ON THE EQUIVALENCE OF THREE DEFINITIONS OF COMPACT INFRA-SOLVMANIFOLDS

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ABSTRACT. We explain the equivalence of three definitions of compact infra-solvmanifolds that appear in various math literatures.

The following are three definitions of compact infra-solvmanifolds appearing in various math literatures.

Def 1: Let G be a connected, simply connected solvable Lie group, K be a maximal compact subgroup of the group $\mathrm{Aut}(G)$ of automorphisms of G, and Γ be a cocompact, discrete subgroup of $E(G) = G \rtimes K$. If the action of Γ on G is free and $[\Gamma : G \cap \Gamma] < \infty$, the orbit space $\Gamma\backslash G$ is called an *infra-solvmanifold modeled on* G. See [6, Definition 1.1].

Def 2: A *compact infra-solvmanifold* is a manifold of the form $\Gamma\backslash G$, where G is a connected, simply connected solvable Lie group, and Γ is a torsion-free cocompact discrete subgroup of $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ which satisfies: the closure of $hol(\Gamma)$ in $\mathrm{Aut}(G)$ is compact where $hol : \mathrm{Aff}(G) \to \mathrm{Aut}(G)$ is the holonomy projection. See [1, Definition 1.1].

Def 3: A *compact infra-solvmanifold* is a double coset space $\Gamma\backslash G/K$ where G is a virtually connected and virtually solvable Lie group, K is a maximal compact subgroup of G and Γ is a torsion-free, cocompact, discrete subgroup of G. See [2, Definition 2.10].

A *virtually connected* Lie group is a Lie group with finitely many connected components.

Remark 1: If we remove the "cocompact" in Def 2, we may get noncompact infra-solvmanifolds in general, which are vector bundles over some compact infra-solvmanifolds (see [7, Theorem6]).

The main purpose of this note is to explain why the above three definitions of compact infra-solvmanifolds are equivalent. The reason should be known to many people. But since we did not find any formal proof of this equivalence, so we write a proof here for the convenience of future reference.

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§1. The equivalence of Def 1 and Def 2

Let M be a compact infra-solvmanifold in the sense of Def 1. First of all, any $g \in \Gamma$ can be decomposed as $g = k_g u_g$ where $k_g \in K \subset \mathrm{Aut}(G)$ and $u_g \in G$. The holonomy projection $hol : G \rtimes \mathrm{Aut}(G) \to \mathrm{Aut}(G)$ sends g to k_g. Since hol is a group homomorphism, its image $hol(\Gamma)$ is a subgroup of K. By assumption, $|hol(\Gamma)| = [\Gamma : G \cap \Gamma]$ is finite, so $hol(\Gamma)$ is compact. In addition, since G is a connected simply-connected solvable Lie group, so G is diffeomorphic to an Euclidean space. Then by Smith fixed point theorem ([5, Theorem I]), Γ acting on G freely implies that Γ is torsion-free. So M satisfies Def 2.

Conversely, if M satisfies Def 2, then [7, Theorem 3 (a) $\Rightarrow$ (f)] tells us that there exists a connected, simply-connected solvable Lie group G' so that the Γ (which defines M) can be thought of as discrete cocompact subgroup of $G' \rtimes F$ where F is a finite subgroup of $\mathrm{Aut}(G')$. Moreover, there exists an equivariant diffeomorphism from G to G' with respect to the action of Γ (by [7, Theorem 1 and Theorem 2]). Hence

$$M = \Gamma \backslash G \underset{\text{diff.}}{\cong} \Gamma \backslash G'.$$

Then $[\Gamma : \Gamma \cap G'] = |hol_{G'}(\Gamma)| \leq |F|$ is finite.

It remains to show that the action of Γ on G' is free. Since F is finite, we can choose an F-invariant Riemannian metric on G' (in fact we only need F to be compact). If the action of an element $g \in \Gamma$ on G' has a fixed point, say h_0. Let L_{h_0} be the left translation of G' by h_0. Then $L_{h_0}^{-1} g L_{h_0}$ fixed the identity element, which implies $L_{h_0}^{-1} g L_{h_0} \in F$. So there exists a compact neighborhood U of h_0 so that $g \cdot U = U$ and g acts isometrically on U with respect to the Riemannian metric just as the element $L_{h_0}^{-1} g L_{h_0} \in F$ acts around e. So $A = Dg : T_{h_0}U \to T_{h_0}U$ is an orthogonal transformation. Then A is conjugate in $O(n)$ to the block diagonal matrix of the form

$$\begin{pmatrix} B(\theta_1) & & 0 \\ & \ddots & \\ 0 & & B(\theta_m) \end{pmatrix}$$

where $B(0) = 1$, $B(\pi) = -1$, and $B(\theta_j) = \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}$, $1 \leq j \leq m$.

