ON EXTENDIBILITY OF A MAP INDUCED BY BERS ISOMORPHISM

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Abstract. Let S be a closed Riemann surface of genus $g(\geqq 2)$ and set $\dot{S} = S \setminus \{\hat{z}_0\}$. Then we have the composed map $\varphi \circ r$ of a map $r : T(S) \times U \to F(S)$ and the Bers isomorphism $\varphi : F(S) \to T(\dot{S})$, where $F(S)$ is the Bers fiber space of S, $T(X)$ is the Teichmüller space of X and U is the upper half-plane.

The purpose of this paper is to show the map $\varphi \circ r : T(S) \times U \to T(\dot{S})$. has a continuous extension to some subset of the boundary $T(S) \times \partial U$.

1. Introduction

1.1. **Teichmüller space.** Let S be a closed Riemann surface of genus $g(\geqq 2)$. Consider any pair (R, f) of a closed Riemann surface R of genus g and a quasiconformal map $f : S \to R$. Two pairs (R_1, f_1) and (R_2, f_2) are said to be *equivalent* if $f_2 \circ f_1^{-1} : R_1 \to R_2$ is homotopic to a biholomorphic map $h : R_1 \to R_2$. Let $[R, f]$ be the equivalence class of such a pair (R, f). We set

$$T(S) = \{[R, f] \mid f : S \to R : \text{quasiconformal}\}$$

and call $T(S)$ the *Teichmüller space* of S.

For any $p_1 = [R_1, f_1]$, $p_2 = [R_2, f_2] \in T(S)$, the *Teichmüller distance* is defined to be

$$d_T(p_1, p_2) = \frac{1}{2} \inf_g \log K(g)$$

where g runs over all qusiconformal maps from R_1 to R_2 homotopic to $f_2 \circ f_1^{-1}$ and $K(g)$ means the maximal dilatation of g. The Teichmüller space is topologized with the Teichmüller distance.

It is known that S can be represented as U/G where U is the upper half-plane and G is a torsion free Fuchsian group. Let $L_\infty(U, G)_1$ be the space of measurable functions μ on U satisfying

(1) $\|\mu\|_\infty = \sup_{z \in U} |\mu(z)| < 1$,

(2) $(\mu \circ g)\dfrac{\overline{g}'}{g}$ for all $g \in G$.

For any $\mu \in L_\infty(U, G)_1$, there is a unique quasiconformal map w of U onto U satisfying normalization conditions $w(0) = 0, w(1) = 1$ and $w(\infty) = \infty$. Let $Q(G)$ be the be the set of all normalized quasiconformal map w such that wGw^{-1} is also

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Fushsian. We write $w = w_\mu$. Two maps $w_1, w_2 \in Q(G)$ are said to be *equivalent* if $w_1 = w_2$ on the real axis $\mathbb{R}$. Let $[w]$ be the equivalence class of $w \in Q(G)$. We set

$$T(G) = \{[w] \mid w \in Q(G)\}$$

and call $T(G)$ the *Teichmüller space* of G.

Then we have a canonical bijection

$$T(G) \ni [w_\mu] \mapsto [U/G_\mu, f_\mu] \in T(S) \tag{1.1}$$

where $G_\mu = w_\mu G w_\mu^{-1}$ and f_μ is the map induced by $w_\mu : U \to U$. Throughout this paper, we always identify $T(G)$ with $T(S)$ via the bijection (1.1).

1.2. **Bers fiber space.** For any $\mu \in L_\infty(U, G)_1$, there is a unique quasiconformal map w^μ of $\hat{\mathbb{C}}$ with $w^\mu(0) = 0, w^\mu(1) = 1, w^\mu(\infty) = \infty$ such that w^μ satisfies the Beltrami equation $w_{\bar{z}} = \mu w_z$ on U, and is conformal on the lower half-plane L. The *Bers fiber space* $F(G)$ over $T(G)$ is defined by

$$F(G) = \{([w_\mu], z) \in T(G) \times \hat{\mathbb{C}} \mid [w_\mu] \in T(G),\ z \in w^\mu(U)\}.$$

