COUNTING GENERALIZED DYCK PATHS

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Abstract. The Catalan number has a lot of interpretations and one of them is the number of Dyck paths. A Dyck path is a lattice path from $(0, 0)$ to (n, n) which is below the diagonal line $y = x$. One way to generalize the definition of Dyck path is to change the end point of Dyck path, i.e. we define (generalized) Dyck path to be a lattice path from $(0, 0)$ to $(m, n) \in \mathbb{N}^2$ which is below the diagonal line $y = \frac{n}{m}x$, and denote by $C(m, n)$ the number of Dyck paths from $(0, 0)$ to (m, n). In this paper, we give a formula to calculate $C(m, n)$ for arbitrary m and n.

1. Introduction

The Catalan number $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ is one of the most fascinating numbers, and it is known that the Catalan number has more than 200 interpretations. (See [2].) For example, the number of ways to dissect a convex $(n+2)$-gon into triangles, that of binary trees with $(n+1)$ leaves, and that of standard tableaux on the young diagram (n, n) are C_n. Moreover, one of the most famous interpretations of the Catalan number is the number of Dyck paths from $(0, 0)$ to (n, n). A sequence of lattice points in $\mathbb{Z}^2$

$$P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_k, y_k)\}$$

is a lattice path if and only if P satisfies the followings for any $i = 1, 2, \cdots, k$:

$$(x_i, y_i) = (x_{i-1}, y_{i-1} + 1) \text{ or } (x_{i-1} + 1, y_{i-1}).$$

If a lattice path $P = \{(0, 0), (x_1, y_1), \cdots, (n, n)\}$ lies in the domain $y \leq x$, P is called a Dyck path. There are $\binom{2n}{n}$ lattice paths from $(0, 0)$ to (n, n), and C_n of them are Dyck paths.

It is known that the Catalan numbers satisfy the recurrence relation that

$$C_0 = 1 \quad \text{and} \quad C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}. \tag{1.1}$$

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This recurrence relation also has many interpretations. Hereafter, if a lattice path P from $(0,0)$ to $(m,n) \in \mathbb{N}^2$ lies in the domain $y \leq \frac{n}{m}x$, we call P a Dyck path, and we denote the number of Dyck paths from $(0,0)$ to (m,n) by $C(m,n)$. We have a natural question: how many Dyck paths from $(0,0)$ to (m,n) are there for any positive integers m and n? The followings are answers to this question for special values of m and n. N. Fuss ([1]) found

$$C(kn,n) = \frac{1}{kn+1}\binom{(k+1)n}{n}. \tag{1.2}$$

$C(kn,n)$ also has the following recurrence relation:

$$C(kn,n) = \sum_{(n_1,n_2,\cdots,n_{k+1})} \prod_{i=1}^{k+1} C(kn_i,n_i), \tag{1.3}$$

where the sum is taken over all sequences of non-negative integers $(n_1,n_2,\cdots,n_{k+1})$ such that $\sum_{i=1}^{k+1} n_i = n-1$. $C(kn,n)$ also appears in various counting problems, like the Catalan number. For instance, the number of ways to dissect a convex $(kn+2)$-gon into $(k+2)$-gons is $C(kn,n)$. Actually, N. Fuss gave the formula (1.2) of $C(kn,n)$ by counting the number of ways to dissect a convex $(kn+2)$-gon into $(k+2)$-gons in [1]. P. Duchon [4] also gave a formula counting the number of Dyck path from $(0,0)$ to $(2\ell,3\ell)$, namely

$$C(2\ell,3\ell) = \sum_{i=0}^{5} \frac{1}{5\ell+i+1}\binom{5\ell+1}{n-i}\binom{5\ell+2i}{i}. \tag{1.4}$$

In this paper, we count $C(m,n)$ for any positive integers m and n. Let $A_{(m,n)} = \frac{1}{m+n}\binom{n+m}{n}$. For any m and n, $C(m,n)$ is given by the following theorem.

Theorem 1.1. [1] *Let* $d = \gcd(m,n)$*, then*

$$C(m,n) = \sum_{a} \prod_{i=1}^{d} \left(\frac{1}{a_i!}A^{a_i}_{(\frac{i}{d}m,\frac{i}{d}n)}\right), \tag{1.5}$$

where the sum $\sum_a$ *is taken over all sequences of non-negative integers* $a = (a_1,a_2,\cdots)$ *such that* $\sum_{i=1}^{\infty} ia_i = d$*. When* $\gcd(m,n) = 1$*,* (1.5) *reduces to the following:*

$$C(m,n) = \frac{1}{m+n}\binom{m+n}{n}. \tag{1.6}$$

[1]When the author almost finished writing this paper, she found a paper [5] which proves Theorem 1.1. But our proof is different from that in [5].

