_3 = 2a_3$, then by $\det A_f = \pm 1$ we have $a_3 = 0 = c_3$ and $b_3 = \epsilon_2 = \pm 1$. Then the matrix A_f is equal

$$\begin{pmatrix} \epsilon_1 & 0 & 2\epsilon_1 \\ a_2 & b_2 & 0 \\ 0 & \epsilon_2 & 0 \end{pmatrix}.$$

This gives a contradiction to $\det A_f = \pm 1$. Therefore, $b_3 = 0$, and hence $\alpha_1 b_2 = 0$. If $b_2 = 0$ then this gives a contradiction to $\det A_f = \pm 1$. Hence, we have $b_2 \neq 0$, and hence $\alpha_1 = \alpha_2 = 0$.

(3-ii) If $c_1 = 0$ and $c_2 \neq 0$, then $a_1b_1 = 0$, $c_2 = 2a_2$ and $b_2 = \alpha_2 = 0$. If $b_1 = 0$, then it is easy to check this gives a contradiction to $\det A_f = \pm 1$. Hence, $a_1 = 0$ and $b_1 = \pm 1$. Because $c_2 = 2a_2$ and $\det A_f = \pm 1$, we have $c_3 - 2a_3 = \pm 1$. By using (4.13), we also have the equation $c_3(c_3 - 2a_3) = 0$. Therefore, $c_3 = 0$, and hence $2a_3 = \pm 1$. This gives a contradiction to $a_3 \in \mathbb{Z}$.

Therefore $c_1 = c_2 = 0$. Since $\det A_f = \pm 1$ and $c_1 = c_2 = 0$, we can put $c_3 = \epsilon_3 = \pm 1$. Then, we can easily see that $a_1 + 2a_3 = \epsilon_3$ by (4.13) and $b_1 = -2b_3$ by (4.14). Therefore, by using $a_1b_1 = a_2b_2 = 0$ and $\det A_f = \pm 1$, we have that $b_1 = b_3 = 0$, $b_2 = \epsilon_2 = \pm 1$ and $a_2 = 0$, $a_1 = \epsilon_1 = \pm 1$. Hence, by using (4.12), we have $\alpha_2 = \pm\alpha_1$. This establishes $(3) - (b)$. Consequently, we have proved the implication $(2) \Rightarrow (3)$.

Finally, we prove $(3) \Rightarrow (1)$. Consider the diffeomorphism $f = \mathrm{id} \times \mathrm{conj} : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ defined by $(p, q) \mapsto (p, \overline{q})$. Because f changes the orientation on $\mathbb{C}P^1 \times \mathbb{C}P^1$, the Euler class $e(f^*\eta_{(s,r,\alpha)})$ coincides with $-e(\eta_{(s,r,\alpha)})$. Because of the definition of Chern class, $e(f^*\eta_{(s,r,\alpha)}) = c_2(f^*\eta_{(s,r,\alpha)}) = -c_2(\eta_{(s,r,\alpha)}) = -\alpha$. Because x and y are the first Chern classes of the tautological line bundles of the first and the second factor of $\mathbb{C}P^1 \times \mathbb{C}P^1$, we have $c_1(f^*\eta_{(s,r,\alpha)}) = f^*(sX + rY) = sx - ry$. Hence, by Lemmas 2.2 and 4.1, we have

$$f^*\eta_{(s,0,\alpha)} \equiv \eta_{(s,0,-\alpha)};$$
$$f^*\eta_{(1,1,\alpha)} \otimes \gamma_2 \equiv \eta_{(1,-1,-\alpha)} \otimes \gamma_2 \equiv \eta_{(1,1,1-\alpha)},$$

where γ_2 is the pull back of the tautological line bundle over $\mathbb{C}P^1$ along the projection $\pi_2 : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1$ to the second factor. This implies that $P(\eta_{(s,r,\alpha)}) \cong P(\eta_{(s,r,-\alpha)})$ for $(s, r) = (0, 0)$ or $(1, 0)$ (or $(0, 1)$) and $P(\eta_{(1,1,\alpha)}) \cong P(\eta_{(1,1,1-\alpha)})$ for $(s, r) = (1, 1)$. This proves the implication $(3) \Rightarrow (1)$. □

CASE II: $P(\xi_{(s,r,\beta)})$ **with the base space** H_1**.** We prove the cohomological rigidity for $P(\xi_{(s,r,\beta)})$ in the following proposition.

Proposition 4.5. The following statements are equivalent.

(1) Two manifolds $P(\xi_{(s_1,r_1,\beta_1)})$ and $P(\xi_{(s_2,r_2,\beta_2)})$ are diffeomorphic.
(2) Two cohomology rings $H^*(P(\xi_{(s_1,r_1,\beta_1)}))$ and $H^*(P(\xi_{(s_2,r_2,\beta_2)}))$ are isomorphic.

(3) Either $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2)$, or one of the following holds:
(a) $(s_1, r_1, \beta_1) = (0, 0, \beta)$ and $(s_2, r_2, \beta_2) = (0, 0, -\beta)$ $(\beta \neq 0)$;
(b) $(s_1, r_1, \beta_1) = (1, 0, \beta)$ and $(s_2, r_2, \beta_2) = (1, 0, -\beta)$;
(c) $\{(s_1, r_1, \beta_1),\ (s_2, r_2, \beta_2)\} = \{(0, 1, \beta),\ (1, 1, -\beta)\}$,
for some $\beta \in \mathbb{Z}$.

By using Proposition 4.4 and 4.5 and Lemma 4.3, we have Theorem 4.2. Let us prove Proposition 4.5.

Proof. (1) $\Rightarrow$ (2) is trivial. We first prove (2) $\Rightarrow$ (3). By (2.1) we have the isormophisms

$$\begin{aligned} H^*(P(\xi_{(s_1,r_1,\beta_1)})) &\simeq \mathbb{Z}[X,Y,Z]/\langle X^2,\ Y^2+XY,\ Z^2+s_1ZX+r_1ZY+\beta_1XY\rangle, \text{ and} \\ H^*(P(\xi_{(s_2,r_2,\beta_2)})) &\simeq \mathbb{Z}[x,y,z]/\langle x^2,\ y^2+xy,\ z^2+s_2zx+r_2zy+\beta_2xy\rangle. \end{aligned}$$

Assume there is a ring isomorphism $f : H^*(P(\xi_{(s_1,r_1,\beta_1)})) \simeq H^*(P(\xi_{(s_2,r_2,\beta_2)}))$, and put the matrix representation of $f : H^2(P(\xi_{(s_1,r_1,\beta_1)})) \simeq H^2(P(\xi_{(s_2,r_2,\beta_2)}))$ as

$$A_f = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

Note that $\det A_f = \pm 1$. Let $\epsilon_i = \pm 1$ $(i = 1,\ 2,\ 3)$. Because of $X^2 = 0 \in H^*(P(\xi_{(s_1,r_1,\beta_1)}))$, we have

$$\begin{aligned} &2a_1b_1 - b_1^2 - c_1^2\beta_2 = 0, \\ &2a_1c_1 - c_1^2 s_2 = 0, \\ &2b_1c_1 - c_1^2 r_2 = 0. \end{aligned}$$

By using these equations and $\det A_f = \pm 1$, it is easy to check that for $\epsilon = \pm 1$

Case 1: if $c_1 \neq 0$, then there are the following two sub-cases:
- $(s_2, r_2) = (0, 0)$ with $(a_1, b_1, c_1) = (0, 0, \epsilon)$ and $\beta_2 = 0$;
- $(s_2, r_2) = (1, 0)$ with $(a_1, b_1, c_1) = (\epsilon, 0, 2\epsilon)$ and $\beta_2 = 0$,

Case 2: if $c_1 = 0$, then $(a_1, b_1) = (\epsilon, 0)$ or $(\epsilon, 2\epsilon)$.