Take a point $z_0 \in U$ and denote by A the set of all points $g(z_0)$, $g \in G$. Let

$$v : U \to U - A$$

be a holomorphic universal covering map. We define

$$\dot{G} = \{h \in \mathrm{Aut}\ U \mid v \circ h = g \circ v \text{ for some } g \in G\ \}.$$

We see that $U/\dot{G} = U/G - \{\pi(z_0)\}$, where $\pi : U \to S = U/G$ is the natural projection. Set $\dot{S} = U/\dot{G}$. By Lemma 6.3 of Bers [2], every point in $F(G)$ is represented as a point $([w_\mu], w^\mu(z_0))$ for some $\mu \in L_\infty(U, G)_1$. For $\mu \in L_\infty(U, G)_1$, we define $\nu \in L_\infty(U, \dot{G})_1$ by

$$\mu(v(z))\frac{\overline{v'(z)}}{v'(z)} = \nu(z).$$

Then, Bers' isomorphism theorem asserts that the map

$$\varphi : ([w_\mu], w^\mu(z_0)) \mapsto [w_\nu]$$

is a biholomorphic bijection map (cf. Theorem 9 of [2]). Moreover we define a map $r : T(G) \times U \to F(G)$ by

$$([w_\mu], z) \mapsto ([w_\mu], h_{[w_\mu]}(z)).$$

where U is the universal covering of S and $h_{[w_\mu]} : U \to w^\mu(U)$ is the Teichmüller mapping in the class of w^μ. We remark that our definition of r is different from Bers' one. See the proof of Lemma 6.4 of [2]. This map r is not real analytic, but it is a homeomorphism. This difference does not influence our purpose.

Via the bijection (1.1), the Bers fiber space $F(S)$ over $T(S)$ is defined by

$$F(S) = \{([R_\mu, f_\mu], z) \in T(S) \times \hat{\mathbb{C}} \mid [R_\mu, f_\mu] \in T(S),\ z \in w^\mu(U)\}$$

with the projection

$$F(S) \ni ([R_\mu, f_\mu], z) \mapsto [R_\mu, f_\mu] \in T(S).$$

Similarly, we have the isomorphism $F(S) \to T(\dot{S})$ and the homeomorphism $T(S) \times U \to F(S)$, and we denote them by the same symbols φ and r, respectively.

1.3. **The Bers embedding.** The Teichmüller space $T(S)$ can be regarded canonically as a bounded domain of a complex Banach space $B_2(L,G)$ in the following way: Let $B_2(L,G)$ consist of all holomorphic functions ϕ defined on L such that

$$\phi(g(z))g'(z)^2 = \phi(z) \text{ for } g \in G \text{ and } z \in L$$

and

$$\|\phi\|_\infty = \sup_{z\in L} |(\mathrm{Im} z)^2\phi(z)| < \infty.$$

For any $\mu \in L_\infty(U,G)_1$, we denote by ϕ^μ the Schwarzian derivative of w^μ on L, that is,

$$\phi^\mu(z) = \{w^\mu, z\} = \frac{(w^\mu)'''(z)}{(w^\mu)'(z)} - \frac{3}{2}\left(\frac{(w^\mu)''(z)}{(w^\mu)'(z)}\right)^2 \text{ for } z \in L.$$

If $\mu \in L_\infty(U,G)_1$, then $\phi^\mu \in B_2(L,G)$ and the *Bers embedding* $T(S) \ni [R_\mu, f_\mu] \mapsto \phi^\mu \in B_2(L,G)$ is a biholomorphic bijection of $T(S)$ onto a holomorphically bounded domain in $B_2(L,G)$. From now on, we will identify $T(S)$ with its image in $B_2(L,G)$.

Similarly, we define the Bers embedding of $T(\dot{S})$ into $B_2(L,\dot{G})$. Since $F(S)$ is a domain of $B_2(L,G) \times \hat{\mathbb{C}}$ and $T(\dot{S})$ is a bounded domain in $B_2(L,\dot{G})$, we define the topological boundaries of them naturally. Let $\overline{F(G)}$ denote the closure of $F(G)$ in $B_2(L,G) \times \hat{\mathbb{C}}$.

