$SH(3)$-MOVE AND OTHER LOCAL MOVES ON KNOTS

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Abstract. An $SH(3)$-move is an unknotting operation on oriented knots introduced by Hoste, Nakanishi and Taniyama. We consider some relationships to other local moves such as a band surgery, Γ_0-move, and Δ-move, and give some criteria for estimating the $SH(3)$-unknotting number using the Jones, HOMFLYPT, Q polynomials. We also show a table of $SH(3)$-unknotting numbers for knots with up to 9 crossings.

1. Introduction

An $SH(3)$-move is a local change for an oriented link diagram which preserves the number of components as shown in Fig. 1. This move is a special case of an $SH(n)$-move introduced by Hoste, Nakanishi and Taniyama [6], where n is odd. Then they showed that each of these moves are unknotting operation, that is, any knot can be deformed into a trivial knot by a sequence of $SH(n)$-moves.@So, we may define an $SH(n)$-Gordian distance between two knots and an $SH(n)$-unknotting number for a knot. In this paper, we mainly consider an $SH(3)$-Gordian distance and an $SH(3)$-unknotting number.

For an $SH(3)$-Gordian distance, Taniyama and Yasuhara [30] have given some interpretations (Proposition 2.1), which suggest the importance of the $SH(3)$-move particularly from 4-dimensional point of view and also give several estimations of an $SH(3)$-Gordian distance (Propositions 2.2, 2.3, 2.5, 2.6). Since an $SH(3)$-move is realized by a sequence of two coherent band surgeries (Fig. 2), we may apply some criteria by the Jones, Q, and HOFMLYPT polynomials for a band surgery ([9, 10]) to obtain some criteria on an $SH(3)$-Gordian distance (Theorem 3.1).

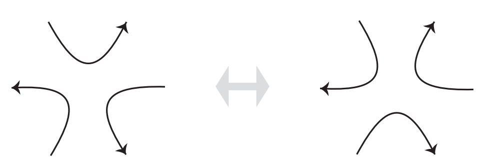

Figure 1. An $SH(3)$-move.

We then consider some relations of an $SH(3)$-move with other local moves such as a crossing change, a Γ_0-move, and a Δ-move. They are also unknotting operation, and we

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may obtain several relations among a usual Gordian distance, Γ_0-Gordian distance, Δ-Gordian distance, and $SH(3)$-Gordian distance (Propositions 4.1, 4.3). These relations are useful to decide an $SH(3)$-unknotting number, and are also efficient to give a lower bound of an Δ-unknotting number (Example 4.8). Lastly, we give a table of an $SH(3)$- and Γ_0-unknotting numbers for knots with up to 9 crossings (Tables 1, 2); we can decide the $SH(3)$-unknotting numbers completely, but cannot decide the Γ_0-unknottings number for 12 knots.

This paper is organized as follows: In Sec. 2, we present some interpretations for an $SH(3)$-Gordian distance due to Taniyama and Yasuhara. In Sec. 3, we give some criteria for an $SH(3)$-Gordian distance by the special values of the Jones, Q, and HOMFLYPT polynomials. In Sec. 4, we give some relations among several Gordian distances by a crossing change, a Γ_0-move, a Δ-move, and an $SH(3)$-move. In Sec. 5, we give a table of an $SH(3)$- and Γ_0-unknotting numbers for knots with up to 9 crossings. In Sec. 6, we remak that our method is also acceptable for the pass move and sharp move.

For knots and links we use Rolfsen notations in Appendix C in [27].

2. Some interpretations of an $SH(3)$-Gordian distance

For an $SH(3)$-Gordian distance Taniyama and Yasuhara [30] have given some interpretations, which is useful to give several estimations of an $SH(3)$-Gordian distance and an $SH(3)$-unknotting number.

Let K and K' be two oriented knots in S^3. The *C-distance* of K and K', $\mathrm{d}_C(K, K')$, is the minimal genus of an embedded oriented surface in S^3 whose boundary is the two knots K and K'. In other words, the C-distance of K and K' is the minimal genus over all 2-components links whose components are K and $-K'$, where $-K'$ is the knot K' with reversed orientation. The *concordance distance* of K and K', $\mathrm{c}(K, K')$, is half of the least number of elementary critical points on an oriented surface in $S^3 \times [0, 1]$ connecting K in $S^3 \times \{0\}$ to K' in $S^3 \times \{1\}$, that is, a concordance between the two knots, whose projection to $[0, 1]$ is a Morse function. The following is the main theorem of [30].

Proposition 2.1. *The $SH(3)$-Gordian distance, the C-distance, and the concordance distance between two knots are equal;* $\mathrm{sd}_3(K, K') = \mathrm{d}_C(K, K') = \mathrm{c}(K, K')$.

The C-distance is an interpretation for the $SH(3)$-Gordian distance from a 3-dimensional point of view, which implies the following ([6, Theorems 3*(2)], [30, Theorems 3.1]):

Proposition 2.2. *For a knot K, we have:*

$$\mathrm{su}_3(K) \leq \mathrm{g}(K), \tag{1}$$

where $\mathrm{g}(K)$ *is the genus of* K.

The concordance distance is an interpretation for the $SH(3)$-Gordian distance from a 4-dimensional point of view, which implies Propositions 2.3 and 2.5 below. For two oriented knots K and K', we define the *coherent band-Gordian distance*, $\mathrm{d}_{\mathrm{band}}(K, K')$, to be the minimum number of band surgeries needed to deform K into K'. We define the *coherent* *band unknotting number* of K to be the coherent band-Gordian distance of K and the trivial knot U, $\mathrm{u}_{\mathrm{band}}(K) = \mathrm{d}_{\mathrm{band}}(K, U)$. Then we have:

Proposition 2.3. *For two oriented knots K and K', we have:*

$$\mathrm{d}_{\mathrm{band}}(K, K') = 2\mathrm{sd}_3(K, K'). \tag{2}$$

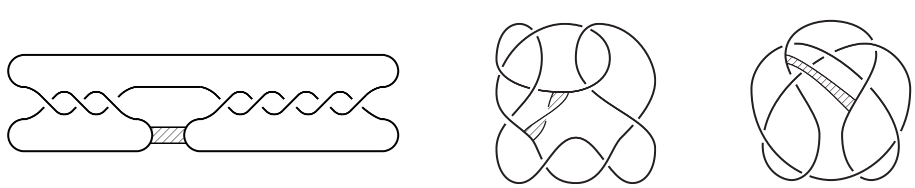
In particular, we have:

$$\mathrm{u}_{\mathrm{band}}(K) = 2\mathrm{su}_3(K). \tag{3}$$

Proof. An $SH(3)$-move is realized by a sequence of two coherent band surgeries as shown in Fig. 2; see [6, Fig. 4], and so $\mathrm{d}_{\mathrm{band}}(K, K') \leq 2\mathrm{sd}_3(K, K')$. Conversely, suppose that there exist a sequence of oriented links $L_0 = K$, $L_1, \ldots, L_{n-1}$, $L_n = K'$ such that L_i is obtained from L_{i-1} by a coherent band surgery for each i, $1 \leq i \leq n$. Then there exists an oriented surface in $S^3 \times [0, 1]$ connecting K in $S^3 \times \{0\}$ to K' in $S^3 \times \{1\}$, the number of whose elementary critical points is n. Then $2\mathrm{c}(K, K') \leq n$, and so by Proposition 2.1 we have $2\mathrm{sd}_3(K, K') \leq \mathrm{d}_{\mathrm{band}}(K, K')$, completing the proof. $\square$

FIGURE 2. An $SH(3)$-move is realized by a sequence of two coherent band surgeries.