In fact, we prove (1.6) first and then (1.5). Actually, a sequence of non-negative integers $a = (a_1, a_2, \cdots)$ in (1.5) characterizes the "form" of a Dyck path, and it is interesting that $C(m, n)$ is given by using these sequences. We will see this in the last section. The description of $C(m, n)$ in Theorem 1.1 is completely different from that of $C(kn, n)$ in (1.2) and that of $C(2\ell, 3\ell)$ in (1.4), and we could not deduce (1.2) and (1.4) from (1.5) by direct computation. (However we will see that (1.2) is a corollary of Theorem 1.5.)

The paper is organized as follows. In Section 2 we will treat the case $\gcd(m, n) = 1$ and prove (1.6). In Section 3 we state a recurrence relation generalizing (1.1) and see that Theorem 1.1 follows from the recurrence relation. The recurrence relation follows from three lemmas and we prove them in Section 4.

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2. The description of $C(m, n)$ when $\gcd(m, n) = 1$.

We begin with some notations about a lattice path. We can regard any lattice path P from $(0, 0)$ to (m, n) as a sequence of m x's and n y's. For example, the lattice path from $(0, 0)$ to $(5, 3)$ in Figure 1 is $xyxxyxyx$. Hereafter

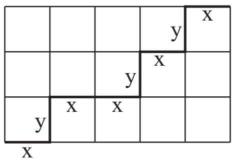

Figure 1

we will treat a lattice path from $(0, 0)$ to (m, n) as a sequence of m x's and n y's. The number of lattice paths from $(0, 0)$ to (m, n) is $\binom{m+n}{n}$.

Definition 2.1. *Two lattice paths $P = u_1 u_2 \cdots u_{m+n}$ and $Q = v_1 v_2 \cdots v_{m+n}$ from $(0, 0)$ to (m, n) are equivalent if and only if there is some $1 \le i \le m + n$ such that $u_{i+1} \cdots u_{m+n} u_1 \cdots u_i = v_1 v_2 \cdots v_{m+n}$, and we denote the equivalence class of P as $[P]$.*

For any lattice path $P = u_1 u_2 \cdots u_{m+n}$ its equivalence class is given by

$$[P] = \{P_s := u_{s+1} u_{s+2} \cdots u_{m+n} u_1 u_2 \cdots u_s \mid s = 1, 2, \cdots, m + n\}.$$

For example, when $P = xyxxy$, the elements in $[P]$ are the following five lattice paths:

$$P_1 = yxxyx, \;\; P_2 = xxyxy, \;\; P_3 = xyxyx, \;\; P_4 = yxyxx \;\; P = P_5 = xyxxy.$$

We define **the period of** P, denoted by $\mathrm{per}(P)$, to be the smallest number r $(1 \le r \le m+n)$ such that $P = P_r$.

Lemma 2.2. *For a lattice path P from $(0,0)$ to (m,n),*

$$\sharp[P] = \mathrm{per}(P) = (m+n)/q\ ,$$

where q is a divisor of $\gcd(m,n)$*. In particular* $\sharp[P] = m+n$ *if* $\gcd(m,n) = 1$.

Proof. Since the lemma clearly holds if $\mathrm{per}(P) = m+n$, we assume that $\mathrm{per}(P) < m+n$, and let $m+n = \mathrm{per}(P)q + r$ $(0 < q,\ 0 \le r < \mathrm{per}(P))$.

Assume that r is zero. Then P is a sequence arranged $u_1 u_2 \cdots u_{\mathrm{per}(P)}$ q times and $\mathrm{per}(P) = (m+n)/q$. If the number of x (resp. y) in $\{u_1, u_2, \cdots, u_{\mathrm{per}(P)}\}$ is b (resp. $\mathrm{per}(P) - b$), then we have $m = bq,\ n = (\mathrm{per}(P) - b)q$. Thus, q is a common divisor of m and n.

Assume that r is not zero. Let $\bar{P}$ be a sequence arranged P infinitely many times and we treat its indexes as consecutive numbers, namely

$$\bar{P} = u_1 \cdots u_{m+n} u_{m+n+1} \cdots u_{2(m+n)} u_{2(m+n)+1} \cdots u_{3(m+n)} \cdots$$

where $u_i = u_{(m+n)b+i}$. There are positive integers a and b which satisfy $\mathrm{per}(P)a - (m+n)b = d$, where $d = \gcd(\mathrm{per}(P), m+n)$. Therefore, we get the following equation:

$$u_i = u_{i+\mathrm{per}(P)a} = u_{i+d+(m+n)b} = u_{i+d},$$

and this equation means $P = P_d$. Since r is not zero, $d < \mathrm{per}(P)$. So it is a contradiction to the minimality of $\mathrm{per}(P)$. □

Lemma 2.3. *For any lattice path P from $(0,0)$ to (m,n), $[P]$ has at least one Dyck path, and if* $\gcd(m,n) = 1$*, $[P]$ has a unique Dyck path.*

Proof. We may assume that $m \ge n$, because $C(m,n) = C(n,m)$. We define a function r for any pair of positive numbers s and t, and any lattice path Q:

$$r(s,t,Q) = t(\text{the number of } x \text{ in } Q) - s(\text{the number of } y \text{ in } Q)$$