1.4. **Main theorem.** Zhang [17] proved the Bers isomorphism φ cannot be continuously extended to $\overline{F(S)}$ if the dimension of $T(S)$ is greater than zero. Then we have the following question: Is there some subset of $\overline{F(S)} - F(S)$ to which φ can be continuously extended ?

To do this, we compose the isomorphism $\varphi : F(S) \to T(\dot{S})$ and the map $r : T(S) \times U \to F(S)$, then we obtain new map $\varphi \circ r : T(S) \times U \to T(\dot{S})$. Let $\mathbb{A}$ be a subset of ∂U consisting of all points filling S (cf. §3.3). Our main theorem is as follows.

Theorem 4.1 *The map $\varphi \circ r : T(S) \times U \to T(\dot{S})$ has a continuous extension to $T(S) \times \mathbb{A}$.*

The idea of proof of Theorem 4.1 is as follows. For any sequence $\{(p_m, z_m)\}_{m=1}^\infty$ in $T(S)\times U$ converging to $(p_\infty, z_\infty) \in T(S)\times\mathbb{A}$, we put $q_m = \varphi\circ r(p_m, z_m) \in T(\dot{S})$. We need to prove that the sequence $\{q_m\}_{m=1}^\infty$ converges without depending on the choice of a convergent sequence to $(p_\infty, z_\infty) \in T(S) \times \mathbb{A}$.

Let q_0 be the basepoint of $T(\dot{S})$. It is known that the image of the Bers embedding is canonically identified with the slice $T(\dot{S}) \times \{\overline{q}_0\}$ in the quasifuchsian space which is biholomorphic to $T(\dot{S}) \times T(\overline{\dot{S}})$ (cf. Chapter 8 of Bers [3]). For each pair $(q_m, \overline{q}_0) \in T(\dot{S}) \times T(\overline{\dot{S}})$, there is a unique quasifuchsian group Γ_m up to conjugation such that the conformal boundaries of a hyperbolic manifold $N_m = \mathbb{H}^3/\Gamma_m$ correspond to the pair $(q_m, \overline{q}_0)$.

We assume throughout the paper that quasifuchsian groups Γ_m and manifolds N_m are marked by a homomorphism and homotopy equivalence, respectively.

For our purpose, it is sufficient to show that a limit Γ_∞ of the sequence $\{\Gamma_m\}_{m=1}^\infty$ is uniquely determined. To do this, we show the following key lemma.

Lemma 4.1 *Given $z_\infty \in \mathbb{A}$, there exists a filling lamination λ with the following property. For any sequence $\{z_m\}_{m=1}^\infty$ with $\lim_{m\to\infty} z_m = z_\infty$ and $q_m = \varphi\circ r(p_m, z_m)$*

as above, there exists a sequence of simple closed curves $\{\alpha_m\}_{m=1}^{\infty}$ with the following properties:

(1) *The lengths $\ell_{N_m}(\alpha_m)$ of α_m in N_m are bounded, and*
(2) *the sequence $\{\alpha_m\}_{m=1}^{\infty}$ converges to λ in $\overline{\mathcal{C}}(\dot{S})$.*

Here the definition of $\overline{\mathcal{C}}(\dot{S})$ will be given in §2 and §3. We remark that λ is identified with an ending lamination by Klarreich's work in [9].

From this lemma, we see that the limit Γ_∞ of $\{\Gamma_m\}_{m=1}^{\infty}$ is singly degenerate Kleinian group, that is, the the region of discontinuity of Γ_∞ is simply connected. Then by using Ending lamination theorem for surface groups of [6], Γ_∞ is uniquely determined by $(\lambda, \overline{q}_0)$ up to conjugation, and it is the only possible limit.

2. Gromov-hyperbolic spaces

In this section, we shall give the boundary at infinity of hyperbolic space. For details, see Klarreich [9].

Let (Δ, d) be a metric space. If Δ is equipped with a basepoint 0, we define the *Gromov product* $\langle x|y\rangle$ of points x and y in Δ by

$$\langle x|y\rangle = \langle x|y\rangle_0 = \frac{1}{2}\{d(x,0) + d(y,0) - d(x,y)\}.$$

For $\delta \geqq 0$, the metric space Δ is said to be *δ-hyperbolic* if

$$\langle x|y\rangle \geqq \min\{\langle x|z\rangle, \langle y|z\rangle\} - \delta$$

holds for every $x, y, z \in \Delta$. We say that Δ is *hyperbolic in the sense of Gromov* if Δ is δ-hyperbolic for some $\delta \geqq 0$.

