Continuum spectrum for the linearized extremal eigenvalue problem with boundary reactions

Futoshi Takahashi
Department of Mathematics, Osaka City University
& Osaka City University Advanced Mathematical Institute
Sumiyoshi-ku, Osaka, 558-8585, Japan
Tel: (+81)(0)6-6605-2508
E-mail: futoshi@sci.osaka-cu.ac.jp

Abstract: We study the semilinear problem with the boundary reaction

$$-\Delta u + u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = \lambda f(u) \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$, $N \ge 2$, is a smooth bounded domain, $f : [0, +\infty) \to (0, +\infty)$ is a smooth, strictly positive, convex, increasing function with superlinear at $+\infty$, and $\lambda > 0$ is a parameter. It is known that there exists an extremal parameter $\lambda^* > 0$ such that a classical minimal solution exists for $\lambda < \lambda^*$, and there is no solution for $\lambda > \lambda^*$. Moreover there is a unique weak solution u^* corresponding to the parameter $\lambda = \lambda^*$. In this paper, we continue to study the spectral properties of u^* and show a phenomenon of continuum spectrum for the corresponding linearized eigenvalue problem.

Keywords: continuum spectrum, extremal solution, boundary reaction.

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1 Introduction

In this paper, we consider the boundary value problem with the boundary reaction:

$$-\Delta u + u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = \lambda f(u) \quad \text{on } \partial\Omega \tag{1.1}$$

where $\lambda > 0$ and $\Omega \subset \mathbb{R}^N$, $N \ge 2$ is a smooth bounded domain. Throughout the paper, we assume

$$f : [0, +\infty) \to (0, +\infty) \text{ is smooth, convex, increasing, } f(0) > 0, \tag{1.2}$$

and superlinear at $+\infty$ in the sense that

$$\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty. \tag{1.3}$$

Then maximum principle implies that solutions are positive on $\overline{\Omega}$.

It is known that there exists an extremal parameter $\lambda^* \in (0,\infty)$ such that

(i) for every $\lambda \in (0,\lambda^*)$, $(1.1)_\lambda$ has a positive, classical, minimal solution $u_\lambda \in C^2(\overline{\Omega})$ which is strictly stable in the sense that

$$\int_\Omega (|\nabla\varphi|^2 + \varphi^2)dx > \lambda \int_{\partial\Omega} f'(u_\lambda)\varphi^2 ds_x \tag{1.4}$$

for every $\varphi \in C^1(\overline{\Omega}), \varphi \not\equiv 0$,

(ii) for $\lambda = \lambda^*$, the pointwise limit

$$u^*(x) = \lim_{\lambda\uparrow\lambda^*} u_\lambda(x), \quad x \in \overline{\Omega}, \tag{1.5}$$

becomes a weak solution of $(1.1)_{\lambda^*}$,

(iii) for $\lambda > \lambda^*$, there exists no solution of $(1.1)_\lambda$, even in the weak sense.

Here, we call a function $u = (u_1, u_2) \in L^1(\Omega) \times L^1(\partial\Omega)$ a *weak solution* to $(1.1)_\lambda$ if $f(u_2) \in L^1(\partial\Omega)$ and

$$\int_\Omega (-\Delta\zeta + \zeta)u_1 dx = \int_{\partial\Omega} (\lambda f(u_2)\zeta - \frac{\partial\zeta}{\partial\nu}u_2)ds_x \tag{1.6}$$

holds for any $\zeta \in C^2(\overline{\Omega})$. The statement (ii) says, under the assumption (1.3), $u^* = (u^*|_\Omega, u^*|_{\partial\Omega})$ is a weak solution in the sense above. If a weak solution u to (1.1) in the sense above satisfies $u \in W^{1,q}(\Omega)$, then $u_1 = u|_\Omega$ and $u_2 = u|_{\partial\Omega}$ where $u|_{\partial\Omega} \in W^{1-\frac{1}{q},q}(\partial\Omega) \subset L^{\frac{(N-1)q}{N-q}}(\partial\Omega)$ is the usual trace of $W^{1,q}$ function u on $\partial\Omega$. For the facts (ii), (iii), we refer the reader to [7]. In the following, we call u^* the *extremal solution* of (1.1). In [7], the author obtained several properties such as regularity and uniqueness of the extremal solution u^*. This paper is a sequel to [7]. For a well-studied problem

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

where f satisfies (1.2), (1.3), see [1], [2], [3], [4], [5], [6], and the references therein.