Example 2.4. Let $K = 3_1!\#5_1$. Then K is deformed into H_-, the negative Hopf link, by a band surgery along the band as shown in Fig. 3. Since H_- is band-trivializable, that is, it can be deformed into the trivial knot by a band surgery, by Proposition 2.3 we have $\mathrm{su}_3(K) = \mathrm{u}_{\mathrm{band}}(K)/2 = 1$. Similarly, we have $\mathrm{su}_3(9_8) = \mathrm{su}_3(9_{31}) = 1$. In fact, the knots 9_8 and 9_{31} are deformed into $4_1^2!$ with linking number -2 and $5_2\#H_+$, respectively by a band surgery along the bands as shown in Fig. 3, where $4_1^2!$ with linking number -2 is the torus link of type $(2, 4)$ with anti-parallel orientation and H_+ is the positive Hopf link; these links are easily seen to be band-trivializable; cf. [10, Lemma 4.3].



$3_1!\#5_1$ 9_8 9_{31}

FIGURE 3. Knots with $SH(3)$-unknotting number one.

We define the 4-*distance* of two oriented knots K and K', $\mathrm{d}_4(K,K')$, to be the minimum genus of a concordance in $S^3\times[0,1]$ between K and K'. In particular, the 4-distance of K and the trivial knot U is the 4-*ball genus* of K, $\mathrm{g}^*(K)=\mathrm{d}_4(K,U)$. Then $\mathrm{d}_4(K,K')=\mathrm{g}^*(K\#(-K'))$. ???references??? We obtain the following ([30, Theorems 1.2 and 3.1]):

Proposition 2.5. *For two oriented knots K and K', we have:*

$$\mathrm{d}_4(K,K')\le \mathrm{sd}_3(K,K'). \tag{4}$$

In particular, we have:

$$\mathrm{g}^*(K)\le \mathrm{su}_3(K). \tag{5}$$

A knot with 4-ball genus zero is usually called a *slice knot*. Namely, a slice knot K in S^3 bounds a properly embedded locally flat disk in $S^3\times[0,\infty)$. A *ribbon knot* is a slice knot bounding a disk in $S^3\times[0,\infty)$ whose critical points consist of maximum and saddle points. We define the ribbon-fusion number of a ribbon knot to be the least number of such saddle points. More precisely, a ribbon knot of m-fusions is a knot obtained from a trivial $(m+1)$-component link by doing band surgery along m bands. So, it has the form

$$S_0^1\cup S_1^1\cup\cdots\cup S_m^1\cup\bigcup_{i=1}^m f_i(\partial I\times I)-\mathrm{int}\bigcup_{i=1}^m f_i(I\times\partial I), \tag{6}$$

where $S_0^1\cup S_1^1\cup\cdots\cup S_m^1$ is a trivial m-component link and $f_i: I\times I\to S^3$ $(i=1,2,\dots,m)$ are disjoint embeddings such that

$$f_i(I\times\partial I)\cap S_j=\begin{cases} f_i(I\times\{0\}), & \text{if } j=0;\\ f_i(I\times\{1\}), & \text{if } j=i;\\ \emptyset, & \text{otherwise.}\end{cases} \tag{7}$$

By a ribbon knot we mean a ribbon knot of m-fusions for some m; see [31]. The least number of such m is the *ribbon-fusion number* of K, which we denote by $\mathrm{rf}(K)$; see [9]. Then the following is immediate from the equation $\mathrm{sd}_3(K,K')=\mathrm{c}(K,K')$, which is a generalization of Example 3.2(1) in [30].

Proposition 2.6. *If K is a ribbon knot, then*

$$\mathrm{su}_3(K)\le \mathrm{rf}(K). \tag{8}$$

In particular, if K is a ribbon knot with ribbon-fusion number one, then $\mathrm{su}_3(K)=1$.

Example 2.7. Let $K=9_{41}$, which is a ribbon knot with 1-fusion [13, Appendix F.5], and so by Proposition (2.6), we obtain $\mathrm{su}_3(K)=1$. Note that the inequalities in Proposition 4.5 below do not work to decide this.

We denote by $\Sigma_m(L)$ the m-fold cyclic covering space of S^3 branched over an oriented link L in S^3, and by $\tilde\Sigma(K)$ the infinite cyclic covering space of the complement of an oriented knot K in S^3. Let $e_m(L)$ be the minimum number of generators of $H_1(\Sigma_m(L);\boldsymbol{Z})$, and $e(K)$ the minimum number of generators of $H_1(\tilde\Sigma(K);\boldsymbol{Z})$ as a $\boldsymbol{Z}[t,t^{-1}]$-module. Then

$e(K)$ is equal to the Nakanishi index of K; see [13, p. 72]. For an $SH(3)$-Gordian distance we have the following:

Proposition 2.8. *For two oriented knots K and K' in S^3, we have the following:*

$$|e_m(K) - e_m(K')|/2(m-1) \leq \mathrm{sd}_3(K, K'). \tag{9}$$

$$|e(K) - e(K')|/2 \leq \mathrm{sd}_3(K, K'). \tag{10}$$

Proof. Eq. (9) can be proved in a similar way to [6, Theorem 4*], and Eq. (10) is ginve in [30, Theorem 1.2], which is essentially due to Nakanishi [22]. □

For an $SH(3)$-unknotting number we have the following:

Proposition 2.9. *For an oriented knot K in S^3, we have the following:*

$$e_m(K)/2(m-1) \leq e(K)/2 \leq \mathrm{su}_3(K). \tag{11}$$

Proof. The first inequality, $e_m(K)/(m-1) \leq e(K)$, is due to Kinoshita [14], and the second one follows from Eq. (10), which is also given in [30, Theorem 3.1]. □

Remark 2.10. The inequality $e_m(K)/2(m-1) \leq \mathrm{su}_3(K)$ is given in [6, Theorem 4*].

3. Special values of some polynomial invariants of knots

We consider the special values of the Jones, Q, and HOMFLYPT polynomials of knots, which allow us to estimate an $SH(3)$-Gordian distance and an $SH(3)$-unknotting number in some cases. First we remember the definitions of several polynomials.