Because $Y^2 = -XY$ in $H^*(P(\xi_{(s_1,r_1,\beta_1)}))$, we also have

$$2a_2b_2 - b_2^2 - c_2^2\beta_2 = -a_1b_2 - b_1a_2 + b_1b_2 + c_1c_2\beta_2, \tag{4.15}$$

$$2a_2c_2 - c_2^2 s_2 = -a_1c_2 - c_1a_2 + c_1c_2s_2, \tag{4.16}$$

$$2b_2c_2 - c_2^2 r_2 = -b_1c_2 - c_1b_2 + c_1c_2r_2. \tag{4.17}$$

Case 1: $c_1 \neq 0$**.** If $(s_2, r_2) = (0, 0)$, then, by using (4.16), (4.17) and $(a_1, b_1, c_1) = (0, 0, \epsilon_3)$, we can easily show that $a_2 = b_2 = 0$; however, because $\det A_f = \pm 1$, this gives a contradiction. Therefore, $(s_2, r_2, \beta_2) = (1, 0, 0)$ and $(a_1, b_1, c_1) = (\epsilon_1, 0, 2\epsilon_1)$. Note that $\det A_f(a_2b_3 - a_3b_2)$ is the $(1, 3)$-entry of the matrix A_f^{-1}. Therefore, by a similar argument to the above, we can see that if $a_2b_3 - a_3b_2 \neq 0$ then $(s_1, r_1) = (1, 0)$ and $\beta_1 = 0$. This means that if we get $a_2b_3 - a_3b_2 \neq 0$ then we have $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$, i.e., the statement of this proposition holds.

By (4.17), we may divide the case when $c_1 \neq 0$ into two sub-cases: **(1-i)** $b_2 = 0$ and **(1-ii)** $b_2 \neq 0$ and $c_2 = -\epsilon_1$.

(1-i) If $b_2 = 0$, then it easily follows from (4.16) that $c_2 = 2a_2$ or $-\epsilon_1$. Moreover, by using $\det A_f = \pm 1$ and $(a_1, b_1, c_1) = (\epsilon_1, 0, 2\epsilon_1)$, we have that $(a_2, b_2, c_2) = (0, 0, -\epsilon_1)$ or $(-\epsilon_1, 0, -\epsilon_1)$, and $b_3 = \epsilon_2$. If $(a_2, b_2, c_2) = (-\epsilon_1, 0, -\epsilon_1)$, then $a_2b_3 - a_3b_2 = -\epsilon_1\epsilon_2 \neq 0$. Therefore, by the argument explained above, we have $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$. Hence, this satisfies the statement of this proposition. Suppose $(a_2, b_2, c_2) = (0, 0, -\epsilon_1)$. Since $Z^2 = -s_1XZ - r_1YZ - \beta_1XY$, we have

$$\begin{aligned} &(2a_3\epsilon_2 - 1)xy + 2\epsilon_2c_3yz + (2a_3c_3 - c_3^2)xz \\ = \;& -s_1(\epsilon_1x + 2\epsilon_1z)(a_3x + \epsilon_2y + c_3z) + r_1\epsilon_1z(a_3x + \epsilon_2y + c_3z) + \beta_1(\epsilon_1x + 2\epsilon_1z)\epsilon_1z. \end{aligned}$$

So, we have

$$\begin{aligned}
&2a_3\epsilon_2 - 1 = -s_1\epsilon_1\epsilon_2;\\
&2a_3c_3 - c_3^2 = -2s_1\epsilon_1 a_3 + s_1\epsilon_1 c_3 + r_1\epsilon_1 a_3 - r_1\epsilon_1 c_3 - \beta_1;\\
&2\epsilon_2 c_3 = -2s_1\epsilon_1\epsilon_2 + r_1\epsilon_1\epsilon_2.
\end{aligned}$$

It easily follows from these equations that $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$.

(1-ii) If $b_2 \neq 0$ and $c_2 = -\epsilon_1$, then we have that $b_2 = 2a_2 + \epsilon_1$ by (4.15). Since $(a_1, b_1, c_1) = (\epsilon_1, 0, 2\epsilon_1)$, we have

$$\det A_f = (2\epsilon_1 a_2 + 1)(b_3 + c_3 - 2a_3) = \pm 1$$

Therefore, either **(1-ii-a)** $(a_2, b_2, c_2) = (0, \epsilon_1, -\epsilon_1)$, or **(1-ii-b)** $(-\epsilon_1, -\epsilon_1, -\epsilon_1)$ and $b_3 + c_3 - 2a_3 = \pm 1$.

(1-ii-a) Suppose $(a_2, b_2, c_2) = (0, \epsilon_1, -\epsilon_1)$, then $a_2b_3 - b_2a_3 = -\epsilon_1 a_3$. As before, if $a_3 \neq 0$ then $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$. This satisfies the statement of proposition. If $a_3 = 0$, then $b_3 + c_3 = \pm 1$ by the equation above. From the relation $Z^2 = -s_1XZ - r_1YZ - \beta_1XY$, we have

$$-b_3^2 = -s_1\epsilon_1 b_3 + r_1\epsilon_1 b_3 - \beta_1, \tag{4.18}$$

$$-c_3^2 = s_1\epsilon_1 c_3 - r_1\epsilon_1 c_3 - \beta_1, \tag{4.19}$$

$$2b_3c_3 = -2s_1\epsilon_1 b_3 - r_1\epsilon_1 c_3 + r_1\epsilon_1 b_3 - 2\beta_1. \tag{4.20}$$

From these equations, we get

$$(b_3 + c_3)^2 = 1 = -s_1\epsilon_1(b_3 + c_3).$$

Hence, $s_1 = 1$ and $b_3 + c_3 = -\epsilon_1$. By (4.18), we have

$$-1 + 2\epsilon_1 c_3 - c_3^2 = -\epsilon_1(-\epsilon_1 - c_3) + r_1\epsilon_1(-\epsilon_1 - c_3) - \beta_1.$$

Substituting (4.19) into this equation, we have

$$-1 + 2\epsilon_1 c_3 + \epsilon_1 c_3 - r_1\epsilon_1 c_3 - \beta_1 = -\epsilon_1(-\epsilon_1 - c_3) + r_1\epsilon_1(-\epsilon_1 - c_3) - \beta_1.$$

Hence,

$$2(2\epsilon_1 c_3 - 1) = r_1 = 0.$$

But this is impossible. Therefore the case (1-ii-a) can not occur.

(1-ii-b) Suppose $(a_2, b_2, c_2) = (-\epsilon_1, -\epsilon_1, -\epsilon_1)$, then $a_2b_3 - b_2a_3 = -\epsilon_1(b_3 - a_3)$. With the method similar to that demonstrated above, if $a_3 \neq b_3$ then $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$. Hence, we may assume $a_3 = b_3$. Because $\det A_f = c_3 + b_3 - 2a_3 = \pm 1$, we also have $c_3 - b_3 = \pm 1$. From the relation $Z^2 = -s_1XZ - r_1YZ - \beta_1XY$, we have

$$b_3^2 = -s_1\epsilon_1 b_3 + r_1\epsilon_1 b_3 + \beta_1, \tag{4.21}$$

$$2b_3c_3 - c_3^2 = -2s_1\epsilon_1 b_3 + s_1\epsilon_1 c_3 + r_1\epsilon_1 b_3 + 2\beta_1, \tag{4.22}$$

$$2b_3c_3 = -2s_1\epsilon_1 b_3 + r_1\epsilon_1 c_3 + r_1\epsilon_1 b_3 + 2\beta_1. \tag{4.23}$$

By using (4.22) and (4.23), we have

$$c_3(r_1\epsilon_1 - c_3 - s_1\epsilon_1) = 0.$$

Therefore, we have either $c_3 = 0$, or $c_3 \neq 0$ and $r_1\epsilon_1 - c_3 - s_1\epsilon_1 = 0$, i.e., $c_3 = \epsilon_1(r_1 - s_1)$ with $r_1 \neq s_1$.

We claim $c_3 \neq 0$. If $c_3 = 0$, then by using $\det A_f = \pm 1$ and $a_3 = b_3$, we may put $b_3 = \epsilon_2$. By using (4.22) and (4.23) again, we have that

$$-2s_1\epsilon_1\epsilon_2 + r_1\epsilon_1\epsilon_2 + 2\beta_1 = 0.$$

Hence, it is easy to check that $(s_1, r_1, \beta_1) = (0, 0, 0)$ or $(1, 0, \epsilon_1\epsilon_2)$. However, using (4.21), both of the cases give contradictions. Consequently, $c_3 \neq 0$, i.e., $c_3 = \epsilon_1(r_1 - s_1)$ with $r_1 \neq s_1$.

Because $r_1 \neq s_1$, there are two cases: $(s_1, r_1) = (1, 0)$ and $(0, 1)$. We first assume that $(s_1, r_1) = (1, 0)$. In this case, $c_3 = -\epsilon_1$. By using (4.22), we have $\beta_1 = 0$. Therefore, this case gives $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$. We next assume that $(s_1, r_1) = (0, 1)$. In this case, $c_3 = \epsilon_1$. Similarly, we have that $\epsilon_1 b_3 - 1 = 2\beta_1$. This also gives the equation

$$\epsilon_1 b_3 - 1 = \epsilon_1(b_3 - \epsilon_1) = 2\beta_1.$$

Recall that $b_3 - c_3 = \pm 1$ and $c_3 = \epsilon_1$. This gives a contradiction. This finishes Case 1.