Since Γ is torsion-free, g is an infinite order element. This implies that at least one θ_j is irrational. Then for any $v \neq 0 \in T_{h_0}G'$, $|\{A^n v\}_{n\in\mathbb{Z}}| = \infty$. So there exists $h \in U$ so that $|\{g^n \cdot h\}_{n\in\mathbb{Z}}| = \infty$. Then the set $\{g^n \cdot h\}_{n\in\mathbb{Z}} \subset U$ has at least one accumulation point. This contradicts the fact that the orbit space $\Gamma\backslash G'$ is Hausdorff (since $\Gamma\backslash G' = M$ is a manifold).

Combing the above arguments, M is an infra-solvmanifold modeled on G' in the sense of Def 1. □

§2. Def 1 ⇒ Def 3

Let M be an infra-solvmanifold in the sense of Def 1. Let $hol(\Gamma)$ be the image of the the holonomy projection $hol : \Gamma \to \mathrm{Aut}(G)$. Then $hol(\Gamma)$ is a finite subgroup of $\mathrm{Aut}(G)$.

Define $\widetilde{G} = G \rtimes hol(\Gamma)$ which is virtually solvable. It is easy to see that $hol(\Gamma)$ is a maximal compact subgroup of $\widetilde{G}$ and, Γ is a cocompact, discrete subgroup of $\widetilde{G}$. Then $M \cong \Gamma\backslash\widetilde{G}/K$. So M satisfies Def 3. □

§3. Def 3 ⇒ Def 2

Let $M = \Gamma\backslash G/K$ be a compact infra-solvmanifold in the sense of Def 3. Since K is a maximal compact subgroup of G, so G/K is contractible. Note that K is not necessarily a normal subgroup of G, so G/K may not directly inherit a group structure from G.

Let G_0 be the connected component of G containing the identity element. Then by [3, Theorem 14.1.3 (ii)], $K_0 = K \cap G_0$ is connected and K_0 is a maximal compact subgroup of G_0. Moreover, K intersects each connected component of G and $K/K_0 \cong G/G_0$. By the classical Lie theory, the Lie algebra of a compact Lie group is a direct product of an abelian Lie algebra and some simple Lie algebras. Then since the Lie algebra $\mathrm{Lie}(G)$ of G is solvable and K is compact, the Lie algebra $\mathrm{Lie}(K) \subset \mathrm{Lie}(G)$ must be abelian. This implies that K_0 is a torus and hence a maximal torus in G_0.

In addition, since G is virtually solvable, G_0 is actually solvable. This is because the radical R of G is a normal subgroup of G_0 and $\dim(R) = \dim(G_0)$ (since G is virtually solvable). So G_0/R is discrete. Then since G_0 is connected, G_0 must equal R.

Let $Z(G)$ be the center of G and define $C = Z(G) \cap K$. Then C is clearly a normal subgroup of G. Let $G' = G/C$ and $K' = K/C$ and let $\rho : G \to G'$ be the quotient map. Then since $\Gamma \cap K = \{1\}$, $\Gamma \cong \rho(\Gamma) \subset G'$, we can think of Γ as a subgroup of G'. So we have

$$M = \Gamma\backslash G/K \cong \Gamma\backslash G'/K'. \tag{1}$$

Let $G'_0 = \rho(G_0)$ be the identity component of G'. Then G'_0 is a finite index normal subgroup of G' and G'_0 is solvable.

$$\mathrm{Lie}(G'_0) = \mathrm{Lie}(G') = \mathrm{Lie}(G)/\mathrm{Lie}(C) = \mathrm{Lie}(G_0)/\mathrm{Lie}(C). \tag{2}$$

Besides, let $K'_0 = K' \cap G'_0$ which is a maximal torus of G'_0 and we have

$$\mathrm{Lie}(K'_0) = \mathrm{Lie}(K') = \mathrm{Lie}(K)/\mathrm{Lie}(C) = \mathrm{Lie}(K_0)/\mathrm{Lie}(C). \tag{3}$$

Claim-1: G'_0 is linear and so G' is linear.