Let $sub_i(P)$ be a subsequence of $P = u_1 \cdots u_{m+n}$ given by $u_1 u_2 \cdots u_{j_i}$ $(i = 1, 2, \cdots, n)$, where u_{j_i} is the i^{th} y in P from the left. Since the Dyck path is a lattice path which is below the diagonal line $y = \frac{n}{m}x$, the definition of Dyck path can be described in terms of the function r as follows:

$$r(m,n,sub_i(P)) \ge 0 \qquad \text{for any } 1 \le i \le n.$$

Suppose that a lattice path P from $(0,0)$ to (m,n) is not a Dyck path and let k be the positive integer such that the function $r(m,n,\cdot)$ takes the minimum value on $sub_k(P)$, then P_k is a Dyck path. To prove this, we should confirm that $r(m,n,sub_i(P_k)) \ge 0$ (for any i), but we can see this easily. See Figure 2. This is the figure of a part of lattice path $\bar{P}$ and the line with the slope $\frac{n}{m}$ which is over $\bar{P}$ and touches $\bar{P}$ at only lattice points. For any lattice path P, there is a unique such line. To observe P_s for any P is same as to observe some subsequence with length $m + n$ of $\overline{P}$. Choose two common

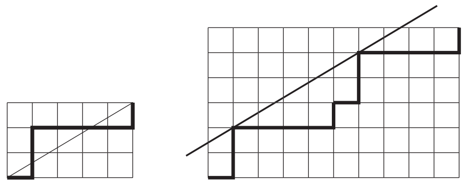

FIGURE 2. $(m, n) = (5, 3)$

points such that the difference of x-coordinates is m, and regard those two points as $(0, 0)$ and (m, n) from the left, that lattice path is a Dyck path. Therefore any lattice path P has at least one Dyck path in their equivalence class. k is one of x-coordinates of common points of $\overline{P}$ and the line. When $\gcd(m, n) = 1$, the difference of x-coordinates of two adjacent lattice points on that line is m, so the Dyck path in $[P]$ is unique, as desired. □

Theorem 2.4. *If* $\gcd(m, n) = 1$, *then*

$$C(m, n) = \frac{1}{m+n}\binom{m+n}{n}.$$

Proof. We can choose some lattice paths $P^1, P^2, \cdots, P^t$ from $(0, 0)$ to (m, n) such that the set of all lattice paths from $(0, 0)$ to (m, n) can be written as the following:

$$\{\text{All lattice paths from } (0,0) \text{ to } (m,n)\} = [P^1] \sqcup [P^2] \sqcup \cdots \sqcup [P^t] \tag{2.1}$$

Lemma 2.3 says that each $[P^i]$ has a unique Dyck path if $\gcd(m, n) = 1$, so $t = C(m, n)$. Comparing the number of elements in both side of (2.1),

$$\begin{aligned}\binom{m+n}{n} &= |\, [P^1] \sqcup [P^2] \sqcup \cdots \sqcup [P^t] \,| \\ &= (m+n)t \qquad (\because \text{ Lemma 2.2}) \\ &= (m+n)C(m, n).\end{aligned}$$

Therefore we have

$$C(m, n) = \frac{1}{m+n}\binom{m+n}{n},$$

and Theorem 2.4 is proven. □

It is easy to show that (1.2) is given as a corollary of Theorem 2.4.

Corollary 2.5.

$$C(kn, n) = \frac{1}{kn+1}\binom{(k+1)n}{n}.$$

Proof. Note that $C(n,kn) = C(n,kn+1)$ holds. In fact, since $\gcd(kn+1,n) = 1$, the lattice points in the domain $\{(x,y) \mid y \le \frac{kn+1}{n}x,\ y \ge kx,\ 0 \le x \le n\}$ are on the line $y = kx$ or $(n, kn+1)$. Namely, all Dyck paths from $(0,0)$ to $(n, kn+1)$ are made by connecting the lattice path from (n,kn) to $(n,kn+1)$ with a lattice path from $(0,0)$ to (n,kn). Therefore, we have

$$\begin{aligned} C(kn,n) &= C(n,kn) = C(n,kn+1) = \frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n} \\ &= \frac{1}{kn+1}\binom{(k+1)n}{n}. \end{aligned}$$

and this is (1.2). □

3. The description of $C(m,n)$ for any positive integers m and n.

In this section, we describe the formula of $C(m,n)$ for any positive integers m and n. Let $A_{(m,n)} = \frac{1}{m+n}\binom{m+n}{n}$, then the following Proposition holds.

Proposition 3.1. *Let* $d = \gcd(m,n)$*, then*

$$C(m,n) = \sum_{i=1}^{d} \frac{i}{d} A_{(\frac{i}{d}m, \frac{i}{d}n)} C\left(\frac{d-i}{d}m, \frac{d-i}{d}n\right). \tag{3.1}$$

The proof of the proposition will be given later.