Case 2: $c_1 = 0$**.** In this case we divided into two sub-cases: **(2-i)** $(a_1, b_1, c_1) = (\epsilon_1, 0, 0)$, and **(2-ii)** $(a_1, b_1, c_1) = (\epsilon_1, 2\epsilon_1, 0)$.

(2-i) Assume $(a_1, b_1, c_1) = (\epsilon_1, 0, 0)$. Then, it follows from (4.15), (4.16) and (4.17) that

$$2a_2 b_2 - b_2^2 - c_2^2 \beta_2 = -\epsilon_1 b_2; \tag{4.24}$$

$$2a_2 c_2 - c_2^2 s_2 = -\epsilon_1 c_2; \tag{4.25}$$

$$2b_2 c_2 - c_2^2 r_2 = 0. \tag{4.26}$$

By (4.25) and (4.26), either **(2-i-a)** $c_2 \neq 0$ and $2a_2 = c_2 s_2 - \epsilon_1$, $2b_2 = c_2 r_2$, or **(2-i-b)** $c_2 = 0$.

(2-i-a) First assume $c_2 \neq 0$. Then, by $2a_2 = c_2 s_2 - \epsilon_1$, we have $s_2 = 1$ and $c_2 = 2a_2 + \epsilon_1$. By substituting this equation into (4.26),we have that $r_2 = 0 = b_2$. Hence, by (4.24), $\beta_2 = 0$, i.e., $(s_2, r_2, \beta_2) = (1, 0, 0)$. Because $\det A_f = \pm 1$, we may put $b_3 = \epsilon_2$. Moreover, we have $\det A_f = -\epsilon_1 \epsilon_2 (2a_2 + \epsilon_1) = \pm 1$; therefore, $a_2 = 0$ or $-\epsilon_1$. If $a_2 = -\epsilon_1$, then $a_2 b_3 - a_3 b_2 = -\epsilon_1 \epsilon_2 \neq 0$. Hence, with the method similar to that demonstrated in Case 1, we have $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$. Thus, we may assume $a_2 = 0$, i.e.,

$$A_f = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & 0 & \epsilon_1 \\ a_3 & \epsilon_2 & c_3 \end{pmatrix}.$$

By using $Z^2 = -s_1 XZ - r_1 YZ - \beta_1 XY$ and $(s_2, r_2, \beta_2) = (1, 0, 0)$, it is easy to get that

$$\begin{aligned} &2a_3 \epsilon_2 - 1 = -s_1 \epsilon_1 \epsilon_2; \\ &2\epsilon_2 c_3 = -r_1 \epsilon_1 \epsilon_2; \\ &(2a_3 - c_3)c_3 = -s_1 \epsilon_1 c_3 - r_1 \epsilon_1 a_3 + r_1 c_3 \epsilon_1 - \beta_1. \end{aligned}$$

By using the first and second equations, we have $s_1 = 1$, $r_1 = 0$ and $c_3 = 0$. Therefore, by the third equation, we have that $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$. Consequently, if $(a_1, b_1, c_1) = (\epsilon_1, 0, 0)$ and $c_2 \neq 0$, then $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$.

(2-i-b) We next assume $c_2 = 0$. Because $\det A_f = \epsilon_1 b_2 c_3 = \pm 1$, we may put $b_2 = \epsilon_2$ and $c_3 = \epsilon_3$, i.e.,

$$A_f = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ a_2 & \epsilon_2 & 0 \\ a_3 & b_3 & \epsilon_3 \end{pmatrix}.$$

Then, it follows from (4.24) that $2a_2 \epsilon_2 - 1 = -\epsilon_1 \epsilon_2$, i.e., $a_2 = \frac{-\epsilon_1 + \epsilon_2}{2}$. By using $Z^2 = -s_1 XZ - r_1 YZ - \beta_1 XY$, it is easy to get that

$$\begin{aligned} &2a_3 b_3 - b_3^2 - \beta_2 = -s_1 \epsilon_1 b_3 - r_1(a_2 b_3 + a_3 \epsilon_2 - \epsilon_2 b_3) - \beta_1 \epsilon_1 \epsilon_2; \\ &2b_3 \epsilon_3 - r_2 = -r_1 \epsilon_2 \epsilon_3; \\ &2a_3 \epsilon_3 - s_2 = -s_1 \epsilon_1 \epsilon_3 - r_1 a_2 \epsilon_3. \end{aligned}$$

If $\epsilon_1 = \epsilon_2$, then $a_2 = 0$ and

$$\begin{aligned} &2a_3 b_3 - b_3^2 - \beta_2 = -s_1 \epsilon_1 b_3 - r_1(a_3 \epsilon_1 - \epsilon_1 b_3) - \beta_1; \\ &2b_3 \epsilon_3 - r_2 = -r_1 \epsilon_1 \epsilon_3; \\ &2a_3 \epsilon_3 - s_2 = -s_1 \epsilon_1 \epsilon_3. \end{aligned}$$

By using the second and third equations, we have that $(s_1, r_1) = (s_2, r_2)$. Therefore, if $\epsilon_1 = \epsilon_3$, then we also have $b_3 = a_3 = 0$. Using the first equation, we have $\beta_1 = \beta_2$, i.e., $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2)$. Suppose $\epsilon_1 \neq \epsilon_3$, i.e., $\epsilon_3 = -\epsilon_1$. In this case, if $s_1 = s_2 = 0$ (resp. $s_1 = s_2 = 1$) then $a_3 = 0$ (resp. $a_3 = -\epsilon_1$) by using the third equation. Similarly by using the second equation, if $r_1 = r_2 = 0$ (resp. $r_1 = r_2 = 1$) then $b_3 = 0$ (resp. $b_3 = -\epsilon_1$). Therefore, by using the first equation, it is easy to check that $\beta_1 = \beta_2$. Consequently, in the case when $\epsilon_1 = \epsilon_2$, hence $(a_2, b_2, c_2) = (0, \epsilon_1, 0)$, we have $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2)$, i.e., this case satisfies the statement of proposition.

If $-\epsilon_1 = \epsilon_2$, then $a_2 = -\epsilon_1$ and

$$\begin{aligned} &2a_3b_3 - b_3^2 - \beta_2 = -s_1\epsilon_1 b_3 + r_1 a_3\epsilon_1 + \beta_1;\\ &2b_3\epsilon_3 - r_2 = r_1\epsilon_1\epsilon_3;\\ &2a_3\epsilon_3 - s_2 = -s_1\epsilon_1\epsilon_3 + r_1\epsilon_1\epsilon_3. \end{aligned}$$

By using the second equation, we have that $r_1 = r_2$. If $r_1 = r_2 = 0$, then $b_3 = 0$ by the second equation and $s_1 = s_2$ by the third equation. Moreover, by using the first equation, we have $(s_1, 0, \beta_1) = (s_2, 0, -\beta_2)$. This implies that $(3) - (a)$ and $(3) - (b)$ in the statement of the proposition. If $r_1 = r_2 = 1$, then $b_3 = \frac{\epsilon_1+\epsilon_3}{2}$ by the second equation and $s_1 \neq s_2$ by the third equation. We first assume $(s_1, s_2) = (1, 0)$. Then, by the third equation, we have that $a_3 = 0$. Therefore, the first equation gives

$$-\frac{1+\epsilon_1\epsilon_3}{2} - \beta_2 = -\frac{1+\epsilon_1\epsilon_3}{2} + \beta_1.$$

Therefore, $\beta_1 = -\beta_2$, i.e., (s_1, r_1, β_1) and (s_2, r_2, β_2) are the pair $(1, 1, r)$ and $(0, 1, -r)$. This implies that $(3) - (c)$ in the statement of the proposition. We next assume $(s_1, s_2) = (0, 1)$. Then, by the second and third equations, we have that $a_3 = b_3$. Therefore, the first equation gives

$$\frac{1+\epsilon_1\epsilon_3}{2} - \beta_2 = \frac{1+\epsilon_1\epsilon_3}{2} + \beta_1.$$

Therefore, $\beta_1 = -\beta_2$, i.e., (s_1, r_1, β_1) and (s_2, r_2, β_2) are the pair $(0, 1, r)$ and $(1, 1, -r)$. This implies that $(3) - (c)$ in the statement of the proposition. Consequently, if $(a_1, b_1, c_1) = (\epsilon_1, 0, 0)$ and $c_2 = 0$, then the statement holds. Therefore the first sub-case **(2-i)** is done.