If Δ is a hyperbolic space, we can define a boundary of Δ in the following way: We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of points in Δ *converges at infinity* if it satisfies $\lim_{m,n\to\infty}\langle x_m|x_n\rangle = \infty$. Given two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ that converge at infinity, they are called to be *equivalent* if $\lim_{m,n\to\infty}\langle x_m|y_n\rangle = \infty$. Since Δ is a hyperbolic, we see that this is an equivalence relation ($\sim$). We set

$$\partial_\infty\Delta = \{\{x_n\}_{n=1}^{\infty} \mid \{x_n\}_{n=1}^{\infty}\text{converges at infinity}\}/\sim$$

and call $\partial_\infty\Delta$ the *boundary at infinity* of Δ. If $\xi \in \partial_\infty\Delta$, then we say that a sequence of points in Δ *converges to ξ* if the sequence belongs to the equivalence class ξ. We put

$$\overline{\Delta} = \Delta \cup \partial_\infty\Delta.$$

3. Leininger, Mj and Schleimer's work

3.1. **The Curve Complex.** Let $S = U/G$ be a closed Riemann surface of genus $g(\geqq 2)$ and $\pi : U \to S$ be the natural projection. We take a point z_0 in U and set $\hat{z}_0 = \pi(z_0)$. Put $\dot{S} = S \setminus \{\hat{z}_0\}$.

The curve complex $\mathcal{C}(S)$ is a simplicial complex which is defined as follows. The vertices of $\mathcal{C}(S)$ are homotopy classes of nontrivial simple closed curves on S. Two curves are connected by an edge if they can be realized disjointly on S, and in general a collection of curves spans a simplex if the curves can be realized disjointly on S. We define $\mathcal{C}(\dot{S})$ similarly, with vertices consisting of nontrivial, non-peripheral simple closed curves on $\dot{S}$.

We give $\mathcal{C}(S)$(resp $\mathcal{C}(\dot{S})$) a metric structure by making every simplex a regular Euclidean simplex whose edges have length 1, and define the distance $d_{\mathcal{C}(S)}$(resp $d_{\mathcal{C}(\dot{S})}$) by taking shortest paths.

Theorem 3.1 (Masur and Minsky [12], Theorem 1.1)**.** *The spaces $\mathcal{C}(S)$ and $\mathcal{C}(\dot{S})$ are δ-hyperbolic for some $\delta > 0$.*

We put $\overline{\mathcal{C}}(S) = \mathcal{C}(S) \cup \partial_\infty \mathcal{C}(S)$ and $\overline{\mathcal{C}}(\dot{S}) = \mathcal{C}(\dot{S}) \cup \partial_\infty \mathcal{C}(\dot{S})$, respectively.

3.2. **Definition of Φ.** Denote by $\mathrm{Diff}^+(S)$ the group of all orientation preserving diffeomorphisms of S onto itself. Let $\mathrm{Diff}_0(S)$ be a group which consists of all elements in $\mathrm{Diff}^+(S)$ isotopic to the identity map id.

We define the evaluation map

$$\mathrm{ev} : \mathrm{Diff}^+(S) \to S$$

by $\mathrm{ev}(f) = f(\hat{z}_0)$. A theorem of Earle and Eells asserts that $\mathrm{Diff}_0(S)$ is contractible. Hence, for the map $\mathrm{ev}|\mathrm{Diff}_0(S)$, there is a unique lift

$$\tilde{\mathrm{ev}} : \mathrm{Diff}_0(S) \to U$$

satisfying the condition that $\tilde{\mathrm{ev}}(id) = z_0$.