This recurrence relation (3.1) is a generalization of the recurrence relation (1.1). In fact, when $m = n$, $d = n$ and (3.1) reduces to

$$C_n = \sum_{i=1}^{n} \frac{i}{n} A_{(i,i)} C_{n-i}.$$

Here a part of the right hand side, $\frac{i}{n}A_{(i,i)}C_{n-i} + \frac{n-i+1}{n}A_{(n-i+1,n-i+1)}C_{i-1}$, is equal to $2C_{n-i}C_{i-1}$, since

$$\begin{aligned} &\frac{i}{n}A_{(i,i)}C_{n-i} + \frac{n-i+1}{n}A_{(n-i+1,n-i+1)}C_{i-1} \\ = \ & \frac{i}{n}\frac{1}{2i}\binom{2i}{i}C_{n-i} + \frac{n-i+1}{n}\frac{1}{2(n-i+1)}\binom{2(n-i+1)}{n-i+1}C_{i-1} \\ = \ & \frac{1}{2n}\frac{1}{i}\binom{2(i-1)}{i-1}\frac{2i(2i-1)}{i}C_{n-i} + \frac{1}{2n}\frac{1}{n-i+1}\binom{2(n-i)}{n-i}\frac{2(n-i+1)(2n-2i+1)}{n-i+1}C_{i-1} \\ = \ & \left(\frac{2i-1}{n} + \frac{2n-2i+1}{n}\right)C_{n-i}C_{i-1} \\ = \ & 2C_{n-i}C_{i-1}. \end{aligned}$$

So, (3.1) is a generalization of (1.1).

Definition 3.2. *For a sequence of non-negative integers $a = (a_1, a_2, \cdots)$, we define $||a||$ by*

$$||a|| = \sum_{i=1}^{\infty} i a_i.$$

The recurrence relation (3.1) leads to the main theorem.

Theorem 3.3. *Let $d = \gcd(m, n)$, then*

$$C(m,n) = \sum_{a;||a||=d} \prod_{i=1}^{d} \left(\frac{1}{a_i!} A^{a_i}_{(\frac{i}{d}m, \frac{i}{d}n)} \right). \tag{3.2}$$

Example 3.4. *Let $m = n$, then we have*

$$C(n,n) = \sum_{a;||a||=n} \prod_{i=1}^{n} \left(\frac{1}{a_i!} A^{a_i}_{(i,i)} \right)$$

by (3.2)*. For instance, in the case of $n = 3$, the sequences of non-negative integers $a = (a_1, a_2, \cdots)$ with $||a|| = 3$ are the following three sequences.*

$$(3, 0, 0, \cdots), \quad (1, 1, 0, 0, \cdots), \quad (0, 0, 1, 0, 0, \cdots).$$

Thus,

$$\begin{aligned} C(3,3) &= \sum_{a;||a||=3} \prod_{i=1}^{3} \left(\frac{1}{a_i!} A^{a_i}_{(i,i)} \right) \\ &= \frac{1}{3!} A^3_{(1,1)} \\ &\quad + \frac{1}{1!} A^1_{(1,1)} \cdot \frac{1}{1!} A^1_{(2,2)} + \frac{1}{1!} A^1_{(3,3)} \\ &= \frac{1}{6} + \frac{3}{2} + \frac{20}{6} = 5. \end{aligned}$$

As this example shows, each factor $\prod_{i=1}^{d} \left(\frac{1}{a_i!} A^{a_i}_{(\frac{i}{d}m, \frac{i}{d}n)} \right)$ in the sum is not necessarily an integer.

For simplicity, we rewrite Proposition 3.1 and Theorem 3.3.

Proposition 3.5 (Proposition 3.1)**.** *Let p and q be two positive integers with* $\gcd(p, q) = 1$*. We denote the number of Dyck paths from* $(0, 0)$ *to* (dp, dq) *(namely, $C(dp, dq)$) by $\widetilde{C}_d$. Likewise, we abbreviate $A_{(dp,dq)}$ as A_d. Then, we have*

$$\widetilde{C}_d = \sum_{i=1}^{d} \frac{i}{d} A_i \widetilde{C}_{d-i}.$$

Theorem 3.6 (Theorem 3.3)**.** *Under the same assumption of Propsition 3.5, we have*

$$\widetilde{C}_d = \sum_{a;\|a\|=d} \prod_{i=1}^{d} \left(\frac{1}{a_i!} A_i^{a_i} \right).$$

In the rest of this section, we show that Theorem 3.6 follows from Proposition 3.5, state three lemmas and prove Proposition 3.5 using them. For that we need some notations.

Definition 3.7. *For a sequence of non-negative integers $a = (a_1, a_2, \cdots,)$, $| a |$, $\ell(a)$ and $h(a)$ are defined by*

$$| a |= \sum_{i=1}^{\infty} a_i, \quad \ell(a) = \sharp\{i \mid a_i \neq 0\}, \quad \textit{and} \quad h(a) = \frac{| a |!}{\prod_{i=1}^{\infty} a_i!}$$

respectively.