(2-ii) Assume $(a_1, b_1, c_1) = (\epsilon_1, 2\epsilon_1, 0)$. Then, it follows from (4.15), (4.16) and (4.17) that

$$2a_2b_2 - b_2^2 - c_2^2\beta_2 = \epsilon_1 b_2 - 2\epsilon_1 a_2; \tag{4.27}$$

$$2a_2c_2 - c_2^2 s_2 = -\epsilon_1 c_2; \tag{4.28}$$

$$2b_2c_2 - c_2^2 r_2 = -2\epsilon_1 c_2. \tag{4.29}$$

By (4.28) and (4.29), either **(2-ii-a)** $c_2 \neq 0$ and $2a_2 = c_2s_2 - \epsilon_1$, $2b_2 = c_2r_2 - 2\epsilon_1$, or **(2-ii-b)** $c_2 = 0$.

(2-ii-a) We first assume $c_2 \neq 0$. Then, by $2a_2 = c_2s_2 - \epsilon_1$, we have $s_2 = 1$ and $c_2 = 2a_2 + \epsilon_1$. Substituting this equation into $2b_2 = c_2r_2 - 2\epsilon_1$, we have $r_2 = 0$ and $b_2 = -\epsilon_1$. Therefore, $\beta_2 = 0$ by (4.27). By using $Z^2 = -s_1XZ - r_1YZ - \beta_1XY$ and $(s_2, r_2, \beta_2) = (1, 0, 0)$, it is easy to get that

$$2a_3b_3 - b_3^2 = -s_1(-\epsilon_1 b_3 + 2\epsilon_1 a_3) - r_1(a_2b_3 - \epsilon_1 a_3 + \epsilon_1 b_3) - \beta_1(1 + 2\epsilon_1 a_2); \tag{4.30}$$

$$2b_3c_3 = -2s_1\epsilon_1 c_3 - r_1(-\epsilon_1 c_3 + 2a_2b_3 + \epsilon_1 b_3) - \beta_1(4a_2\epsilon_1 + 2); \tag{4.31}$$

$$(2a_3 - c_3)c_3 = -s_1\epsilon_1 c_3 - r_1(-a_2c_3 + 2a_2a_3 + \epsilon_1 a_3 - \epsilon_1 c_3) - \beta_1(2a_2\epsilon_1 + 1). \tag{4.32}$$

Because $\det A_f = (2a_2\epsilon_1 + 1)(2a_3 - b_3 - c_3) = \pm 1$, either **(2-ii-a-I)** $a_2 = 0$ or **(2-ii-a-II)** $a_2 = -\epsilon_1$, and we may put $2a_3 - b_3 - c_3 = \epsilon_3$.

(2-ii-a-I) Assume $a_2 = 0$. With the method similar to that demonstrated in Case 1, if $a_2b_3 - a_3b_2 = a_3 \neq 0$ then $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$. Therefore, we may assume $a_3 = 0$ and $-b_3 - c_3 = \epsilon_3$. Hence, by the above equations, we have that

$$-b_3^2 = s_1\epsilon_1 b_3 - r_1\epsilon_1 b_3 - \beta_1; \tag{4.33}$$

$$2b_3c_3 = -2s_1\epsilon_1 c_3 - r_1(-\epsilon_1 c_3 + \epsilon_1 b_3) - 2\beta_1; \tag{4.34}$$

$$-c_3^2 = -s_1\epsilon_1 c_3 + r_1\epsilon_1 c_3 - \beta_1. \tag{4.35}$$

This implies that

$$-(b_3 + c_3)^2 = -1 = s_1\epsilon_1(b_3 + c_3) = -s_1\epsilon_1\epsilon_3.$$

Therefore, we have $s_1 = 1 = \epsilon_1\epsilon_3$ and $c_3 = -b_3 - \epsilon_1$. By substituting these equations into the third equation, we have

$$-b_3^2 - 2\epsilon_1 b_3 - 1 = \epsilon_1(b_3 + \epsilon_1) - r_1\epsilon_1(b_3 + \epsilon_1) - \beta_1.$$

Because of the first equation, we have

$$2\epsilon_1 b_3 + 2 = r_1.$$

This implies that $r_1 = 0$ and $b_3 = -\epsilon_1$. Hence $c_3 = -b_3 - \epsilon_1 = 0$. Therefore, from (4.34), we have $\beta_1 = 0$. Therefore, $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$. This satisfies the statement of proposition, and the case **(2-ii-a-I)** is done.

(2-ii-a-II) Assume $a_2 = -\epsilon_1$ With the method similar to that demonstrated in Case 1, if $a_3 \neq b_3$ then $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$. Therefore, we may assume $a_3 = b_3$ and $a_3 - c_3 = \epsilon_3$. By the above equations (4.30), (4.31), and (4.32), we have

$$\begin{aligned}
&a_3^2 = -s_1\epsilon_1 a_3 + r_1\epsilon_1 a_3 + \beta_1;\\
&2a_3c_3 = -2s_1\epsilon_1 c_3 - r_1(-\epsilon_1 c_3 - \epsilon_1 a_3) + 2\beta_1;\\
&(2a_3 - c_3)c_3 = -s_1\epsilon_1 c_3 + r_1\epsilon_1 a_3 + \beta_1.
\end{aligned}$$

This implies that

$$\begin{aligned}
(a_3 + c_3)(-a_3 + c_3) &= s_1\epsilon_1 a_3 - s_1\epsilon_1 c_3 + r_1\epsilon_1 c_3 - r_1\epsilon_1 a_3\\
&= \epsilon_1(r_1 - s_1)(-a_3 + c_3).
\end{aligned}$$

Because $a_3 - c_3 = \epsilon_3$, we have that $a_3 + c_3 = \epsilon_1(r_1 - s_1)$; therefore, $r_1 \neq s_1$. If $(s_1, r_1) = (0, 1)$, then $2a_3c_3 = 1 + 2\beta_1$ by the second equation above. This gives a contradiction. Hence, $(s_1, r_1) = (1, 0)$. In this case, $a_3 = \frac{-\epsilon_1+\epsilon_3}{2}$ and $c_3 = \frac{-\epsilon_1-\epsilon_3}{2}$. If $\epsilon_1 = \epsilon_3$, then $a_3 = 0$ and $c_3 = -\epsilon_1$. In this case, by using the first equation, $\beta_1 = 0$. However, by using the second equation, we also have $\beta_1 = -1$. This gives a contradiction and we have $\epsilon_1 = -\epsilon_3$, i.e., $a_3 = -\epsilon_1$ and $c_3 = 0$. It is easy to check that $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$. Consequently, if $(a_1, b_1, c_1) = (\epsilon_1, 2\epsilon_1, 0)$ and $c_2 \neq 0$, then $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2) = (1, 0, 0)$. This satisfies the statement of proposition. This finishes the proof for **(2-ii-a)**.

(2-ii-b) We next assume $c_2 = 0$, i.e.,

$$A_f = \begin{pmatrix} \epsilon_1 & 2\epsilon_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

Since $\det A_f = \pm 1$, we have $c_3 = \pm 1 =: \epsilon_3$. By (4.27)

$$2a_2b_2 - b_2^2 = \epsilon_1 b_2 - 2\epsilon_1 a_2.$$

Hence,

$$(2a_2 - b_2)(b_2 + \epsilon_1) = 0.$$

Therefore, $b_2 = 2a_2$ or $-\epsilon_1$. If $b_2 = 2a_2$, then $\det A_f = 0$, which is contradiction. Therefore, $b_2 = -\epsilon_1$. Hence, $\det A_f = \epsilon_3(-1 - 2\epsilon_1 a_2) = \pm 1$; therefore,

- $a_2 = 0$ or
- $a_2 = -\epsilon_1$.