The *Conway polynomial* $\nabla(L; z) \in \boldsymbol{Z}[z]$ [3], the *Jones polynomial* $V(L; t) \in \boldsymbol{Z}[t^{\pm 1/2}]$ [7], and the *HOMFLYPT polynomial* $P(L; v, z) \in \boldsymbol{Z}[v^{\pm 1}, z^{\pm 1}]$ [4, 26] are invariants of the isotopy type of an oriented link L, which are defined by the following formulas:

$$\nabla(U; z) = 1; \tag{12}$$

$$\nabla(L_+; z) - \nabla(L_-; z) = z\nabla(L_0; z); \tag{13}$$

$$V(U; t) = 1; \tag{14}$$

$$t^{-1}V(L_+; t) - tV(L_-; t) = \left(t^{1/2} - t^{-1/2}\right) V(L_0; t); \tag{15}$$

$$P(U; v, z) = 1; \tag{16}$$

$$v^{-1}P(L_+; v, z) - vP(L_-; v, z) = zP(L_0; v, z), \tag{17}$$

where U is the unknot and (L_+, L_-, L_0) is a skein triple, that is, three links that are identical except near one point where they are as in Fig. 4.

For an oriented link L, the Conway and Jones polynomials are related to the HOMFLYPT polynomial by:

$$\nabla(L; z) = P\left(L; 1, z\right); \tag{18}$$

$$V(L; t) = P\left(L; t, t^{1/2} - t^{-1/2}\right). \tag{19}$$

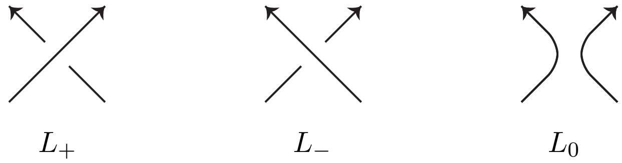


FIGURE 4. A skein triple.

The Conway polynomial of a knot K is of the form

$$\nabla(K;z) = 1 + \sum_{k=1}^{n} a_{2k}(K) z^{2k},$$

where $a_{2k}(K) \in \boldsymbol{Z}$.

The *Q polynomial* $Q(L;z) \in \boldsymbol{Z}[z^{\pm 1}]$ [1, 5] is an invariant of the isotopy type of an unoriented link L, which is defined by the following formulas:

$$Q(U;z) = 1; \tag{20}$$

$$Q(L_+;z) + Q(L_-;z) = z\left(Q(L_0;z) + Q(L_\infty;z)\right), \tag{21}$$

where U is the unknot and L_+, L_-, L_0, L_∞ are four unoriented links that are identical except near one point where they are as in Fig. 5.

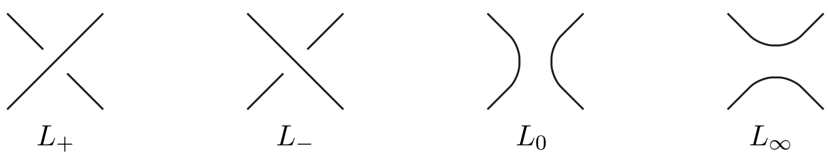


FIGURE 5. An unoriented skein quadruple.

Some special values of these polynomials are closely related with some finite cyclic covering spaces of S^3 branched over a link. Let $\Sigma_m(L)$ be the m-fold cyclic cover of S^3 branched over an oriented link L, $c(L)$ the number of the components of L, $d = \dim H_1(\Sigma_2(L); \boldsymbol{Z}_3)$, $f = \dim H_1(\Sigma_2(L); \boldsymbol{Z}_5)$, and $h = \dim H_1(\Sigma_3(L); \boldsymbol{Z}_2)$. Further, put $\omega = e^{i\pi/3}$ and $\rho(L) = Q\left(L; (\sqrt{5}-1)/2\right)$; we consider L an oriented link in $\rho(L)$. Then we have:

$$V(L;\omega) = \pm i^{c(L)-1}(i\sqrt{3})^d; \tag{22}$$

$$\rho(L) = \pm\sqrt{5}^f; \tag{23}$$

$$P(P;i,i) = (i\sqrt{2})^h, \tag{24}$$

where $V(L;\omega)$ means the value of $V(L;t)$ at $t^{1/2} = e^{i\pi/6}$, whence $t^{1/2} - t^{-1/2} = i$; see [8, 15]; cf. [16, Table 16.3].

Theorem 3.1. *If two knots K and K' are related by an $SH(3)$-move, then:*

$$V(K;\omega)/V(K';\omega) \in \left\{ \pm 1,\ \pm i\sqrt{3}^{\pm 1},\ 3^{\pm 1} \right\}; \tag{25}$$

$$\rho(K)/\rho(K') \in \left\{ \pm 1, \pm\sqrt{5}^{\pm 1},\ 5^{\pm 1} \right\}; \tag{26}$$

$$P(K;i,i)/P(K';i,i) \in \{1,\ -2^{\pm 1},\ 4^{\pm 1}\}. \tag{27}$$

In particular, if $\mathrm{su}_3(K) = 1$*, then:*

$$V(K;\omega) \in \left\{ \pm 1,\ \pm i\sqrt{3},\ 3 \right\}; \tag{28}$$

$$\rho(K) \in \left\{ \pm 1, \pm\sqrt{5},\ 5 \right\}; \tag{29}$$

$$P(K;i,i) \in \{1,\ -2,\ 4\}. \tag{30}$$

Proof. If two knots K and K' are related by an $SH(3)$-move, then there is a 2-component link L such that L is obtained from each of K and K' by a coherent band surgery. Then by Theorems 2.2 and 3.1 in [9] and Proposition 2.4 in [10] we have:

$$V(K;\omega)/V(L;\omega),\ V(L;\omega)/V(K';\omega) \in \left\{ \pm i,\ -\sqrt{3}^{\pm 1} \right\}; \tag{31}$$

$$\rho(K)/\rho(L),\ \rho(L)/\rho(K') \in \left\{ \pm 1, \sqrt{5}^{\pm 1} \right\}; \tag{32}$$

$$P(K;i,i)/P(L;i,i),\ P(L;i,i)/P(K';i,i) \in \{1,\ -2^{\pm 1}\}, \tag{33}$$

which imply Eqs. (25), (26), (27), respectively. This completes the proof. □

Example 3.2. Let $K = 9_{37}$, 9_{48} or $3_1!\#6_1$. Then $V(K;\omega) = -3$, and so by Theorem 3.1, we have $\mathrm{su}_3(K) > 1$. On the other hand, since $\mathrm{u}(K) = 2$ (see [2, 29]), by Eq. (37) we obtain $\mathrm{su}_3(K) = \mathrm{u}_{\Gamma_0}(K) = 2$. Notice that since $|\sigma(K)| = \mathrm{g}^*(K) = 1$, Eq. (5) does not work. (These knots and $3_1\#3_1$ are the only knots with $V(K;\omega) = -3$ up to 9 crossings.)