If $\|a\| = d < \infty$, $h(a) = \dfrac{| a |!}{\prod_{i=1}^{d} a_i!}$. Hereafter we always assume a satisfies $\ell(a) < \infty$.

Definition 3.8. *For two sequences of non-negative integers $a = (a_1, a_2, \cdots)$ and $c = (c_1, c_2, \cdots)$, we define*

$$c \geq a \Leftrightarrow c_i \geq a_i \quad \textit{for any } i = 1, 2, \cdots$$

$$c > a \Leftrightarrow c \geq a \quad \textit{and} \quad c_i \neq a_i \ \textit{ for some } \ i = 1, 2, \cdots .$$

Moreover, for $0 \leq j <| c |$, let B_c^j be

$$B_c^j := \{a \mid a \leq c, | a |=| c | - j\}.$$

Definition 3.9. *Suppose that p and q are a pair of positive integers with* $\gcd(p, q) = 1$, *and d is any positive integer. Let $D(dp, dq)$ be the number of Dyck paths from $(0, 0)$ to (dp, dq) which is strictly below the diagonal $y = \frac{p}{q} x$ except at $(0, 0)$ and (dp, dq), and $D(0, 0) = 1$. For any sequence of non-negative integers a with $\|a\| < \infty$, we set*

$$D^a_{p,q} := \prod_{i=1}^{\infty} D(ip, iq)^{a_i}.$$

If p and q are clear from the context, we abbreviate $D^a_{p,q}$ as D^a and abbreviate $A_{(dp,dq)}$ as A_d as before.

Lemma 3.10.

$$\widetilde{C}_d = \sum_{a;\|a\|=d} h(a) D^a. \tag{3.3}$$

Lemma 3.11.

$$A_d = \sum_{a;\|a\|=d} \frac{1}{|\, a\, |} h(a) D^a. \tag{3.4}$$

Lemma 3.12. *For any sequence of non-negative integers* $c = (c_1, c_2, \cdots)$ *and any* j *with* $0 \leq j \leq |\, c\, | -1$*, the following holds.*

$$\sum_{a \in B_c^j} \frac{\|\, a\, \|}{|\, a\, |} h(a) h(c-a) = \frac{\|\, c\, \|}{|\, c\, |} h(c).$$

We will give the proofs of these lemmas in the next section. Here we assume that Lemma 3.10, Lemma 3.11, and Lemma 3.12 are correct and give the proof of Proposition 3.5.

Proof of Proposition 3.5. Now we fix the pair of positive integers p and q with $\gcd(p, q) = 1$. Substituting (3.3) and (3.4) for the right hand side of Proposition 3.5, we have

$$\begin{aligned}
\sum_{i=1}^{d} \frac{i}{d} A_i \widetilde{C}_{d-i} &= \frac{1}{d} \sum_{i=1}^{d} \left(\sum_{\|a\|=i} \frac{i}{|\, a\, |} h(a) D^a \right) \left(\sum_{\|b\|=d-i} h(b) D^b \right) \\
&= \frac{1}{d} \sum_{i=1}^{d} \left(\sum_{\|a\|=i} \sum_{\|b\|=d-i} \frac{i}{|\, a\, |} h(a) h(b) D^{a+b} \right) \\
&= \frac{1}{d} \sum_{\|c\|=d} \left(\sum_{a \leq c} \frac{\|\, a\, \|}{|\, a\, |} h(a) h(c-a) \right) D^c, \qquad (3.5)
\end{aligned}$$

where the last equality in (3.5) is given by substituting $c = a+b$. Calculating the factor in the right hand side of (3.5):

$$\begin{aligned}
\sum_{a \leq c} \frac{\|\, a\, \|}{|\, a\, |} h(a) h(c-a) &= \sum_{j=0}^{|c|-1} \sum_{a \in B_c^j} \frac{\|\, a\, \|}{|\, a\, |} h(a) h(c-a) \\
&= \sum_{j=0}^{|c|-1} \frac{\|\, c\, \|}{|\, c\, |} h(c) \\
&= \|\, c\, \| h(c).
\end{aligned}$$

The second equality above follows from Lemma 3.12, thus we have

$$\begin{aligned}
\sum_{i=1}^{d} \frac{i}{d} A_i \widetilde{C}_{d-i} &= \frac{1}{d} \sum_{\|c\|=d} \left(\sum_{a \leq c} \frac{\|\, a\, \|}{|\, a\, |} h(a) h(c-a) \right) D^c \\
&= \sum_{c;\|c\|=d} h(c) D^c \\
&= \widetilde{C}_d
\end{aligned}$$

by Lemma 3.10. Therefore Proposition 3.5 is proven. □

Theorem 3.6 follows from Proposition 3.5.