By using $Z^2 = -s_1XZ - r_1YZ - \beta_1XY$, it is easy to get that

$$\begin{aligned}
&2a_3b_3 - b_3^2 - \beta_2 = -s_1(-\epsilon_1 b_3 + 2\epsilon_1 a_3) - r_1(a_2b_3 - \epsilon_1 a_3 + \epsilon_1 b_3) - \beta_1(1 + 2\epsilon_1 a_2);\\
&2b_3\epsilon_3 - r_2 = -2s_1\epsilon_1\epsilon_3 + r_1\epsilon_1\epsilon_3;\\
&2a_3 - s_2\epsilon_3 = -s_1\epsilon_1 - r_1 a_2.
\end{aligned}$$

By the second equation, we have that $r_1 = r_2$. If $r_1 = r_2 = 0$, by the second and third equations, we have that $b_3 = -s_1\epsilon_1$ and $s_1 = s_2$, respectively. It follows easily from the first equation that $\beta_1 = \beta_2$ for $a_2 = 0$ and $\beta_1 = -\beta_2$ for $a_2 = -\epsilon_1$. This implies that $(3) - (a)$ and $(3) - (b)$ and (3) with $(s_1, 0, \beta_1) = (s_2, 0, \beta_2)$ in the statement of the proposition. If $r_1 = r_2 = 1$, then by the above equations, we have that

$$\begin{aligned}
&2a_3b_3 - b_3^2 - \beta_2 = -s_1(-\epsilon_1 b_3 + 2\epsilon_1 a_3) - a_2b_3 + \epsilon_1 a_3 - \epsilon_1 b_3 - \beta_1(1 + 2\epsilon_1 a_2);\\
&2b_3\epsilon_3 - 1 = -2s_1\epsilon_1\epsilon_3 + \epsilon_1\epsilon_3;\\
&2a_3 - s_2\epsilon_3 = -s_1\epsilon_1 - a_2.
\end{aligned}$$

When $a_2 = 0$, then by the third equation we have that $s_1 = s_2$. If $s_1 = s_2 = 0$, then by the third equation we have $a_3 = 0$; therefore by the first and second equations we have

$$-\frac{1+\epsilon_1\epsilon_3}{2} - \beta_2 = -\frac{1+\epsilon_1\epsilon_3}{2} - \beta_1$$

Hence, $\beta_1 = \beta_2$. This implies that (3) with $(0,1,\beta_1) = (0,1,\beta_2)$ in the statement of the proposition. If $s_1 = s_2 = 1$, then by the second and third equations, we have that $a_3 = b_3 = \frac{-\epsilon_1+\epsilon_3}{2}$. Using the first equation, we have $\beta_1 = \beta_2$. This implies that (3) with $(1,1,\beta_1) = (1,1,\beta_2)$ in the statement of the proposition.

When $a_2 = -\epsilon_1$, then by the third equation we have that $s_1 \neq s_2$. If $(s_1, s_2) = (1, 0)$, then it follows from the third equation that $a_3 = 0$; therefore by the first and second equations we have

$$-\frac{1-\epsilon_1\epsilon_3}{2} - \beta_2 = -\frac{1-\epsilon_1\epsilon_3}{2} + \beta_1$$

Hence, $\beta_1 = -\beta_2$. If $(s_1, s_2) = (0, 1)$, then by the second and third equations, we have that $a_3 = b_3 = \frac{\epsilon_1+\epsilon_3}{2}$. Using the first equation, we have $\beta_1 = -\beta_2$. This implies that (3) – (c) in the statement of the proposition. Consequently, if $(a_1, b_1, c_1) = (\epsilon_1, 2\epsilon_1, 0)$ and $c_2 = 0$, then the statement holds. Therefore **(2-ii-b)** is finished, and this establishes the statement (2) $\Rightarrow$ (3).

Finally, we prove (3) $\Rightarrow$ (1). If $(s_1, r_1, \beta_1) = (s_2, r_2, \beta_2)$, then the statement is trivial. Assume $(s_1, r_1, \beta_1) \neq (s_2, r_2, \beta_2)$. Recall that $H_1 \cong S^3 \times_{T^1} P(\mathbb{C}_1 \oplus \mathbb{C})$. Let $f : H_1 \to H_1$ be the diffeomorphism which is induced from the composition of the diffeomorphisms

$$S^3 \times_{T^1} P(\mathbb{C}_1 \oplus \mathbb{C}) \xrightarrow{g} S^3 \times_{T^1} P(\mathbb{C}_{-1} \oplus \mathbb{C}) \xrightarrow{h} S^3 \times_{T^1} P(\mathbb{C}_1 \oplus \mathbb{C}),$$

where g is the diffeomorphism induced from the orientation reversing of the fibers and h is the diffeomorphism induced from the tensor product of the tautological line bundle on $\gamma_{-1} \oplus \epsilon$. Then, it is easy to check that the induced homomorphism f^* is $f^*(X) = x$ and $f^*(Y) = -x - y$, where $H^*(H_1) \simeq \mathbb{Z}[x, y]/\langle x^2, y^2 + xy\rangle$. Then, we can easily check the following isomorphisms;

$$\begin{aligned} f^*\xi_{(0,0,\beta)} &\equiv \xi_{(0,0,-\beta)}; \\ f^*\xi_{(1,0,\beta)} &\equiv \xi_{(1,0,-\beta)}; \\ f^*\xi_{(0,1,\beta)} &\equiv \xi_{(-1,-1,-\beta)}. \end{aligned}$$

Because of Lemma 2.2, we have

$$\gamma_{x+y} \otimes \xi_{(-1,-1,-\beta)} \equiv \xi_{(1,1,-\beta)},$$

where γ_{x+y} is the line bundle over H_1 induced from $x + y \in H^2(H_1)$. This establishes that

$$\begin{aligned} P(\xi_{(0,0,\beta)}) &\cong P(\xi_{(0,0,-\beta)}); \\ P(\xi_{(1,0,\beta)}) &\cong P(\xi_{(1,0,-\beta)}); \\ P(\xi_{(0,1,\beta)}) &\cong P(\xi_{(1,1,-\beta)}). \end{aligned}$$

□

Consequently, using Theorem 3.1 and 4.2, we have Theorem 1.1.

5. Cohomological non-rigidity of 8-dimensional $\mathbb{C}P$-tower

In this section, we classify all 2-stage $\mathbb{C}P$-towers whose first stage is $\mathbb{C}P^3$. We first introduce the following classification result of complex 2-dimensional vector bundles over $\mathbb{C}P^3$ by Atiyah and Rees [**AtRe**]. Let $\mathrm{Vect}_2(\mathbb{C}P^3)$ be the set of complex 2-dimensional vector bundles over $\mathbb{C}P^3$ up to bundle isomorphisms.

Theorem 5.1 (Atiyah-Rees). *There exist an injective map $\phi : \mathrm{Vect}_2(\mathbb{C}P^3) \to \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ such that $\phi(\xi) = (\alpha(\xi), c_1(\xi), c_2(\xi))$, where $c_1(\xi)$ and $c_2(\xi)$ are the first and the second Chern classes of ξ, and $\alpha(\xi)$ is a mod 2 element which is 0 when $c_1(\xi)$ is odd.*

By Theorem 5.1, any element in $\mathrm{Vect}_2(\mathbb{C}P^3)$ can be denoted by $\eta_{(\alpha,c_1,c_2)}$, where $(\alpha,c_1,c_2)\in\mathbb{Z}_2\oplus\mathbb{Z}\oplus\mathbb{Z}$ such that $\alpha\equiv 0 \pmod 2$ when $c_1\equiv 1 \pmod 2$. The goal of this section is to classify the topological types of $P(\eta_{(\alpha,c_1,c_2)})$ up to diffeomorphisms.

Because $P(\eta_{(\alpha,c_1,c_2)})$ is diffeomrphic to $P(\eta_{(\alpha,c_1,c_2)}\otimes\gamma)$ for any line bundle γ over $\mathbb{C}P^3$ by Lemma 2.1, we may assume $c_1\in\{0,1\}$. Therefore, in order to classify all $P(\eta_{(\alpha,c_1,c_2)})$ up to diffeomorphisms, it is enough to classify the following:

$$\begin{aligned} M_0(u) &= P(\eta_{(0,0,u)});\\ M_1(u) &= P(\eta_{(1,0,u)});\\ N(u) &= P(\eta_{(0,1,u)}), \end{aligned}$$

where $u\in\mathbb{Z}$. In the following three lemmas, we classify the cohomology rings of the above three types of manifolds up to graded ring isomorphisms.