4. Relations with other local moves of knots

A Γ_0-move is a local change in an oriented link diagram as shown in Fig. 6, which was introduced by Shibuya [28]. This move is an unknotting operation. In fact, a crossing change is realized by a Γ_0-move as shown in Fig. 7. Then for oriented knots K and K', we may define the Γ_0-*Gordian distance* from K to K', $\mathrm{d}_{\Gamma_0}(K,K')$, and Γ_0-*unknotting number* of K, $\mathrm{u}_{\Gamma_0}(K)$, in a usual way. Then since a Γ_0-move is realized by two crossing changes, we obtain:

Proposition 4.1. *Let K and K' be knots. Then we have:*

$$\mathrm{d}_{\Gamma_0}(K,K') \le \mathrm{d}(K,K') \le 2\mathrm{d}_{\Gamma_0}(K,K'). \tag{34}$$

In particular, we have:

$$\mathrm{u}_{\Gamma_0}(K) \le \mathrm{u}(K) \le 2\mathrm{u}_{\Gamma_0}(K). \tag{35}$$

FIGURE 6. A Γ_0-move.

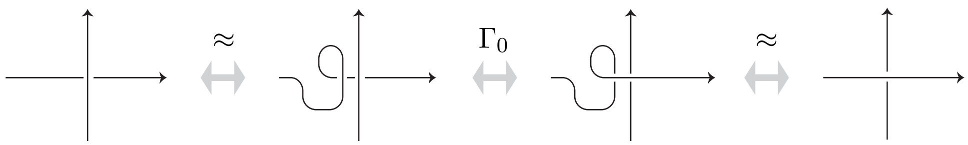


FIGURE 7. A crossing change is realized by a Γ_0-move.

Shibuya [28] also introduced a similar local change as shown in Fig. 8, which is called a Γ_0'-move. Then Γ_0'-move is equivalent to Γ_0-move, that is, a Γ_0'-move is realized by a Γ_0-move, and vice-versa as shown in Fig. 9

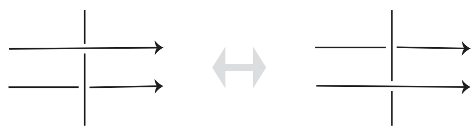


FIGURE 8. A Γ_0'-move.

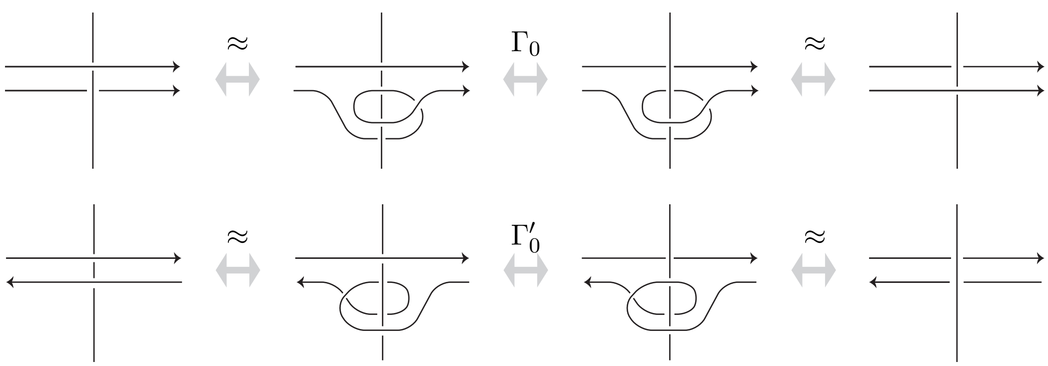


FIGURE 9. A Γ_0'-move is realized by a Γ_0-move, and vice-versa.

Murakami and Nakanishi [20] and Matveev [17] introduced a local change in an oriented link diagram called a Δ-*move* as shown in Fig. 10, where the orientations of strings are irrelevant.

A Δ-move is known to be an unknotting operation; see [20, Lemma 1.1]. Then for oriented knots K and K', we may define the Δ-*Gordian distance* from K to K', $\mathrm{d}_\Delta(K, K')$,

FIGURE 10. A Δ-move.

and Δ-*unknotting number* of K, $\mathrm{u}_\Delta(K)$, in a usual way. A Δ-move has the following properties, where (i) is deduced from Theorem 2.3 in [20] and (ii) is Theorem 1.1 in [23]:

Proposition 4.2. (i) *For two knots K and K', we have:* $\mathrm{d}_\Delta(K,K') \equiv a_2(K) - a_2(K') \pmod 2$.
(ii) *If two knots K and K' are related by a single Δ-move, then* $|a_2(K) - a_2(K')| = 1$.
In particular, we have $\mathrm{u}_\Delta(K) \equiv a_2(K) \pmod 2$ *and* $\mathrm{u}_\Delta(K) \geq |a_2(K)|$.

Proposition 4.3. *Let K and K' be knots in S^3. Then we have:*

$$\mathrm{sd}_3(K,K') \leq \mathrm{d}_{\Gamma_0}(K,K') \leq \mathrm{d}_\Delta(K,K'). \tag{36}$$

In particular, we have:

$$\mathrm{su}_3(K) \leq \mathrm{u}_{\Gamma_0}(K) \leq \mathrm{u}_\Delta(K). \tag{37}$$

Proof. The first inequality of Eq. (36) is deduced from [30, Proposition 2.1]. In fact, a Γ_0-move is realized by an $SH(3)$-move as shown in Fig. 11; cf. [30, Fig. 2.7]. The second inequality of Eq. (36) is due to Shibuya [28, Theorem 1.3]. In fact, a Δ-move is realized by a Γ_0-move as shown in Fig. 12 [28, Fig. 6]. $\square$

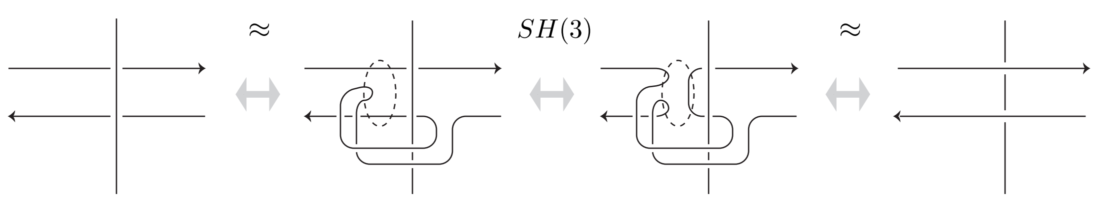


FIGURE 11. A Γ_0-move is realized by an $SH(3)$-move.