A group is called *linear* if it admits a faithful finite-dimensional representation. By [3, Theorem 16.2.9 (b)], a connected solvable Lie group S is linear if and only if $\mathfrak{t} \cap [\mathfrak{s}, \mathfrak{s}] = \{0\}$ where $\mathfrak{s}$ and $\mathfrak{t}$ are Lie algebras of S and its maximal torus T_S, respectively. And for a general connected solvable group S, the Lie subalgebra $\mathfrak{t} \cap [\mathfrak{s}, \mathfrak{s}]$ is always central in $\mathfrak{s}$. So for our G_0 and its maximal torus K_0, we have $\mathrm{Lie}(K_0) \cap [\mathrm{Lie}(G_0), \mathrm{Lie}(G_0)]$ is central in $\mathrm{Lie}(G_0) = \mathrm{Lie}(G)$. So $\mathrm{Lie}(K_0) \cap [\mathrm{Lie}(G_0), \mathrm{Lie}(G_0)] \subset \mathrm{Lie}(Z(G))$ and

$$\mathrm{Lie}(K_0) \cap [\mathrm{Lie}(G_0), \mathrm{Lie}(G_0)] \subset \mathrm{Lie}(K) \cap \mathrm{Lie}(Z(G)) = \mathrm{Lie}(C) \tag{4}$$

Then for the Lie group G_0' and its maximal torus K_0', we have

$$\mathrm{Lie}(K_0') \cap [\mathrm{Lie}(G_0'), \mathrm{Lie}(G_0')] = \frac{\mathrm{Lie}(K_0)}{\mathrm{Lie}(C)} \cap \left[\frac{\mathrm{Lie}(G_0)}{\mathrm{Lie}(C)}, \frac{\mathrm{Lie}(G_0)}{\mathrm{Lie}(C)}\right] = 0. \tag{5}$$

So by [3, Theorem 16.2.9 (b)], G_0' is linear. Moreover, suppose V is a faithful finite-dimensional representation of G_0'. Then $\mathbb{R}[G'] \otimes_{\mathbb{R}[G_0']} V$ is a faithful finite-dimensional representation of G' where $\mathbb{R}[G']$ and $\mathbb{R}[G_0']$ are the group rings of G' and G_0' over $\mathbb{R}$, respectively. So the Claim-1 is proved.

From the Claim-1 and (1), we can just assume that our group G is linear at the beginning. Under this assumption, G_0 is a connected linear solvable group. So there exists a simply connected solvable normal Lie subgroup S of G_0 so that $G_0 = S \rtimes K_0$ and $[G_0, G_0] \subset S$ (see [3, Lemma 16.2.3]). So

$$\mathrm{Lie}(G_0) = \mathrm{Lie}(K_0) \oplus \mathrm{Lie}(S).$$

More specifically, we can take $S = p^{-1}(V)$ where $p : G_0 \to G_0/[G_0, G_0]$ is the quotient map and V is a vector subgroup of the abelian group $G_0/[G_0, G_0]$ so that $G_0/[G_0, G_0] \cong p(K_0) \times V$ (see the proof of [3, Theorem 16.2.3]). Note that the vector subgroup V is not unique, so S is not unique either.

Claim-2: We can choose S to be normal in G and so $G \cong S \rtimes K$.

Indeed since K is compact, we can choose a metric on $\mathrm{Lie}(G_0)$ which is invariant under the adjoint action of K. Then we can choose V properly so that $\mathrm{Lie}(S)$ is orthogonal to $\mathrm{Lie}(K_0)$ in $\mathrm{Lie}(G_0)$. Then because K_0 is normal in K, the adjoint action of K on $\mathrm{Lie}(G_0)$ preserves $\mathrm{Lie}(K_0)$, so it also preserves the orthogonal complement $\mathrm{Lie}(S)$ of $\mathrm{Lie}(K_0)$. This implies that S is preserved under the adjoint action of K.