Lemma 5.2. *Two cohomology rings $H^*(M_\alpha(u))$ and $H^*(N(u'))$ are not isomorphic for any $u,\ u'\in\mathbb{Z}$.*

Proof. By the Borel-Hirzebruch formula (2.1), we have ring isomorphisms

$$\begin{aligned} H^*(M_\alpha(u)) &\simeq \mathbb{Z}[X,Y]/\langle X^4,\ uX^2+Y^2\rangle, \text{ and}\\ H^*(N(u')) &\simeq \mathbb{Z}[x,y]/\langle x^4,\ u'x^2+xy+y^2\rangle. \end{aligned}$$

Assume that there is an isomorphism map $f: H^*(M_\alpha(u))\to H^*(N(u'))$. Then we may put

$$\begin{aligned} f(X) &= ax+by, \text{ and}\\ f(Y) &= cx+dy, \end{aligned}$$

for some $a,\ b,\ c,\ d\in\mathbb{Z}$ such that $ad-bc=\epsilon=\pm 1$. By taking the inverse of f, we also have

$$\begin{aligned} f^{-1}(x) &= d\epsilon X - b\epsilon Y, \text{ and}\\ f^{-1}(y) &= -c\epsilon X + a\epsilon Y. \end{aligned}$$

From the ring structures of $H^*(M_\alpha(u))$ and $H^*(N(u'))$, we have $f(uX^2+Y^2)=0$ and $f^{-1}(y^2+xy+u'x^2)=0$. Therefore we have the following equations:

$$u(a^2-u'b^2)+(c^2-u'd^2)=0; \tag{5.1}$$

$$u(2ab-b^2)+(2cd-d^2)=0; \tag{5.2}$$

$$c^2-a^2u-cd+abu+u'd^2-b^2uu'=0; \tag{5.3}$$

$$-2ac+cb+ad-2bdu'=0. \tag{5.4}$$

Because $f^{-1}(x^4)=(dX-bY)^4=0$, we also have

$$bd(d^2-ub^2)=0.$$

Therefore $bd=0$, or otherwise $d^2=ub^2$. We first assume $bd=0$. Then, there are two cases: $b=0$ and $d=0$. If $b=0$, then $|a|=|d|=1$. However, by using (5.2), we have $2cd=1$. This gives a contradiction. If $d=0$, then $|b|=|c|=1$. By using (5.4), we have $c(-2a+b)=0$, i.e., $b=2a$ by $|c|=1$. However, this contradicts to $|b|=1$. Hence, $bd\neq 0$ and $d^2=ub^2$, i.e., $|d|=\sqrt{|u|}|b|$. In this case, because $ad-bc=\epsilon=\pm1$, we have $|b|=1$ and $d^2=u$. Let $b=\epsilon'=\pm1$ and $d=\sqrt{u}\epsilon''$, where $\epsilon''=\pm 1$. Then, it follows from $ad-bc=\epsilon$ that $c=-\epsilon\epsilon'+a\sqrt{u}\epsilon''\epsilon'$. Therefore, by using (5.1), we have the following equation:

$$\begin{aligned} & u(a^2-u'b^2)+(c^2-u'd^2)\\ = & u(a^2-u')+(-\epsilon\epsilon'+a\sqrt{u}\epsilon''\epsilon')^2-u'u\\ = & 2ua^2-2uu'+1-2a\sqrt{u}\epsilon\epsilon''=0. \end{aligned}$$

However, this gives the equation $1=2(-ua^2+uu'+a\sqrt{u}\epsilon\epsilon'')$, which is a contradition. Hence, $H^*(M_\alpha(u))\not\simeq H^*(N(u'))$ for all $u,\ u'\in\mathbb{Z}$. ∎

Lemma 5.3. *The following two statements are equivalent.*

(1) *$H^*(M_\alpha(u))\simeq H^*(M_{\alpha'}(u'))$ where $\alpha,\ \alpha'\in\{0,1\}$.*

(2) $u = u' \in \mathbb{Z}$

Proof. Because (2) $\Rightarrow$ (1) is trivial, it is enough to show (1) $\Rightarrow$ (2). Assume there is an isomorphism $f : H^*(M_\alpha(u)) \simeq H^*(M_{\alpha'}(u'))$ where

$$H^*(M_\alpha(u)) \simeq \mathbb{Z}[X,Y]/\langle X^4,\ uX^2+Y^2\rangle;$$
$$H^*(M_{\alpha'}(u')) \simeq \mathbb{Z}[x,y]/\langle x^4,\ u'x^2+y^2\rangle.$$

We may use the same representation for f as in the proof of Lemma 5.2. Note that $f(uX^2+Y^2)=0$ and $f^{-1}(u'x^2+y^2)=0$. By using the representation of f, we have the following equations:

$$ua^2 - uu'b^2 + c^2 - u'd^2 = 0; \tag{5.5}$$
$$uab + cd = 0; \tag{5.6}$$
$$u'd^2 - uu'b^2 + c^2 - a^2u = 0; \tag{5.7}$$
$$u'bd + ac = 0. \tag{5.8}$$

By (5.5) and (5.7), we have

$$c^2 = b^2uu'; \tag{5.9}$$
$$ua^2 = u'd^2. \tag{5.10}$$

Because $X^4 = 0$, we also have that

$$ab(a^2 - b^2u') = 0.$$

We first assume $ab \neq 0$. Then

$$a^2 = b^2u'$$

by this equation. Together with (5.9) and (5.10), we have that

$$c^2b^2 = b^4uu' = b^2a^2u = b^2d^2u' = a^2d^2.$$

This implies that

$$(ad-bc)(ad+bc) = \epsilon(ad+bc) = 0.$$

Hence, $ad=-bc$. However this gives a contradiction because $ad-bc = 2ad = \epsilon = \pm 1$. Consequently, we have $ab=0$. Since $ad-bc=\epsilon$, if $a=0$ then $|b|=|c|=1$; therefore, we have $u=u'=\pm 1$ by (5.9); if $b=0$ then $|a|=|d|=1$; therefore, we have $u=u'$ by (5.10). This establishes the statement. $\square$

Lemma 5.4. *The following two statements are equivalent.*

(1) $H^*(N(u)) \simeq H^*(N(u'))$

(2) $u = u' \in \mathbb{Z}$

Proof. Because (2) $\Rightarrow$ (1) is trivial, it is enough to show (1) $\Rightarrow$ (2). Assume there is an isomorphism $f : H^*(N(u)) \simeq H^*(N(u'))$ where

$$H^*(N(u)) \simeq \mathbb{Z}[X,Y]/\langle X^4,\ uX^2+xy+Y^2\rangle;$$
$$H^*(N(u')) \simeq \mathbb{Z}[x,y]/\langle x^4,\ u'x^2+xy+y^2\rangle.$$

Again, we use the same representation for f as in the proof of Lemma 5.2. Because $f(Y^2+XY+uX^2)=0$ and $f^{-1}(y^2+xy+u'x^2)=0$, we have that

$$c^2 - d^2u' = -ua^2 + b^2uu' - ac + bdu'; \tag{5.11}$$
$$2cd - d^2 = -2abu + b^2u - ad - bc + bd; \tag{5.12}$$
$$c^2 - a^2u = -u'd^2 + b^2uu' + cd - bau; \tag{5.13}$$
$$-2ac - a^2 = 2bdu' + b^2u' - ad - bc - ab. \tag{5.14}$$

Because $f(X^4)=0$ and $f^{-1}(x^4)=0$, there are the following two cases:

(1) $b = 0$;

(2) $b \neq 0$ and $4a^3 - 6a^2b + 4ab^2(1-u') + b^3(2u'-1) = -4d^3 - 6d^2b - 4db^2(1-u) + b^3(2u-1) = 0$.

If $b=0$, then $|a|=|d|=1$. Therefore, by (5.12), $2c=d-a$, i.e., $c=0$ if $d=a$ or $c=-a$ if $d=-a$. Because $c^2-u'=-u-ac$ by (5.11), we have that $u=u'$.