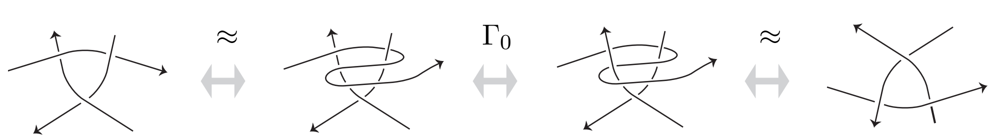


FIGURE 12. A Δ-move is realized by a Γ_0-move.

Remark 4.4. Combining Eqs. (34) and (36), we obatin Theorem 2.1 in [20]: for any knots K and K', the following hold:

$$\mathrm{d}(K,K') \leq 2\mathrm{d}_\Delta(K,K'); \tag{38}$$

$$\mathrm{u}(K) \leq 2\mathrm{u}_\Delta(K). \tag{39}$$

We summarize several estimations on the $SH(3)$-unknotting number.

Proposition 4.5. *For a knot K, we have:*

$$|\sigma(K)|/2 \leq \mathrm{g}^*(K) \leq \mathrm{su}_3(K) \leq \mathrm{g}(K); \tag{40}$$

$$e_m(K)/2(m-1) \leq e(K)/2 \leq \mathrm{su}_3(K) \leq \mathrm{u}_{\Gamma_0}(K) \leq \mathrm{u}(K),\ \mathrm{u}_\Delta(K). \tag{41}$$

Example 4.6. Let $K = 9_{35}$. Then $\mathrm{u}(K) = 3$ (see [2, 25]), and so by Eq. (35), we have $\mathrm{u}_{\Gamma_0}(K) = 2$ or 3. On the other hand, since $\mathrm{g}(K) = 1$, by Eq. (37) we obtain $\mathrm{su}_3(K) = 1$.

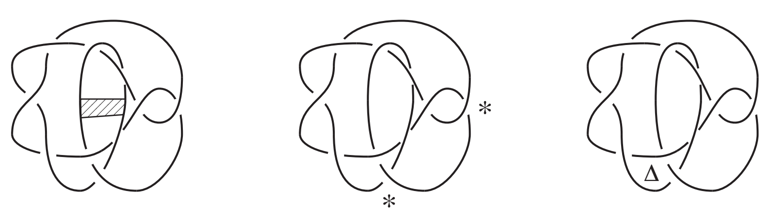
Example 4.7. For the knot $K = 10_{103}$, we have:

$$|\sigma(K)|/2 = \mathrm{g}^*(K) = 1;\ \mathrm{su}_3(K) = \mathrm{u}_{\Gamma_0}(K) = 2;\ \mathrm{g}(K) = \mathrm{u}(K) = \mathrm{u}_\Delta(K) = 3.$$

Proof. From Table F.3 in [13, p. 103] we have $|\sigma(K)|/2 = \mathrm{g}^*(K) = 1$ and $\mathrm{g}(K) = 3$. In fact, performing the band surgery along the band as shown in Fig. 13(a), we obtain the 2-component link $H_-\#8_8!$, the composition of the negative Hopf link and the mirror image of the knot 8_8. Since the 8_8 knot is a ribbon knot, we have $\mathrm{g}^*(K) \leq 1$.

Performing the Γ_0'-move at the 2 crossings near the marks $*$ indicated in Fig. 13(b), K becomes the knot 5_2, which has unknotting number one, and so $\mathrm{u}_{\Gamma_0}(K) \leq 2$. On the other hand, since $\rho(K) = -5$, we have $\mathrm{su}_3(K) > 1$. Therefore, $\mathrm{su}_3(K) = \mathrm{u}_{\Gamma_0}(K) = 2$.

The Δ-unknotting number, $\mathrm{u}_\Delta(K) = 3$, is taken from the table in [21]. In fact, performing the Δ-move around the region containing the mark Δ in Fig. 13(c), K becomes the knot 5_2, which has Δ-unknotting number two [23], and so $\mathrm{u}_\Delta(K) \leq 3$. On the other hand, since $a_2(K) = 3$, $\mathrm{u}_\Delta(K) \geq 3$. □



(a) (b) (c)

FIGURE 13. The knot 10_{103} yields: (a) the 2-component link $H_-\#8_8!$ by a band surgery; (b) the 5_2 knot by a Γ_0'-move; (c) the 5_2 knot by a Δ-move.

The following example shows that the estimation for an Δ-unknotting number by using an $SH(3)$-unknotting number is effective, which is also similar for the composite knot $3_1!\#6_1$; see Example 3.2.

Example 4.8. Let K be the alternating 12 crossing knot $12a177$ as shown in Fig. 14; see [2]. Then we have $\mathrm{u}_\Delta(K)=3$, which cannot be obtained from only the previous methods as in Proposition 4.2 and the inequalities: $|\sigma(K)|/2 \leq \mathrm{g}^*(K) \leq \mathrm{u}_\Delta(K)$, $e(K)/2 \leq \mathrm{u}_\Delta(K)$.

First, we have:

$$\nabla(K;z) = 1 - z^2 - 3z^4 + 2z^6; \tag{42}$$

$$\begin{aligned} V(K;t) = & t^{-9}(1 - 3t + 7t^2 - 14t^3 + 20t^4 - 25t^5 \\ & + 28t^6 - 25t^7 + 21t^8 - 15t^9 + 8t^{10} - 3t^{11} + t^{12}); \end{aligned} \tag{43}$$

$$\sigma(K) = 2. \tag{44}$$

By applying the Δ-move around the region containing the mark Δ, K is deformed into the knot 10_{67}, whose Δ-unknotting number is 2 [21], and so $\mathrm{u}_\Delta(K)\leq 3$. Since $|a_2(K)|=1$, by Proposition 4.2 we have $\mathrm{u}_\Delta(K)\equiv 1 \pmod 2$ and $\mathrm{u}_\Delta(K)\geq 1$. However, since $V(K;\omega) = -3$, by Theorem 3.1 $\mathrm{su}_3(K)>1$, and thus by Proposition 4.3 we obtain $\mathrm{u}_\Delta(K)=3$.

Furthermore, we have $\mathrm{u}(K) = \mathrm{u}_{\Gamma_0}(K) = 2$ and $\mathrm{g}^*(K)=1$. In fact, by changing the 2 crossings near the marks $*$, K is unknotted, and so $\mathrm{u}(K)\leq 2$. Then using the inequalities in Proposition 4.5, we obtain $\mathrm{su}_3(K) = \mathrm{u}_{\Gamma_0}(K) = \mathrm{u}(K) = 2$. Lastly, by performing a band surgery along the band as shown in Fig. 14, we obtain the composite link $6_1\#4_1\#H_-$. Then since 6_1 is a ribbon knot and $4_1\#H_-$ is band-trivializable by [10, Lemma 4.3], the 4-ball genus of this link is zero, and so $\mathrm{g}^*(K)\leq 1$. Since $\sigma(K)=2$, we obtain $\mathrm{g}^*(K)=1$.