For $\lambda \in (0, \lambda^*)$, we denote by $\mu_1(\lambda f'(u_\lambda))$ the first eigenvalue of the following eigenvalue problem

$$-\Delta\varphi + \varphi = 0 \quad \text{in } \Omega, \quad \frac{\partial \varphi}{\partial \nu} = \lambda f'(u_\lambda)\varphi + \mu\varphi \quad \text{on } \partial\Omega.$$

By the variational characterization, we have

$$\mu_1(\lambda f'(u_\lambda)) = \inf_{\varphi \in C^1(\overline{\Omega}), \varphi \not\equiv 0} \frac{\int_\Omega (|\nabla\varphi|^2 + \varphi^2)\, dx - \int_{\partial\Omega} \lambda f'(u_\lambda)\varphi^2 ds_x}{\int_{\partial\Omega} \varphi^2 ds_x}.$$

Note that $\mu_1(\lambda f'(u_\lambda)) > 0$ since the minimal solution u_λ is strictly stable, and decreases as $\lambda \uparrow \lambda^*$. Denote

$$\mu_1^* = \lim_{\lambda \uparrow \lambda^*} \mu_1(\lambda f'(u_\lambda)). \tag{1.7}$$

If u^* is classical, it must hold that $\mu_1^* = 0$ by considering (iii) above. However if $u^* = (u^*|_\Omega, u^*|_{\partial\Omega}) \notin L^\infty(\Omega) \times L^\infty(\partial\Omega)$, it could be happen that μ_1^* is positive. In [7], we proved that even when $\mu_1^* > 0$, there exists a nonnegative weak solution of

$$-\Delta\varphi + \varphi = 0 \quad \text{in } \Omega, \quad \frac{\partial \varphi}{\partial \nu} = \lambda^* f'(u^*)\varphi + \mu\varphi \quad \text{on } \partial\Omega \tag{1.8}$$

for $\mu = 0$. This is a phenomenon of the existence of (L^1-) zero eigenvalue for the eigenvalue problem (1.8). Main purpose of this paper is to prove the following result, which might be seen as a phenomenon of the existence of (L^1-) continuum spectrum for the eigenvalue problem (1.8).

Theorem 1 *Let μ_1^* be defined by (1.7). Then for any $\mu \in [0, \mu_1^*]$, there exists a weak solution φ to (1.8), $\varphi \in W^{1,q}(\Omega)$ $(1 \le q < \frac{N}{N-1})$, $\varphi \ge 0$, in the sense that $f'(u^*)\varphi|_{\partial\Omega} \in L^1(\partial\Omega)$ and*

$$\int_\Omega (-\Delta\zeta + \zeta)\varphi dx = \int_{\partial\Omega} \left\{ (\lambda^* f'(u^*)\varphi|_{\partial\Omega} + \mu\varphi|_{\partial\Omega})\, \zeta - \frac{\partial \zeta}{\partial \nu}\varphi|_{\partial\Omega} \right\} ds_x$$

for all $\zeta \in C^2(\overline{\Omega})$. Here $\varphi|_{\partial\Omega}$ is the usual trace of $\varphi \in W^{1,q}(\Omega)$.

2 Proof of Theorem 1

In this section, we prove Theorem 1. We need the uniqueness theorem from [7], which is an analogue of the result by Y. Martel [6].

Theorem 2 *([7] Theorem 14) Assume* $(1.1)_{\lambda^*}$ *has a weak supersolution* $w = (w_1, w_2) \in L^1(\Omega) \times L^1(\partial\Omega)$*, in the sense that* $f(w_2) \in L^1(\partial\Omega)$ *and*

$$\int_\Omega (-\Delta\zeta + \zeta) w_1 dx \geq \int_{\partial\Omega} \left\{ \lambda^* f(w_2)\zeta - \frac{\partial\zeta}{\partial\nu} w_2 \right\} ds_x$$

for any $\zeta \in C^2(\overline{\Omega})$*,* $\zeta \geq 0$ *on* $\overline{\Omega}$*. Then* $(w_1, w_2) = (u^*|_\Omega, u^*|_{\partial\Omega})$*, where* u^* *is defined by (1.5).*

The following is Lemma 17 in [7].

Lemma 3 *Let* $\{u_n\} \subset C^2(\overline{\Omega})$ *be a sequence of functions such that*

$$-\Delta u_n + u_n = 0 \quad in\, \Omega, \quad \frac{\partial u_n}{\partial\nu} \geq 0 \quad on\, \partial\Omega.$$

Assume $\|u_n\|_{L^1(\partial\Omega)} \leq C$ *for some* $C > 0$ *independent of* n*. Then there exists a subsequence (denoted again by* u_n*) and* $u \in W^{1,q}(\Omega)$ *such that*

$$\begin{aligned} u_n &\rightharpoonup u \quad \textit{weakly in } W^{1,q}(\Omega),\ 1 < q < \frac{N}{N-1}, \\ u_n &\to u \quad \textit{strongly in } L^p(\partial\Omega),\ 1 \leq p < \frac{N-1}{N-2}. \end{aligned}$$

Moreover, for any $1 \leq p < \frac{N-1}{N-2}$*, there exists a constant* $C_p > 0$ *depending only on* p *such that*

$$\|u_n\|_{L^p(\partial\Omega)} \leq C_p \|u_n\|_{L^1(\partial\Omega)}$$

holds true for any $n \in \mathbb{N}$*.*

Now, we prove Theorem 1.