Assume $b\neq 0$. By the equation $4a^3-6a^2b+4ab^2(1-u')+b^3(2u'-1)=0$, we have b is even. Substituting $a=A+\frac{b}{2}$ for some $A\in\mathbb{Z}$ to this equation (i.e., Tschirnhaus's transformation), we have the following equation:

$$\begin{aligned}
& 4(A+\frac{b}{2})^3-6(A+\frac{b}{2})^2b+4(A+\frac{b}{2})b^2(1-u')+b^3(2u'-1) \\
= \;& 4(A^3+3A^2\frac{b}{2}+3A\frac{b^2}{4}+\frac{b^3}{8})-6(A^2+Ab+\frac{b^2}{4})b+4(Ab^2+\frac{b^3}{2})(1-u')+b^3(2u'-1) \\
= \;& 4A^3+6A^2b+3Ab^2+\frac{b^3}{2}-6A^2b-6Ab^2-\frac{3b^3}{2}+4Ab^2+2b^3-4Ab^2u'-2b^3u'+2b^3u'-b^3 \\
= \;& 4A^3+Ab^2-4Ab^2u' \\
= \;& A(4A^2+b^2-b^2u')=0
\end{aligned}$$

Therefore, there are the two cases: $A=0$ or $A\neq 0$. We first assume $A\neq 0$. Then, by using the equation $4A^2+b^2-b^2u'=0$, we have $u'\geq 1$. Now, there is the following commutative diagram:

$$\begin{array}{ccc}
H^2(N(u))=\mathbb{Z}X\oplus\mathbb{Z}Y & \xrightarrow{X} & \mathbb{Z}X^2\oplus\mathbb{Z}XY=H^4(N(u)) \\
f\downarrow & & \downarrow f \\
H^2(N(u'))=\mathbb{Z}x\oplus\mathbb{Z}y & \xrightarrow{ax+by} & \mathbb{Z}x^2\oplus\mathbb{Z}xy=H^4(N(u'))
\end{array}$$

Because X and f are isomorphisms, so is $ax+by$ in the diagram. Using the indicated generators as bases, the determinant of the map $f\circ X: H^2(N(u))\to H^4(N(u'))$ is equal to the determinant of the map $(ax+by)\circ f: H^2(N(u))\to H^4(N(u'))$, which is equal to

$$a^2-ab+b^2u'=\epsilon_1=\pm 1. \tag{5.15}$$

Because $a\in\mathbb{Z}$, the discriminant of this equation satisfies

$$b^2-4(b^2u'-\epsilon_1)=b^2(1-4u')+4\epsilon_1\geq 0$$

Because $u'\geq 1$, we have that

$$0<b^2\leq\frac{4\epsilon_1}{4u'-1}<1.$$

This gives a contradiction to $b\in\mathbb{Z}$. Therefore, we have $A=0$, i.e., $a=\frac{b}{2}$. Because $ad-bc=\epsilon(=\pm 1)$, we also have that $a=\epsilon'=\pm 1$, $b=2\epsilon'$ and $d-2c=\epsilon\epsilon'$. Hence, by (5.15), we have $-1+4u'=\epsilon_1$, i.e., $u'=0$ and $\epsilon_1=-1$. By applying a similar method to the one used to derive (5.15) for $f^{-1}(x)$, we have

$$d^2+db+b^2u=\epsilon_2=\pm 1. \tag{5.16}$$

Substituting (5.15) and (5.16) to (5.13) and (5.14), we have

$$\begin{aligned}
c^2 &= u\epsilon_1-u'd^2+cd=-u+cd; \\
-2ac &= \epsilon_1+2bdu'-ad-bc=-1-(d+2c)\epsilon'.
\end{aligned}$$

By using the second equation above, we also have $d=-\epsilon'$; therefore, by $d-2c=\epsilon\epsilon'$, we have $c=\frac{-\epsilon'-\epsilon\epsilon'}{2}=0$ or $-\epsilon'$. If $c=0$, then $u=0$ by the first equation above; if $c=-\epsilon'$ then we also have $u=0$ by $d=-\epsilon'$ and the first equation above. This implies that $u=u'=0$ for the case $b\neq 0$.

This establishes the statement. □

Therefore, by Theorem 5.1 and Lemma 5.4, we have the following corollary.

Corollary 5.5. The following three statements are equivalent.

(1) Two spaces $N(u)$ and $N(u')$ are diffeomorphic.
(2) Two cohomology rings $H^*(N(u))$ and $H^*(N(u'))$ are isomorphic.
(3) $u=u'\in\mathbb{Z}$.

On the other hand, for $M_\alpha(u)$ we have the following Proposition.

Proposition 5.6. The following two statements are equivalent.

(1) Two spaces $M_\alpha(u)$ and $M_{\alpha'}(u')$ are diffeomorphic.
(2) $(\alpha, u) = (\alpha', u') \in \mathbb{Z}_2 \times \mathbb{Z}$.

In order to prove Proposition 5.6, we first compute the 6-dimensional homotopy group of $M_\alpha(u)$ in Proposition 5.8. Now $M_\alpha(u)$ can be defined by the following pull-back diagram:

$$\begin{tikzcd}
M_\alpha(u) \arrow[r] \arrow[d] & EU(2) \times_{U(2)} \mathbb{C}P^1 \arrow[d] \\
\mathbb{C}P^3 \arrow[r, "\mu_{\alpha,u}"] & BU(2)
\end{tikzcd}$$

Let $\rho : \mathbb{C}P^3 \to S^6$ be the collapsing map of $\mathbb{C}P^2 \subset \mathbb{C}P^3$ to $\{*\} \subset S^6$. Then, due to the argument in [**AtRe**], there is a map $\nu_\alpha \colon S^6 \to BU(2)$ in the following diagram such that the element $[\nu_\alpha] \in \pi_6(BU(2))$ corresponds to $\alpha \in \mathbb{Z}_2$, where $\pi_6(BU(2)) \simeq \mathbb{Z}_2$.

$$\begin{tikzcd}
M_\alpha(u) \arrow[r] \arrow[d] & EU(2) \times_{U(2)} \mathbb{C}P^1 \arrow[d] \\
\mathbb{C}P^3 \arrow[r, "\mu_{\alpha,u}"] \arrow[d, "\rho"] & BU(2) \\
S^6 \arrow[ur, dashed, "\nu_\alpha"] &
\end{tikzcd}$$

Note that the lower triangle of the above diagram is not necessarily commutative.

Let $p : S^7 \to \mathbb{C}P^3$ be the canonical S^1-fibration and $P(\xi_{\alpha,u})$ be the pull-back of $M_\alpha(u)$ along p. Namely, we have the following diagram, which are commutative except for the lower right-hand side triangle.

(5.17)
$$\begin{tikzcd}
P(\xi_{\alpha,u}) \arrow[r] \arrow[d] & M_\alpha(u) \arrow[r] \arrow[d] & EU(2) \times_{U(2)} \mathbb{C}P^1 \arrow[d] \\
S^7 \arrow[r, "p"] \arrow[dr] & \mathbb{C}P^3 \arrow[r, "\mu_{\alpha,u}"] \arrow[d, "\rho"] & BU(2) \\
& S^6 \arrow[ur, dashed, "\nu_\alpha"] &
\end{tikzcd}$$

Lemma 5.7. *For* $* \geq 6$, $\pi_*(P(\xi_{\alpha,u})) \simeq \pi_*(M_\alpha(u))$.

Proof. Because $P(\xi_{\alpha,u})$ is the pull-back of $M_\alpha(u)$, the homotopy exact sequences of $P(\xi_{\alpha,u})$ and $M_\alpha(u)$ satisfy the following commutative diagram:

$$\begin{tikzcd}
\pi_{*+1}(S^7) \arrow[r] \arrow[d] & \pi_*(\mathbb{C}P^1) \arrow[r] \arrow[d] & \pi_*(P(\xi_{\alpha,u})) \arrow[r] \arrow[d] & \pi_*(S^7) \arrow[r] \arrow[d] & \pi_{*-1}(\mathbb{C}P^1) \arrow[d] \\
\pi_{*+1}(\mathbb{C}P^3) \arrow[r] & \pi_*(\mathbb{C}P^1) \arrow[r] & \pi_*(M_\alpha(u)) \arrow[r] & \pi_*(\mathbb{C}P^3) \arrow[r] & \pi_{*-1}(\mathbb{C}P^1)
\end{tikzcd}$$

As is well known, $\pi_*(S^7) \simeq \pi_*(\mathbb{C}P^3)$ for $* \geq 6$. Therefore, by using the 5 lemma, we have the statement. $\square$

Now we may prove the following proposition.

Proposition 5.8. The following two isomorphisms hold.