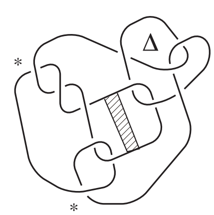


FIGURE 14. The knot $12a177$.

5. TABLES OF $SH(3)$- AND Γ_0-UNKNOTTING NUMBERS OF KNOTS WITH UP TO 9 CROSSINGS

Table 1 lists the 4-ball genus, g^*, $SH(3)$-unknotting number, su_3, Γ_0-unknotting number, u_{Γ_0}, unknotting number, u, Δ-unknotting number, u_Δ, and genus, g, of prime knots with up to 9 crossings. We have a complete list of the 4-ball genus, unknotting number and genus for prime knots with up to 9 crossings in [2]. For the Δ-unknotting number we have a table of prime knots with up to 8 crossings in [23], and one for up to 10 crossings in [21].

For some knots the $SH(3)$- and Γ_0-unknotting numbers are already given in Examples, which are indicated in the last column in Table 1, and for the remaining knots the marks (I), (II) indicate the methods for deciding these numbers. Then we can completely decide the $SH(3)$-unknotting number, but for 11 knots the Γ_0-unknotting number is undecided; in Table 1 1-2 means $\mathrm{u}_{\Gamma_0}(K)=1$ or 2, and 2-3 means $\mathrm{u}_{\Gamma_0}(K)=2$ or 3.

(I) The inequalities $g^* \le su_3 \le u_{\Gamma_0} \le u$, u_Δ and $su_3 \le g$ in Proposition 4.5 give the $SH(3)$- and Γ_0-unknotting numbers. Notice that for a nontrivial knot, $su_3 > 0$. Also, in some case we cannot obtain definite numbers; for example, for the knot 7_4 we have $su_3 = 1$ since $g = 1$, but $u_{\Gamma_0} = 1$ or 2, undecided, since $u = 2$ and $u_\Delta = 4$.

(II) Fig. 15 shows $u_{\Gamma_0}(K) = 1$, where each knot is transformed into the trivial knot by performing a Γ_0'-move at the 2 crossings near the marks $*$.

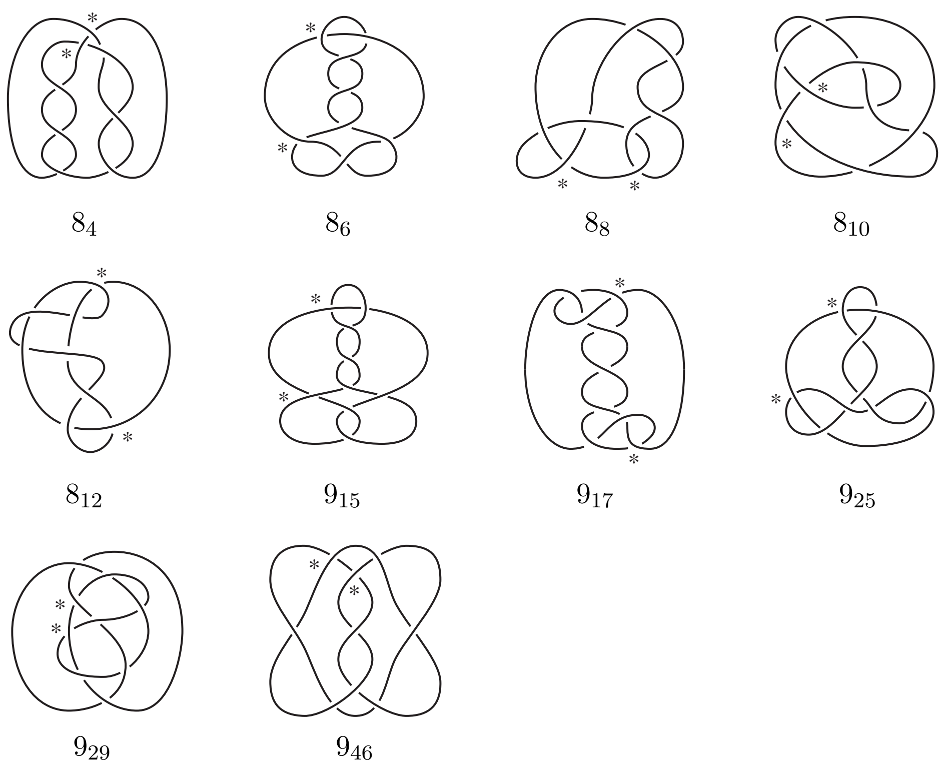


FIGURE 15. Knots with Γ_0-unknotting number one.

Remark 5.1. Recently, Yoshiaki Uchida has pointed out an error in the figure for giving $u_\Delta(9_{29})$ in [24, p. 59], from which the number $u_\Delta(9_{29})$ in [21] is taken. So, in Table 1 we list $u_\Delta(9_{29}) = 1$ or 3.

Table 2 lists the 4-ball genus, the $SH(3)$-unknotting number, the Γ_0-unknotting number, the unknotting number, the Δ-unknotting number, and the genus, together with the half of the absolute value of the signature, $|\sigma|/2$, and the absolute value of the coefficient of z^2 of the Conway polynomial, $|a_2|$, of composite knots with up to 9 crossings and $3_1\#3_1\#4_1$, $3_1!\#3_1\#4_1$ (Example 5.5). The genus, signature, and a_2 are definitely obtained. The unknotting numbers are taken from the table in [29]. An upper bound of the Δ-unknotting

TABLE 1. $SH(3)$- and Γ_0-unknotting numbers of prime knots with up to 9 crossings.