Following Leininger, Mj and Schleimer [10], we will define a map $\tilde{\Phi} : \mathcal{C}(S) \times \mathrm{Diff}_0(S) \to \mathcal{C}(\dot{S})$. To give an idea of the definition of $\tilde{\Phi}$, we consider the case of $\mathcal{C}^0(S) \times \mathrm{Diff}_0(S)$ where $\mathcal{C}^0(S)$ is 0-skeleton of $\mathcal{C}(S)$. Take a point $(v, f) \in \mathcal{C}^0(S) \times \mathrm{Diff}_0(S)$. From now on, if no confusion is possible, we identify the homotopy class v with the geodesic representative. Then there is an isotopy $f_t,\ t \in [0, 1]$, between $f_0 = id$ and $f_1 = f$. Setting $C(t) = f_t(\hat{z}_0)$ for every $t \in [0, 1]$, we have a path C from $\hat{z}_0$ to $f(\hat{z}_0)$ on S. Move a point in S from $f(\hat{z}_0)$ to $\hat{z}_0$ along C and drag v back along the moving point. Then we obtain new simple closed curve on $\dot{S}$ and denote the curve by $f^{-1}(v)$. Thus we define $\tilde{\Phi}(v, f) = f^{-1}(v)$.

However, when $f(\hat{z}_0) \in v$, we can not define $\tilde{\Phi}(v, f)$ as above. We solve this problem in the following way: Now choose $\{\epsilon(v)\}_{v \in \mathcal{C}^0(S)} \subset \mathbb{R}_{>0}$ so that the $\epsilon(v)$-neighborhood $N(v) = N_{\epsilon(v)}$ of v has the following properties:

(i) $N(v)$ is homeomorphic to $S^1 \times [0, 1]$
(ii) $N(v_1) \cap N(v_2) = \emptyset$ if $v_1 \cap v_2 = \emptyset$.

Let $N^\circ(v)$ be the interior of $N(v)$ and $v^\pm$ the boundary components of $N(v)$. Notice that $\epsilon(v)$ is depending only on the length of the geodesic representative of v (cf. [7]).

If $v \subset \mathcal{C}(S)$ is a simplex with vertices $\{v_0, v_1, \cdots, v_k\}$, then we consider the barycentric coordinates for points in v:

$$\{\sum_{j=0}^{k} s_j v_j \mid \sum_{j=0}^{k} s_j = 1 \text{ and } s_j \geq 0, \text{ for } j = 0, 1, \cdots, k\}$$

For a point (v, f) with v a vertex of $\mathcal{C}(S)$, we can define $\tilde{\Phi}$ as follows. If $f(\hat{z}_0) \notin N^\circ(v)$, then we define

$$\tilde{\Phi}(v, f) = f^{-1}(v)$$

as above.

If $f(\hat{z}_0) \in N^\circ(v)$, then $f^{-1}(v^+)$ and $f^{-1}(v^-)$ are not isotopic in $\dot{S}$. We set

$$t = \frac{d(v^+, f(\hat{z}_0))}{2\epsilon(v)},$$

where $d(v^+, f(\hat{z}_0))$ is the distance inside $N(v)$ from $f(\hat{z}_0)$ to v^+. Then we define

$$\tilde{\Phi}(v,f) = tf^{-1}(v^+) + (1-t)f^{-1}(v^-)$$

in barycentric coordinates on the edge $[f^{-1}(v^+), f^{-1}(v^-)]$.

In general, for a point $(x,f) \in \mathcal{C}(S) \times \mathrm{Diff}_0(S)$ with $x = \sum_{j=0}^k s_j v_j$, we define $\tilde{\Phi}(x,f)$ as follows. If $f(\hat{z}_0) \notin \bigcup_{j=0}^k N^\circ(v_j)$, then we define

$$\tilde{\Phi}(x,f) = \sum_j s_j f^{-1}(v_j).$$

If $f(\hat{z}_0) \in N^\circ(v_i)$ for exactly one i, we set

$$t = \frac{d(v^+, f(\hat{z}_0))}{2\epsilon(v_i)},$$

and define

$$\tilde{\Phi}(x,f) = s_i(tf^{-1}(v_i^+) + (1-t)f^{-1}(v_i^-)) + \sum_{j\neq i} s_j f^{-1}(v_j). \tag{3.1}$$

Finally, by Proposition 2.2 in [10], if $\tilde{\mathrm{ev}}(f_1) = \tilde{\mathrm{ev}}(f_2)$ in U, then we see that $\tilde{\Phi}(x,f_1) = \tilde{\Phi}(x,f_2)$. From this, we have a map $\Phi : \mathcal{C}(S) \times U \to \mathcal{C}(\dot{S})$ satisfying $\tilde{\Phi} = \Phi \circ (id \times \tilde{\mathrm{ev}})$.