Proof of Theorem 3.6. Fix the pair of positive integers p and q with $\gcd(p,q) = 1$. We prove Theorem 3.6 by induction on d. When $d = 1$, since $(m,n) = (1\cdot p, 1\cdot q)$, $C(m,n) = \widetilde{C}_1 = A_1$ follows Theorem 2.4, thus Theorem 3.6 holds. Assume that Theorem 3.6 holds for less than or equal to $d-1$. Then we have

$$\begin{aligned} \widetilde{C}_d &= \sum_{i=1}^{d} \frac{i}{d} A_i \widetilde{C}_{d-i} \\ &= \sum_{i=1}^{d} \frac{i}{d} A_i \left(\sum_{a;\|a\|=d-i} \prod_{j=1}^{d-i} \frac{1}{a_j!} A_j^{a_j} \right) \end{aligned} \tag{3.6}$$

by Propsition 3.5 and the induction assumption.

Claim 3.13. *The following equation holds for d variables* $x_1, x_2, \cdots, x_d$*:*

$$\sum_{i=1}^{d} \frac{i}{d} x_i \left(\sum_{\|a\|=d-i} \prod_{j=1}^{d-i} \frac{1}{a_j!} x_j^{a_j} \right) = \sum_{\|a\|=d} \prod_{i=1}^{d} \frac{1}{a_i!} x_i^{a_i}. \tag{3.7}$$

Proof of the claim. For any sequence of non- negative integers

$$b = (b_1, b_2, \cdots, b_d, 0, 0, \cdots),$$

we shall observe the coefficients of $x_1^{b_1} x_2^{b_2} \cdots x_d^{b_d}$ in the left hand side of (3.7).
(1) If $\| b \| = d$, the term which contains $x_1^{b_1} x_2^{b_2} \cdots x_d^{b_d}$ in the left hand side of (3.7) is

$$\sum_{i=1}^{d} \frac{i}{d} x_i \left(\frac{1}{(b_i-1)!} x_i^{b_i-1} \prod_{j\neq i} \frac{1}{b_j!} x_j^{b_j} \right), \tag{3.8}$$

where we understand $\frac{1}{(b_i-1)!} x_i^{b_i-1} = 0$ if $b_i = 0$, and (3.8) is equal to

$$\sum_{i=1}^{d} \frac{ib_i}{d} \left(\prod_{j=1}^{d} \frac{1}{b_j!} x_j^{b_j} \right) = \left(\frac{1}{d} \sum_{i=1}^{d} ib_i \right) \left(\prod_{j=1}^{d} \frac{1}{b_j!} x_j^{b_j} \right) = \prod_{j=1}^{d} \frac{1}{b_j!} x_j^{b_j}.$$

(2) We shall show that any monomial in the left hand side of (3.7) is of the form $x_1^{b_1} x_2^{b_2} \cdots x_d^{b_d}$ with $\|b\| = d$. Any monomial in the left hand side of (3.7) is of the form $x_i \prod_{j=1}^{d-i} x_j^{a_j}$ with a such that $\|a\| = d - i$. Set $b = (a_1, \cdots, a_{i-1}, a_i + 1, a_{i+1}, \cdots)$. Then $\| b \| = i + \|a\| = i + d - i = d$ and $x_i \prod_{j=1}^{d-i} x_j^{a_j} = x_1^{b_1} x_2^{b_2} \cdots x_d^{b_d}$ with $\| b \| = d$. □

By substituting $x_i = A_i$ for the left hand side of (3.7), we have

$$\sum_{i=1}^{d} \frac{i}{d} A_i \left(\sum_{a;\|a\|=d-i} \prod_{j=1}^{d-i} \frac{1}{a_j!} A_j^{a_j} \right) = \sum_{b;\|b\|=d} \prod_{j=1}^{d} \frac{1}{b_j!} A_j^{b_j}.$$

Therefore, by (3.6), we have

$$\widetilde{C}_d = \sum_{i=1}^{d} \frac{i}{d} A_i \left(\sum_{a;\|a\|=d-i} \prod_{j=1}^{d-i} \frac{1}{a_j!} A_j^{a_j} \right) = \sum_{b;\|b\|=d} \prod_{j=1}^{d} \frac{1}{b_j!} A_j^{b_j},$$

and Theorem 3.6 follows. □

4. Proofs of the Lemmas

We fix the pair of positive numbers p and q with $\gcd(p, q) = 1$ as before. Let

$$m = dp, \quad n = dq.$$

To start with, we give the definitions of **shape** $e = (e_1, e_2, \cdots)$ and **type** $a = (a_1, a_2, \cdots)$ **of the Dyck path** from $(0, 0)$ to (m, n). Any Dyck path P touches the diagonal $y = \frac{n}{m}x$ at least one point except at $(0, 0)$, and coordinates of intersection of P and the diagonal can be described as $(\frac{k}{d}m, \frac{k}{d}n) = (kp, kq)$ for some $k \in \mathbb{Z}_{>0}$ because $\gcd(m, n) = d$. Let all intersection points of P and the diagonal be $(0, 0)$, $(k_1 p, k_1 q)$, $(k_2 p, k_2 q)$, $\cdots$, $(k_s p, k_s q)$ from the left. (Namely, $0 < k_1 < k_2 < \cdots < k_s = d$.) Then, the shape $e = (e_i)_{i\in\mathbb{N}}$ of a Dyck path P is defined by $e_i = k_i - k_{i-1}$ for any non-negative integer i, where $k_0 = 0$ and $k_t = 0$ $(t > s)$. Furthermore, the type $a = (a_i)_{i\in\mathbb{N}}$ of a Dyck path P is defined by $a_i = \sharp\{e_j \mid e_j = i\}$ for any $i \geq 1$. We denote the type of P by type(P).