K	g^*	su_3	u_{Γ_0}	u	u_Δ	g	Method
3_1	1	1	1	1	1	1	(I)
4_1	1	1	1	1	1	1	(I)
5_1	2	2	2	2	3	2	(I)
5_2	1	1	1	1	2	1	(I)
6_1	0	1	1	1	2	1	(I)
6_2	1	1	1	1	1	2	(I)
6_3	1	1	1	1	1	2	(I)
7_1	3	3	3	3	6	3	(I)
7_2	1	1	1	1	3	1	(I)
7_3	2	2	2	2	5	2	(I)
7_4	1	1	1-2	2	4	1	(I)
7_5	2	2	2	2	4	2	(I)
7_6	1	1	1	1	1	2	(I)
7_7	1	1	1	1	1	2	(I)
8_1	1	1	1	1	3	1	(I)
8_2	2	2	2	2	2	3	(I)
8_3	1	1	1-2	2	4	1	(I)
8_4	1	1	1	2	3	2	(II)
8_5	2	2	2	2	3	3	(I)
8_6	1	1	1	2	2	2	(II)
8_7	1	1	1	1	2	3	(I)
8_8	0	1	1	2	2	2	(II)
8_9	0	1	1	1	2	3	(I)
8_{10}	1	1	1	2	3	3	(II)
8_{11}	1	1	1	1	1	2	(I)
8_{12}	1	1	1	2	3	2	(II)
8_{13}	1	1	1	1	1	2	(I)
8_{14}	1	1	1	1	2	2	(I)
8_{15}	2	2	2	2	4	2	(I)
8_{16}	1	1	1	2	1	3	(I)
8_{17}	1	1	1	1	1	3	(I)
8_{18}	1	1	1	2	1	3	(I)
8_{19}	3	3	3	3	5	3	(I)
8_{20}	0	1	1	1	2	2	(I)
8_{21}	1	1	1	1	2	2	(I)
9_1	4	4	4	4	10	4	(I)
9_2	1	1	1	1	4	1	(I)
9_3	3	3	3	3	9	3	(I)
9_4	2	2	2	2	7	2	(I)
9_5	1	1	1-2	2	6	1	(I)
9_6	3	3	3	3	7	3	(I)
9_7	2	2	2	2	5	2	(I)
9_8	1	1	1-2	2	2	2	Example 2.4
9_9	3	3	3	3	8	3	(I)
9_{10}	2	2	2-3	3	8	2	(I)
9_{11}	2	2	2	2	4	3	(I)
9_{12}	1	1	1	1	1	2	(I)
9_{13}	2	2	2-3	3	7	2	(I)
9_{14}	1	1	1	1	1	2	(I)
9_{15}	1	1	1	2	2	2	(II)
9_{16}	3	3	3	3	6	3	(I)
9_{17}	1	1	1	2	2	3	(II)
9_{18}	2	2	2	2	6	2	(I)
9_{19}	1	1	1	1	2	2	(I)
9_{20}	2	2	2	2	2	3	(I)
9_{21}	1	1	1	1	3	2	(I)
9_{22}	1	1	1	1	1	3	(I)
9_{23}	2	2	2	2	5	2	(I)
9_{24}	1	1	1	1	1	3	(I)
9_{25}	1	1	1	2	2	2	(II)
9_{26}	1	1	1	1	2	3	(I)
9_{27}	0	1	1	1	2	3	(I)
9_{28}	1	1	1	1	1	3	(I)
9_{29}	1	1	1	2	1/3	3	(II)
9_{30}	1	1	1	1	1	3	(I)
9_{31}	1	1	1-2	2	2	3	Example 2.4
9_{32}	1	1	1	2	1	3	(I)
9_{33}	1	1	1	1	1	3	(I)
9_{34}	1	1	1	1	1	3	(I)
9_{35}	1	1	2-3	3	7	1	Example 4.6
9_{36}	2	2	2	2	3	3	(I)
9_{37}	1	2	2	2	3	2	Example 3.2
9_{38}	2	2	2-3	3	6	2	(I)
9_{39}	1	1	1	1	2	2	(I)
9_{40}	1	1	1	2	1	3	(I)
9_{41}	0	1	1-2	2	2	2	Example 2.7
9_{42}	1	1	1	1	2	2	(I)
9_{43}	2	2	2	2	3	3	(I)
9_{44}	1	1	1	1	2	2	(I)
9_{45}	1	1	1	1	2	2	(I)
9_{46}	0	1	1	2	2	1	(II)
9_{47}	1	1	1	2	1	3	(I)
9_{48}	1	2	2	2	3	2	Example 3.2
9_{49}	2	2	2-3	3	6	2	(I)

number of a composite knot is given by

$$u_\Delta(J\#K) \leq u_\Delta(J) + u_\Delta(K), \tag{45}$$

and we also use Proposition 4.2. In order to find an upper bound of the Γ_0-unknotting number of a composite knot we use the formula:

$$\mathrm{u}_{\Gamma_0}(J\#K) \leq \mathrm{u}_{\Gamma_0}(J) + \mathrm{u}_{\Gamma_0}(K), \tag{46}$$

and also the following proposition, which is trivial, but useful.

Proposition 5.2. *Suppose that a knot J' is obtained from a knot J by changing a positive crossing to a negative one and a knot K' is obtained from a knot K by changing a negative crossing to a positive one. Then the composition $J'\#K'$ is obtained from $J\#K$ by performing a single Γ_0-move.*

In particular, suppose that J and K are unknotting number one knots such that J is unknotted by changing a positive crossing to a negative one and K is unknotted by changing a negative crossing to a positive one. Then $\mathrm{u}_{\Gamma_0}(J\#K) = 1$.

Example 5.3. Let $K = 4_1\#5_1$. Then the knot 5_1 is transformed into 3_1 by changing a negative crossing to a positive one, and 4_1 is unknotted by changing a positive crossing to a negative one, and so by the proposition above K is deformed into 3_1 by a Γ_0-move. Then we have $\mathrm{u}_{\Gamma_0}(K) \leq 2$. Since $\sigma(K)/2 = 2$, we obtain $\mathrm{g}^*(K) = \mathrm{su}_3(K) = \mathrm{u}_{\Gamma_0}(K) = 2$.

Example 5.4. Let $T_{p,q}$ be the composition of p copies of 3_1, the left-hand trefoil, and q copies of $3_1!$, the right-hand trefoil; $T_{p,q} = (\overset{p}{\#}3_1)\#(\overset{q}{\#}3_1!)$. We assume $0 \leq p \leq q$. Then $T_{p,q} = \overset{p}{\#}(3_1\#3_1!)\#(\overset{q-p}{\#}3_1!)$ and $3_1\#3_1!$ is a square knot, which is a ribbon knot. Then we have:

- Since $\sigma(3_1) = 2$, we have $\sigma(T_{p,q}) = -2(q-p)$.
- Since $\mathrm{g}^*(3_1) = 1$, we have $\mathrm{g}^*(T_{p,q}) = q - p$.
- Since the double branched covering space $\Sigma_2(3_1)$ is the lens space of type $(3,1)$, $H_1(\Sigma_2(T_{p,q}); \boldsymbol{Z}) \cong \boldsymbol{Z}_3 \oplus \cdots \oplus \boldsymbol{Z}_3$ ($p+q$ summands); thus, $e_2(T_{p,q}) = p + q$.
- By Proposition 5.2 we have $\mathrm{u}_{\Gamma_0}(3_1\#3_1!) = 1$, and so $\mathrm{u}_{\Gamma_0}(T_{p,q}) \leq q$.

Therefore, by Proposition 4.5 we have

$$\max\{(p+q)/2,\, q-p\} \leq \mathrm{su}_3(T_{p,q}) \leq q. \tag{47}$$

In particular, if $q = p$, $p+1$, $p+2$ or $p = 0$, then $\mathrm{su}_3(T_{p,q}) = q$. In fact, since $V(3_1;\omega) = -i\sqrt{3}$, $V(T_{p,p+2};\omega) = -3^{p+1}$, and so by Theorem 3.1, $\mathrm{su}_3(T_{p,p+2}) = p+2$; see [30, Examples 3.2(3) and 4.3].