3.3. **Extendibility of Φ.** A subsurface of S is said to be an *essential* if it is either a component of the complement of a geodesic multicurve in S, the annular neighborhood $N(v)$ of some geodesic $v \in \mathcal{C}^0(S)$, or else S.

Given an essential subsurface Y, if a point $x \in \partial U$ has the following properties,

(i) for every geodesic ray $r \subset U$ ending at x and for every $v \in \mathcal{C}^0(S)$ which nontrivially intersects an essential subsurface Y, we have $\pi(r) \cap v \neq \emptyset$ and
(ii) there is a geodesic ray $r \subset U$ ending at x such that $\pi(r) \subset Y$,

we call such a point x a *filling point* for Y (or simply, x *fills* Y). We set

$$\mathbb{A} = \{x \in \partial U \mid x \text{ fills } S\}.$$

We have the following result.

Theorem 3.2 ([10], Theorem 1.1 and 3.6)**.** *For any $v \in \mathcal{C}(S)$, the map*

$$\Phi(v,\cdot) : U \to \mathcal{C}(\dot{S})$$

can be continuously extended to

$$\overline{\Phi}(v,\cdot) : U \cup \mathbb{A} \to \overline{\mathcal{C}}(\dot{S}).$$

Moreover for every $z_\infty \in \mathbb{A}$, $\overline{\Phi}(v, z_\infty)$ does not depend on v.

4. Main Theorem

Let γ be a nontrivial simple closed curve on a Riemann surface R. Denote by $\mathrm{Mod}(A)$ the modulus of an annulus in R whose core curve is homotopic in R to γ. We define the extremal length $\mathrm{Ext}(\gamma)$ of γ on R by

$$\mathrm{Ext}_R(\gamma) = \inf_A 1/\mathrm{Mod}(A),$$

where the infimum is over all annuli $A \subset R$ whose core curve is homotopic in R to γ (cf. Chapter 4 of Ahlfors [1]).

Given any point $p = [R, f] \in T(S)$ and a nontrivial simple closed curve α on S, we define the extremal length $\mathrm{Ext}_p(\alpha)$ by

$$\mathrm{Ext}_p(\alpha) = \mathrm{Ext}_R(f(\alpha)).$$

Theorem 4.1. *The map $\varphi \circ r : T(S) \times U \to T(\dot{S})$ has a continuous extension to $T(S) \times \mathbb{A}$.*

Proof. Let $\{(p_m, z_m)\}_{m=1}^{\infty}$ be any sequence in $T(S) \times U$ converging to $(p_\infty, z_\infty) \in T(S) \times \mathbb{A}$. Put $q_m = \varphi \circ r(p_m, z_m)$. We regard $\{q_m\}_{m=1}^{\infty}$ as the sequence $\{(q_m, \overline{q}_0)\}_{m=1}^{\infty}$ in a Bers slice of $T(\dot{S}) \times T(\overline{\dot{S}})$ where q_0 is the base point $(\dot{S}, id)$ of $T(\dot{S})$.

For each pair $(q_m, \overline{q}_0) \in T(\dot{S}) \times \{\overline{q}_0\}$, there is a unique quasifuchsian group Γ_m up to conjugation such that it uniformizes $(q_m, \overline{q}_0)$. For each Γ_m, the quotient space $N_m = \mathbb{H}^3/\Gamma_m$ is a hyperbolic manifold, where $\mathbb{H}^3$ is upper half space.

To prove that $\{q_m\}_{m=1}^{\infty}$ converges, we need the following lemma.