(1) $\pi_6(P(\xi_{0,u})) \simeq \pi_6(M_0(u)) \simeq \mathbb{Z}_{12}$
(2) $\pi_6(P(\xi_{1,u})) \simeq \pi_6(M_1(u)) \simeq \mathbb{Z}_6$

Proof. From the $\mathbb{C}P^1$-fibrations $\mathbb{C}P^1 \to P(\xi_{\alpha,u}) \to S^7$ and $\mathbb{C}P^1 \to EU(2) \times_{U(2)} \mathbb{C}P^1 \cong BT^2 \to BU(2)$ in (5.17), we have the following commutative diagram.

$$\begin{tikzcd}
\pi_7(S^7) \simeq \mathbb{Z} \arrow[r] \arrow[d] & \pi_6(\mathbb{C}P^1) \arrow[r] \arrow[d, "\simeq"] & \pi_6(P(\xi_{\alpha,u})) \arrow[r] \arrow[d] & \pi_6(S^7) = \{0\} \arrow[d] \\
\pi_7(BU(2)) \simeq \mathbb{Z}_{12} \arrow[r, "\simeq"] & \pi_6(\mathbb{C}P^1) \arrow[r] & \pi_6(BT^2) = \{0\} \arrow[r] & \pi_6(BU(2)) \simeq \mathbb{Z}_2
\end{tikzcd}$$

Since $\xi_{\alpha,u}$ is of complex 2-dimension, its total Chern class $c(\xi_{\alpha,u}) = 1 \in H^*(S^7)$. Therefore we may regard that $\widetilde{\mu} := \mu_{\alpha,u} \circ p\colon S^7 \to BU(2)$ can be defined by passing through the map $\nu_\alpha : S^6 \to BU(2)$, i.e., $\widetilde{\mu} = \nu_\alpha \circ \rho \circ p$. Therefore, we have the following relations:

$$\widetilde{\mu}_\# : \pi_7(S^7) \longrightarrow \pi_7(S^6) \simeq \mathbb{Z}_2 \xrightarrow{(\nu_\alpha)_\#} \pi_7(BU(2)) \simeq \mathbb{Z}_{12}.$$

Because ν_0 induces the trivial bundle over S^7, $\widetilde{\mu}_\#$ is the 0-map. Therefore, by using the above commutative diagram and exactness, we have

$$\pi_6(P(\xi_0)) \simeq \pi_6(\mathbb{C}P^1)/\widetilde{\mu}_\#(\pi_7(S^7)) \simeq \mathbb{Z}_{12}/\{[0]_{12}\} \simeq \mathbb{Z}_{12}.$$

On the other hand, ν_1 induces the non-trivial bundle over S^7. Therefore $0 \neq [\widetilde{\mu}] \in \pi_7(BU(2))$. But $[\widetilde{\mu}] = \widetilde{\mu}_\#(1)$ where $1 \in \pi_7(S^7) \simeq \mathbb{Z}$ is a generator. It follows that $\nu_{1\#}\colon \pi_7(S^6) \to \pi_7(BU(2))$ is not the zero map. This implies that $\widetilde{\mu}_\#(\mathbb{Z}) = \mathbb{Z}_2 \subset \mathbb{Z}_{12}$, i.e., $\widetilde{\mu}_\#(1) = [6]_{12}$. Hence, by using the above commutative diagram and exactness, we have that

$$\pi_6(P(\xi_1)) \simeq \pi_6(\mathbb{C}P^1)/\widetilde{\mu}_\#(\pi_7(S^7)) \simeq \mathbb{Z}_{12}/\{[0]_{12}, [6]_{12}\} \simeq \mathbb{Z}_6.$$

By Lemma 5.7, we have the statement. □

Let us prove Proposition 5.6

Proof of Proposition 5.6. By using Theorem 5.1, (2) ⇒ (1) is trivial. We prove (1) ⇒ (2). Assume $M_\alpha(u) \cong M_{\alpha'}(u')$. If $u \neq u'$, then $H^*(M_\alpha(u)) \not\simeq H^*(M_{\alpha'}(u'))$ by Lemma 5.3. Therefore, we have $u = u'$. By Proposition 5.8, $M_0(u) \ncong M_1(u)$. This implies that the statement (1) ⇒ (2) in Proposition 5.6. This establishes Proposition 5.6. □

Consequently, by Lemma 5.3 and Proposition 5.6, we have the following corollary:

Corollary 5.9. The set of 8-dimensional $\mathbb{C}P$-towers does not satisfy the cohomological rigidity.

Note that if we restrict the class of 8-dimensional $\mathbb{C}P$-towers to the 8-dimensional generalized Bott manifolds with height 2, then cohomological rigidity holds by [**CMS10**].

Using Corollary 5.5 and Proposition 5.6, we also have Theorem 1.3.

6. Appendix: $M_1(1)$ is diffeomorphic to $Sp(2)/T^2$

In this appendix, we prove the following proposition.

Proposition 6.1. The following diffeomorphism holds.

$$M_1(1) = P(\eta_{(1,0,1)}) \cong Sp(2)/T^2.$$

Proof. Let $\eta_{\mathbb{H}} = Sp(2) \times_{T^1 \times Sp(1)} \mathbb{H}$, where $T^1 \times Sp(1)$ acts on $\mathbb{H}$ canonically via the projection $T^1 \times Sp(1) \to Sp(1)$. Because $Sp(2)/T^2 = Sp(2) \times_{T^1 \times Sp(1)} \mathbb{C}P^1 = P(\eta_{\mathbb{H}})$ is an 8-dimensional two stage $\mathbb{C}P$-tower, if we have its cohomology ring then we can determine the Chern classes of $\eta_{\mathbb{H}}$ by using the Borel-Hirzebruch formula. As is well known that the cohomology ring of $Sp(2)/T^2$ is isomorphic to the following ring (e.g. see [**Bo**] or [**FIM**]).

$$H^*(Sp(2)/T^2; \mathbb{Z}) \simeq \mathbb{Z}[\tau_1, \tau_2]/\langle \tau_1^2 + \tau_2^2, \tau_1^2\tau_2^2 \rangle,$$

where $\deg \tau_i = 2$ for $i = 1, 2$. Therefore, it is easy to check that the cohomology ring is isomorphic to

$$H^*(Sp(2)/T^2; \mathbb{Z}) \simeq \mathbb{Z}[x, y]/\langle x^2 + y^2, x^4 \rangle.$$

This implies that $c(\eta_{\mathbb{H}}) = 1 + x^2$ where $x \in H^2(\mathbb{C}P^3)$ (we may abuse the elements $x \in H^2(\mathbb{C}P^3)$ and $\pi^*(x) \in H^2(Sp(2)/T^2)$ induced by $\pi : Sp(2)/T^2 \to \mathbb{C}P^3$). Therefore we have $\eta_{\mathbb{H}} \equiv \eta_{(\alpha,0,1)}$ for some $\alpha = 0$ or 1. We claim $\alpha = 1$. Let $p : S^7 \to \mathbb{C}P^3$ be the quotient map by the free S^1-action on S^7. Then, the pull back of $\eta_{\mathbb{H}}$ along p can be denoted by $\xi_{\alpha,1} \equiv Sp(2) \times_{Sp(1)} \mathbb{H}$. Namely, we have the following diagram.

(6.1)
$$\begin{tikzcd}
Sp(2) \times_{Sp(1)} \mathbb{H} \arrow[r] \arrow[d] & \eta_{\mathbb{H}} \arrow[r] \arrow[d] & ESp(1) \times_{Sp(1)} \mathbb{H} \arrow[d] \\
S^7 \arrow[r, "p"] \arrow[dr] & \mathbb{C}P^3 \arrow[r, "\mu_{\alpha,1}"] \arrow[d, "\rho"] & BSp(1) \\
 & S^6 \arrow[ur, dashed, "\nu_\alpha"] &
\end{tikzcd}$$

where $\nu_\alpha : S^6 \to BSp(1) \in \pi_6(BSp(1)) \simeq \pi_6(BU(2)) \simeq \mathbb{Z}_2$. Note that $c(Sp(2) \times_{Sp(1)} \mathbb{H}) = 1$. Therefore, we may also regard $Sp(2)\times_{Sp(1)}\mathbb{H}$ as the pull-back along $\nu_\alpha \circ \rho \circ p$. Because $Sp(2)\times_{Sp(1)}\mathbb{H}$ is non-trivial bundle, we have that $\nu_\alpha \neq 0 \in \pi_6(BSp(1)) \simeq \mathbb{Z}_2$. Therefore, we have $\alpha = 1$. This establishes the statement of this proposition. □

From this proposition and the argument in Section 5, two 8-dimensional $\mathbb{C}P$-towers $Sp(2)/T^2(= M_1(1))$ and $M_0(1)$ have the same cohomology ring but their homotopy types are different. Hence, we have the following corollary.

Corollary 6.2. There is a $\mathbb{C}P$-tower whose cohomology ring is the same with the flag manifold of type C but its topological type is different from the flag manifold of type C.

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