Proof of Lemma 3.10. Suppose that P is a Dyck path form $(0, 0)$ to (m, n) of shape e and type a. Then

$$\begin{aligned} \|a\| &= \sum_{i=1}^{\infty} i a_i = \sum_{i=1}^{\infty} i \sharp\{e_j \mid e_j = i\} \\ &= \sum_{i=1}^{\infty} i \sharp\{k_j \mid k_j - k_{j-1} = i\} \\ &= \sum_{i=1}^{\infty} (k_i - k_{i-1}) = -k_0 + k_s \\ &= d. \end{aligned}$$

Conversely, for any sequence of non-negative integers a with $\|a\| = d$, it is clear that there exists some Dyck path of type a from $(0, 0)$ to (m, n). The number of Dyck paths of shape e is $\prod_{i=1}^{\infty} D_{(e_i p, e_i q)}$. If the shapes of two Dyck paths coincide, their types also coincide; so the number of Dyck paths of type a is

$$\sum_e \prod_{i=1}^{\infty} D_{(e_i p, e_i q)} = \sum_e \prod_{i=1}^{\infty} D_{(ip,iq)}^{a_i} = h(a)D^a,$$

where the sum $\sum_e$ is taken over all e with $\sharp\{e_j \mid e_j = i\} = a_i$ for any i. This proves Lemma 3.10. □

Proof of Lemma 3.11. If $[P]$ has more than one Dyck path, then types of these Dyck paths coincide. Let P be a Dyck path of type a and period $r(\neq m+n)$. Lemma 2.2 says that r is a divisor of $m+n$, and this means that

$$\exists a' = (a'_1, a'_2, \cdots) \quad s.t. \quad a_i = \frac{m+n}{r} a'_i \quad \forall i \in \mathbb{N}.$$

Namely, any a_i is divisible by $(m+n)/r$. Let $cd(a)$ be the set of all r such that $(m+n)/r$ divides all a_i, in other words, r which can be the period of some Dyck path with type a. For any Dyck path P with type a and period r, the number of lattice paths in $[P]$ is r and that of Dyck paths in $[P]$ is $\mid a' \mid = r \mid a \mid /(m+n)$ by Lemma 2.2. Let $E(a,r)$ be the number of Dyck paths with type a and period r. Counting the number of all lattice paths from $(0,0)$ to (m,n), we have

$$\begin{aligned}
(m+n)A_{(m,n)} &= \sum_{a;\|a\|=d} \sum_{r;r\in cd(a)} \sum_{P;\mathrm{type}(P)=a,\mathrm{per}(P)=r} \frac{\sharp[P]}{(\text{Number of Dyck paths in } [P])} \\
&= \sum_{a;\|a\|=d} \sum_{r;r\in cd(a)} E(a,r) \frac{r}{\frac{r|a|}{m+n}} \\
&= \sum_{a;\|a\|=d} \sum_{r;r\in cd(a)} \frac{m+n}{\mid a \mid} E(a,r) \\
&= \sum_{a;\|a\|=d} \frac{m+n}{\mid a \mid} \left(\sum_{r;r\in cd(a)} E(a,r) \right).
\end{aligned}$$

We know $h(a)D^a = \sum_{r;r\in cd(a)} E(a,r)$, thus,

$$(m+n)A_{(m,n)} = \sum_{a;\|a\|=d} \frac{m+n}{\mid a \mid} h(a)D^a.$$

Lemma 3.11 is proved. □

Proof of Lemma 3.12. We begin with the following claim.

Claim 4.1.

$$\sum_{a\in B_c^j} \frac{\| a \|}{\mid a \mid} h(a)h(c-a) = \sum_{c'\in B_c^1} \sum_{a\in B_{c'}^{j-1}} \frac{\| a \|}{\mid a \mid} h(a)h(c'-a)$$