Lemma 4.1. *Given $z_\infty \in \mathbb{A}$, there exists a filling lamination λ with the following property. For any sequence $\{z_m\}_{m=1}^{\infty}$ with $\lim_{m\to\infty} z_m = z_\infty$ and $q_m = \varphi \circ r(p_m, z_m)$ as above, there exists a sequence of simple closed curves $\{\alpha_m\}_{m=1}^{\infty}$ with the following properties:*

(1) *The lengths $\ell_{N_m}(\alpha_m)$ of α_m in N_m are bounded, and*
(2) *the sequence $\{\alpha_m\}_{m=1}^{\infty}$ converges to λ in $\overline{\mathcal{C}}(\dot{S})$.*

Proof of Lemma 4.1. First we pick any simple closed curve α on S and fix it. By Theorem 3.2, $\Phi(\alpha, z_m) \to \lambda$ as $m \to \infty$ in $\overline{\mathcal{C}}(\dot{S})$ and λ does not depend on α.

Next we produce a sequence of curves which satisfies (1) and (2) as follows. Let S_m be the underlying Riemann surface for p_m and $\hat{h}_m$ the Teichmüller map from S onto S_m. Then $p_m = (S_m, \hat{h}_m)$. Take $\{f_m\}_{m=1}^{\infty} \subset \mathrm{Diff}_0(S)$ with $\tilde{\mathrm{ev}}(f_m) = z_m$. Then the point $[S_m - \{\hat{h}_m(\hat{z}_m)\}, \hat{h}_m \circ f_m]$ represents q_m in $T(\dot{S})$ where $\hat{z}_m$ is the image in S of z_m via the projection $U \to S$. We choose α_m to be $\tilde{\Phi}(\alpha, f_m)$ if $\hat{z}_m$ is not contained in $N^\circ(\alpha)$, and otherwise let α_m be a vertex of $\tilde{\Phi}(\alpha, f_m)$ with weight at least $1/2$ in barycentric coordinates on the edge of $\tilde{\Phi}(\alpha, f_m)$ (cf. (3.1)).

We show that the sequence $\{\alpha_m\}_{m=1}^{\infty}$ satisfies (1) and (2). By Theorem 3.2, $\tilde{\Phi}(\alpha, f_m) = \Phi(\alpha, z_m) \to \lambda$ as $m \to \infty$ in $\overline{\mathcal{C}}(\dot{S})$, which implies (2).

To see (1), first we set

$$E_0 = 1/\mathrm{Mod}(N(\alpha)).$$

Suppose that $\hat{z}_m = f_m(\hat{z}_0) \notin N^\circ(\alpha)$. Then the interior of the annulus $N(\alpha)$ is embedded in $S - \{\hat{z}_m\}$. Let p_0 be the besepoint of $T(S)$. Since $\{d_T(p_m, p_\infty)\}_{m=1}^{\infty}$ is a bounded sequence, by using the triangle inequality we see that $\{d_T(p_m, p_0)\}_{m=1}^{\infty}$ is also a bounded sequence. Hence we may assume that $K(\hat{h}_m) < K$ for every m with a sufficiently large $K(>1)$. Since every $\hat{h}_m$ satisfies

$$\mathrm{Mod}(\hat{h}_m(N(\alpha))) \geqq 1/(KE_0),$$

we obtain

$$\mathrm{Ext}_{q_m}(\alpha_m) \leqq KE_0. \tag{4.1}$$

Suppose $\hat{z}_m \in N^\circ(\alpha)$. Let α^* be the core geodesic of $N(\alpha)$ and denote by $\alpha^\pm$ the components of $\partial N(\alpha)$. Take a conformal (not isometric) coordinates

$$g_m : \alpha^* \times [-\epsilon(\alpha), \epsilon(\alpha)] \to N(\alpha)$$

such that $\alpha^* \times \{0\}$ maps to the core geodesic of $N(\alpha)$ and for each t, $\alpha^* \times \{t\}$ is sent to the equidistant circle to the core geodesic. Let $t_m \in [-\epsilon(\alpha), \epsilon(\alpha)]$ such that $\hat{z}_m \in g_m(\alpha^* \times \{t_m\})$. We suppose $t_m > 0$. The case $t_m \leqq 0$ can be dealt with the same manner.