Example 5.5. Let $K = 3_1\#3_1\#4_1$ and $K' = 3_1!\#3_1\#4_1$. Then since $P(3_1;i,i) = P(4_1;i,i) = -2$, we have $P(K;i,i) = P(K';i,i) = -8$, which implies $\mathrm{su}_3(K)$, $\mathrm{su}_3(K') > 1$ by Theorem 3.1, and $\mathrm{u}(K)$, $\mathrm{u}(K') \geq 3$ by [18, Theorem 1.1], which implies $\mathrm{u}(K) = \mathrm{u}(K') = 3$; see [29, Appendix 1]. Since $\mathrm{u}_{\Gamma_0}(3_1\#4_1) = 1$, we have $\mathrm{u}_{\Gamma_0}(K)$, $\mathrm{u}_{\Gamma_0}(K') \leq 2$, and so $\mathrm{su}_3(K) = \mathrm{u}_{\Gamma_0}(K) = \mathrm{su}_3(K') = \mathrm{u}_{\Gamma_0}(K') = 2$. Therefore, we obtain $\mathrm{u}_\Delta(K) = \mathrm{u}_\Delta(K') = 3$ by Eq. (37). Further, since $|\sigma(K)|/2 = 2$, we have $\mathrm{g}^*(K) = 2$, and since $3_1!\#3_1$ is a ribbon knot and $\mathrm{g}^*(4_1) = 1$, we have $\mathrm{g}^*(K') = 1$.

In Table 2 the marks (I'), (II') indicate the methods for deciding the numbers g^*, su_3, u_{Γ_0}, u_Δ as for Table 1.

(I') The inequalities $|\sigma|/2 \leq \mathrm{g}^* \leq \mathrm{su}_3 \leq \mathrm{u}_{\Gamma_0} \leq \mathrm{u}, \mathrm{u}_\Delta$ and $\mathrm{su}_3 \leq \mathrm{g}$ in Proposition 4.5 and Eq. (45) give g^*, su_3, u_{Γ_0}, and u_Δ.

(II') Propsition 5.2 gives $\mathrm{u}_{\Gamma_0}(K) = 1$. Note that the knots $3_1!\#3_1$ and $4_1\#4_1$ are ribbon knots with 1-fusion and the others are not slice because the signature is not zero or the determinant is not a square integer.

TABLE 2. $SH(3)$- and Γ_0-unknotting numbers of composite knots with up to 9 crossings and $3_1\#3_1\#4_1$, $3_1!\#3_1\#4_1$.

K	g^*	su_3	u_{Γ_0}	u	u_Δ	g	$\frac{\lvert\sigma\rvert}{2}$	$\lvert a_2\rvert$	Method
$3_1\#3_1$	2	2	2	2	2	2	2	2	(I')
$3_1!\#3_1$	0	1	1	2	2	2	0	2	(II')
$3_1\#4_1$	1	1	1	2	2	2	1	0	(II')
$3_1\#5_1$	3	3	3	3	4	4	3	4	(I')
$3_1!\#5_1$	1	1	1-2	2-3	4	3	1	4	Example 2.4
$3_1\#5_2$	2	2	2	2	3	2	2	3	(I')
$3_1!\#5_2$	1	1	1	2	3	2	0	3	(II')
$4_1\#4_1$	0	1	1	2	2	2	0	2	(II')
$3_1\#3_1\#3_1$	3	3	3	3	3	3	3	3	(I')
$3_1!\#3_1\#3_1$	1	2	2	3	3	3	1	3	Example 5.4
$3_1\#6_1$	1	1	1	2	1/3	2	1	1	(II')
$3_1!\#6_1$	1	2	2	2	3	2	1	1	Example 3.2
$3_1\#6_2$	2	2	2	2	2	3	2	0	(I')
$3_1!\#6_2$	1	1	1	2	2	3	0	0	(II')
$3_1\#6_3$	1	1	1	2	2	3	1	2	(II')
$4_1\#5_1$	2	2	2	3	2/4	3	2	2	Example 5.3
$4_1\#5_2$	1	1	1	2	1/3	2	1	1	(II')
$3_1\#3_1\#4_1$	2	2	2	3	3	3	2	1	Example 5.5
$3_1!\#3_1\#4_1$	1	2	2	3	3	3	0	1	Example 5.5

6. Final remark

The pass move [12] and the sharp move [19] are other local moves on oriented knots and links as shown in Fig. 16. Since the sharp move is an unknotting operation, we can define a sharp move. However, the pass move is not an unknotting operation; two knots K and K' are related by a sequence of pass moves if and only if $a_2(K) \equiv a_2(K') \pmod 2$. Then, we may define a pass-unknotting number for all knots with even second coefficient of the Conway polynomial (or knots with Arf invariant zero).

Since the pass move is realized by a sequence of two $H(2)$-moves [6] and the sharp move is realized by a sequence of two coherent band moves, we may give a lower bound for a pass-unknotting number and a sharp-unknotting number using an $H(2)$-unknotting number and an $SH(2)$-unknotting number, respectively, which provide a new estimation. Namely, denoting by $\mathrm{u}_\#(K)$, $\mathrm{u}_{\mathrm{pass}}(K)$, $\mathrm{u}_2(K)$ the sharp-, pass-, $H(2)$-unknotting numbers of a knot K, we have $\mathrm{u}_2(K) \leq 2\mathrm{u}_\#(K)$ and $\mathrm{su}_3(K) \leq \mathrm{u}_{\mathrm{pass}}(K)$.

(a) (b)

FIGURE 16. (a) Pass move. (b) Sharp move.

Example 6.1. Let us consider the two knots $K_1 = 10_{103}$ and $K_2 = 10_{74}$. Since $\mathrm{u}_2(K_1) = 3$ [11, p. 453] and $\mathrm{u}_\#(K_1) \equiv a_2(K_1) = 3 \pmod 2$ [19, Theorem 3.5], we obtain $\mathrm{u}_\#(K_1) \geq 3$. Conversely, since we may show $\mathrm{u}_\#(K_1) \leq 3$, we obtain $\mathrm{u}_\#(K_1) = 3$, which cannot be obtained by using the signature [19, Theorem 3.2]. Next, since $\nabla(K_2) = 1 - z^4$, we may consider the pass-unknotting number for K_2. Since $V(K_2, \omega) = -3$ and $\mathrm{g}(K_2) = 2$, we have $\mathrm{su}_3(K_2) = 2$. By the 4-move, we may transform K_2 into the knot 6_1, which is further transformed into the trivial knot by a 4-move, and so $\mathrm{u}_{\mathrm{pass}}(K_2) \leq 2$. Thus we have $\mathrm{u}_{\mathrm{pass}}(K_2) = 2$, which cannot be obtained by using the signature; in general $|\sigma(K)|/2 \leq \mathrm{u}_{\mathrm{pass}}(K)$ for a knot K and $|\sigma(K_2)| = 2$.

In a forthcoming paper we will make a detailed report on these moves.

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