Let G_0, $h_1G_0, \cdots, h_mG_0$ be all the connected components of G. Since K intersects each connected component of G, we can assume $h_i \in K$ for all $1 \leq i \leq m$. Then any element $g \in G$ can be written as $g = g_0h_i$ for some $g_0 \in G_0$ and $h_i \in K$. So $gSg^{-1} = g_0h_iSh_i^{-1}g_0^{-1} \subset g_0Sg_0^{-1} \subset S$. The Claim-2 is proved.

From the semidirect product $G = S \rtimes K$, we can define an injective group homomorphism $\alpha : G \to \mathrm{Aff}(S) = S \rtimes \mathrm{Aut}(S)$ as follows. For any $g \in G$, we can write $g = s_gk_g$ for a unique $s_g \in S$ and $k_g \in K$ since $S \cap K = S \cap K_0 = \{1\}$. Then $\alpha(g) : S \to S$ is the composition of the adjoint action of k_g on S and the left translation on S by s_g, i.e. $\alpha(g) = L_{s_g} \circ \mathrm{Ad}_{k_g}$.

Claim-3: $\alpha(\Gamma)\backslash S$ is diffeomorphic to the double coset space $\Gamma\backslash G/K$.

Notice that each left coset in G/K contains a unique element of S, so we have

$$G/K = \{sK\,;\, s \in S\}.$$

For any $\gamma \in \Gamma$, let $\gamma = s_\gamma k_\gamma$ where $s_\gamma \in S$ and $k_\gamma \in K$, and we have

$$\gamma s K = s_\gamma k_\gamma s K = s_\gamma k_\gamma s k_\gamma^{-1} K = \alpha(\gamma)(s)K,\ \forall\, s \in S.$$

So the natural action of Γ on the left coset space G/K can be identified with the action of $\alpha(\Gamma) \subset \mathrm{Aff}(S)$ on S. The Claim-3 is proved.

Let $\mathrm{Ad} : K \to \mathrm{Aut}(S)$ denote the adjoint action of K on S. Since K is compact and Ad is continuous, so $\mathrm{Ad}(K) \subset \mathrm{Aut}(S)$ is also compact. Notice that $\alpha(\Gamma)$ is a subgroup of $S \rtimes \mathrm{Ad}(K) \subset S \rtimes \mathrm{Aut}(S)$, so the closure $\overline{hol(\alpha(\Gamma))}$ of the holonomy group $hol(\alpha(\Gamma))$ in $\mathrm{Aut}(S)$ is contained in $\mathrm{Ad}(K)$. So $\overline{hol(\alpha(\Gamma))}$ is compact. This implies that $\Gamma\backslash G/K \cong \alpha(\Gamma)\backslash S$ is an infra-solvmanifold in the sense of Def 2. □

Remark 2: A simply-connected solvable Lie group is always linear, but for non-simply-connected solvable Lie groups, this is not always so. A counterexample is the quotient group of the Heisenberg group by an infinite cyclic group (see [4, p.169 Example 5.67]).

Remark 3: The fundamental group Γ of any infra-solvmanifolds is torision-free virtually poly-cyclic (see [2]). It is shown in [1] that such a group Γ determines a virtually solvable real linear algebraic group H_Γ which contains Γ as a discrete and Zariski-dense subgroup (see [1]). H_Γ is called the *real algebraic hull of* Γ. In addition, $H_\Gamma = U \rtimes T$ where T is a maximal reductive subgroup of H_Γ and U is the unipotent radical of H_Γ. The splitting gives an injective group homomorphism $\alpha : H_\Gamma \to \mathrm{Aff}(U)$ and a corresponding affine action of $\Gamma < H_\Gamma$ on U so that $M_\Gamma = \alpha(\Gamma)\backslash U$ is an infra-solvmanifold whose fundamental group is Γ. M_Γ is called the *standard* Γ*-manifold*. Notice that the group U is connected, simply-connected, nilpotent and $M_\Gamma = \alpha(\Gamma)\backslash U \cong \Gamma\backslash H_\Gamma/T$. But here T is not necessarily compact (which is different from the K in Def 3). In addition, it is shown in [1, Theorem 1.4] that any compact infra-solvamanifold is diffeomorphic to some standard Γ-manifold.

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