Let A_m be the component of $N(\alpha) \setminus g_m(\alpha^* \times \{t_m\})$ which is containing α^*. Since g_m is conformal,

$$\mathrm{Mod}(A_m) \geqq \mathrm{Mod}(N(\alpha))/2.$$

Thus

$$\mathrm{Mod}(\hat{h}_m(A_m)) \geqq 1/(2KE_0).$$

By the definition of α_m, we have

$$\mathrm{Ext}_{q_m}(\alpha_m) = \mathrm{Ext}_{q_m}(f_m^{-1}(\alpha^-)) \leqq 2KE_0. \tag{4.2}$$

From (4.1) and (4.2), we conclude that $\mathrm{Ext}_{q_m}(\alpha_m)$ are bounded above. By Maskit's comparizon theorem of [11], we see that $\ell_{q_m}(\alpha_m)$ are bounded above. Here for any point $q = [\dot{R}, \dot{f}] \in T(\dot{S})$ and a nontrivial simple closed curve γ on $\dot{S}$ the symbol $\ell_q(\gamma)$ means the length of the geodesic representative of the homotopy class of $\dot{f}(\gamma)$ in the hyperbolic metric on $\dot{R}$. Therefore by Bers inequality, we have

$$\ell_{N_m}(\alpha_m) \leqq 2\min\{\ell_{q_m}(\alpha_m), \ell_{q_0}(\alpha_m)\},$$

and hence $\ell_{N_m}(\alpha_m)$ are uniformly bounded, which implies (1). ■

We now return to the proof of Theorem 4.1. Consider the normalized sequence $\{\alpha_m/\ell_{q_0}(\alpha_m)\}_{m=1}^{\infty}$. This sequence has a convergent subsequence (represented by the same indices) to a measured lamination ν, which by Theorem 1.4 of [9] has the same support as λ from Lemma 4.1 (2).

For a hyperbolic manifold N with marked homotopy equivalence $\dot{S} \to N$, and a measured lamination ξ on $\dot{S}$, we denote by $\underline{\ell}_N(\xi)$ the extended length of ξ in N (see Brock [5]). Any quasifuchsian group uniformizing $(q_m, \overline{q}_0)$ admits a natural marked homotopy equivalence inherited from that of q_m. By Brock's continuity theorem we get

$$\underline{\ell}_{N_m}\left(\frac{\alpha_m}{\ell_{q_0}(\alpha_m)}\right) \to \underline{\ell}_{N\infty}(\nu) \text{ as } m \to \infty$$

where $N_\infty = \mathbb{H}^3/\Gamma_\infty$ is a marked hyperbolic manifold and Γ_∞ is an algebraic limit of the subsequence $\{\Gamma_m\}_{m=1}^{\infty}$. (cf. Theorem 2 of [5]. See also Lemma 3.1 of Ohshika [16]). On the other hand, from (2) of Lemma 4.1, because α_m tends to infinity in $\mathcal{C}(\dot{S})$, in the fixed metric q_0, we must have $\ell_{q_0}(\alpha_m) \to \infty$ as $m \to \infty$. Therefore, from (1) in Lemma 4.1, we have

$$\begin{aligned} \underline{\ell}_{N_m}\left(\frac{\alpha_m}{\ell_{q_0}(\alpha_m)}\right) &= \frac{1}{\ell_{q_0}(\alpha_m)}\underline{\ell}_{N_m}(\alpha_m) \\ &\to 0 \ (m \to \infty), \end{aligned}$$

and thus the length of ν in N_∞ is zero. Since the support of ν contains λ as its support, the length of λ in N_∞ is also zero. Hence λ is not realizable in N_∞. Since λ is filling, it follows Γ_∞ is a singly degenerate Kleinian group. By using Ending lamination theorem for surface groups of [6], Γ_∞ is uniquely determined by $(\lambda, \overline{q}_0)$ up to conjugation. By Theorem 3.2, λ depends only on z_∞. Thus the sequence $\{q_m\}_{m=1}^{\infty}$ converges without depending on the choice of a convergent sequence to

$(p_\infty, z_\infty) \in T(S) \times \mathbb{A}$. Hence we conclude that the map $\varphi \circ r : T(S) \times U \to T(\dot{S})$ has a continuous extension to $T(S) \times \mathbb{A}$. ∎

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