Proof of the claim. Recall $\ell(c) = \#\{i \mid c_i \neq 0\}$. For any sequence of non-negative integers c, the elements in B_c^1 are the following $\ell(c)$ sequences:

$$1 \leq \forall t \leq \ell(c), \quad c^t := (0, \cdots, 0, c_{s_1}, \cdots, c_{s_t} - 1, 0, \cdots, c_{s_{\ell(c)}}, 0, \cdots)$$

where c_{s_t} is the t^{th} nonzero number in c from the left. Then, we have

$$\begin{aligned}
\sum_{c' \in B_c^1} h(c') &= \sum_{t=1}^{\ell(c)} h(c^t) \\
&= \sum_{t=1}^{\ell(c)} \frac{(\mid c \mid -1)!}{\prod_{i=1}^{\ell(c)} (c_{s_i}!)/c_{s_t}} \\
&= \frac{\mid c \mid!}{\mid c \mid} \frac{1}{\prod_{i=1}^{\ell(c)} (c_{s_i}!)} \sum_{t=1}^{\ell(c)} c_{s_t} \\
&= \frac{\mid c \mid!}{\prod_{i=1}^{\ell(c)} (c_{s_i}!)} = h(c) \qquad (4.1)
\end{aligned}$$

We note that

$$a \in B_c^j \iff a \in B_{c'}^{j-1} \text{for some } \ c' \in B_c^1.$$

Therefore we have

$$\begin{aligned}
\sum_{c' \in B_c^1} \sum_{a \in B_{c'}^{j-1}} \frac{\| a \|}{\mid a \mid} h(a) h(c' - a) &= \sum_{a \in B_c^j} \frac{\|a\|}{\mid a \mid} \left(\sum_{c' \in B_c^1 s.t. c' > a} h(c' - a) \right) h(a) \\
&= \sum_{a \in B_c^j} \frac{\|a\|}{\mid a \mid} \left(\sum_{b \in B_{c-a}^1} h(b) \right) h(a) \\
&= \sum_{a \in B_c^j} \frac{\|a\|}{\mid a \mid} h(c - a) h(a).
\end{aligned}$$

The last equation above holds by (4.1). Therefore Claim 4.1 is proved. □

We go back to the proof of Lemma 3.12. We prove by induction on j. Lemma 3.12 clearly holds for $j = 0$. We have

$$B_c^1 = \{c^t = (0, \cdots, 0, c_{s_1}, \cdots, c_{s_t} - 1, 0, \cdots, c_{s_{\ell(c)}}, 0, \cdots) \mid 1 \leq \forall t \leq \ell(c)\},$$

and then

$$\begin{aligned}
\sum_{a\in B_c^1} \frac{\| a \|}{| a |} h(a)h(c-a) &= \sum_{t=1}^{\ell(c)} \frac{\| c^t \|}{| c^t |} h(c^t)h(c-c^t) \\
&= \sum_{t=1}^{\ell(c)} \frac{\| c \| - s_t}{| c | - 1} \frac{(| c | - 1)!}{(\prod_{i=1}^{\ell(c)} c_{s_i}!)/c_{s_t}} \cdot 1 \\
&= \sum_{t=1}^{\ell(c)} \frac{\|c\| - s_t}{| c | - 1} \frac{| c |!}{| c |} \frac{c_{s_t}}{\prod_{i=}^{\ell(c)} c_{s_i}!} \\
&= \left(\sum_{t=1}^{\ell(c)} c_{s_t}\|c\| - \sum_{t=1}^{\ell(c)} s_t c_{s_t} \right) \frac{1}{| c | (| c | - 1)} h(c) \\
&= (\|c\| \, | c | - \|c\|) \frac{1}{| c | (| c | - 1)} h(c) \\
&= \frac{\|c\|}{| c |} h(c).
\end{aligned}$$

Thus, Lemma 3.12 holds for $j = 1$. Assume that $j \geq 2$ and Lemma 3.12 holds for $j - 1$. By Claim 4.1, we have

$$\begin{aligned}
\sum_{a\in B_c^j} \frac{\| a \|}{| a |} h(a)h(c-a) &= \sum_{c'\in B_c^1} \sum_{a\in B_{c'}^{j-1}} \frac{\| a \|}{| a |} h(a)h(c'-a) \\
&= \sum_{c'\in B_c^1} \frac{\| c' \|}{| c' |} h(c') \\
&= \frac{\| c \|}{| c |} h(c),
\end{aligned}$$

so Lemma 3.12 also holds for j. □

References

[1] N. Fuss, *Solio quastionis, quot modis polygonum n laterum in polyga m laterum, per diagonales resolvi quœat*, Nova acta academiæscientarium Petropolitanæ9 (1791), 243-251.

[2] R. P. Stanley, http://www-math.mit.edu/˜rstan/ec/catadd.pdf

[3] P. Hilton and J. Pedersen, *Catalan Numbers, Their Generalization, and Their Uses*, The Mathematical Intelligencer, Spring 1991, Volume 13, Issue 2, pp 64-75.

[4] P. Duchon, *On the enumeration and generation of generalized Dyck words*, Discrete Mathematics, Volume 225, Issues 1-3, 28 October 2000, pp 121-135.

[5] M. T. L. Bizley, *Derivation of a new formula for the number of minimal lattice path from* $(0,0)$ *to* (km, kn) *having just contacts with the line my = nx and having no points above this line; and a proof of Grossman's formula for the number of paths which may*

touch but do not rise above this line, Journal of the Institute of Actuaries (1954) 80, 31 December 1954, pp 55-62.

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