Proof.
We follow the argument by X. Cabré and Y. Martel [3].

Step 1. For $n \in \mathbb{N}$, define a sequence of functions f_n as

$$f_n(s) = \begin{cases} f(s) & \text{if } s \leq n, \\ f(n) + f'(n)(s-n) & \text{if } s > n, \end{cases}$$

and consider the approximated problem

$$-\Delta u + u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = \lambda f_n(u) \quad \text{on } \partial\Omega. \tag{2.1}$$

Denote $\lambda_n^* = \sup\{\lambda > 0 : (2.1)_\lambda \text{ admits a minimal solution} \in C^2(\overline{\Omega})\}$, and let $u_{n,\lambda} \in C^2(\overline{\Omega})$ be the classical minimal solution to $(2.1)_\lambda$ for $\lambda < \lambda_n^*$. Since $f_n \le f_{n+1} \le f$, we have $u_{n,\lambda} \le u_{n+1,\lambda} \le u_\lambda$ and $\lambda^* \le \lambda_{n+1}^* \le \lambda_n^*$ for any $n \in \mathbb{N}$. Define

$$\mu_1(\lambda f_n'(u_{n,\lambda})) = \inf_{\varphi \in C^1(\overline{\Omega}), \varphi \not\equiv 0} \frac{\int_\Omega (|\nabla \varphi|^2 + \varphi^2)\, dx - \int_{\partial\Omega} \lambda f_n'(u_{n,\lambda}) \varphi^2 ds_x}{\int_{\partial\Omega} \varphi^2 ds_x}. \tag{2.2}$$

Note that $\mu_1(\lambda f_n'(u_{n,\lambda}))$ is continuous with respect to λ by (2.2). Take $0 \le \mu \le \mu_1^*$ where μ_1^* is defined by (1.7). Since u_{n,λ_n^*} is classical (which is because f_n is asymptotic linear) and there is no classical solution of $(2.1)_\lambda$ for $\lambda > \lambda_n^*$, the linearized problem around $(\lambda_n^*, u_{n,\lambda_n^*})$ must have zero eigenvalue. Thus

$$\mu_1(\lambda_n^* f_n'(u_{n,\lambda_n^*})) = 0 \le \mu \le \mu_1^* \le \mu_1(\lambda^* f_n'(u_{n,\lambda^*})),$$

here we have used the fact that $f_n' \le f'$ and $u_{n,\lambda} \le u_\lambda$, which implies $\mu_1(\lambda f'(u_\lambda)) \le \mu_1(\lambda f_n'(u_{n,\lambda}))$. By the Intermediate Value Theorem, there exists $\lambda_n \in [\lambda^*, \lambda_n^*]$ such that

$$\mu_1(\lambda_n f_n'(u_{n,\lambda_n})) = \mu,$$

which in turn implies there exists $\varphi_n > 0$ with $\int_{\partial\Omega} \varphi_n ds_x = 1$ such that

$$-\Delta \varphi_n + \varphi_n = 0 \quad \text{in } \Omega, \quad \frac{\partial \varphi_n}{\partial \nu} = \lambda_n f_n'(u_{n,\lambda_n})\varphi_n + \mu \varphi_n \quad \text{on } \partial\Omega. \tag{2.3}$$

Recall also that u_{n,λ_n} satisfies

$$-\Delta u_{n,\lambda_n} + u_{n,\lambda_n} = 0 \quad \text{in } \Omega, \quad \frac{\partial u_{n,\lambda_n}}{\partial \nu} = \lambda_n f_n(u_{n,\lambda_n}) \quad \text{on } \partial\Omega. \tag{2.4}$$

We claim there exists $n_0 \in \mathbb{N}$ such that

$$\|u_{n,\lambda_n}\|_{L^1(\partial\Omega)} \le C \quad \text{for any } n \ge n_0. \tag{2.5}$$

Indeed, let ψ_1 be the first eigenfunction of the Steklov type eigenvalue problem

$$-\Delta \psi_1 + \psi_1 = 0 \quad \text{in } \Omega, \quad \frac{\partial \psi_1}{\partial \nu} = \kappa_1 \psi_1 \quad \text{on } \partial\Omega \tag{2.6}$$

with the first eigenvalue κ_1, which is normalized as $\int_{\partial\Omega} \psi_1 ds_x = 1$. Multiplying (2.4) by ψ_1 and using Jensen's inequality for f_n, we obtain

$$\begin{aligned} &\kappa_1 \int_{\partial\Omega} \psi_1 u_{n,\lambda_n} ds_x = \lambda_n \int_{\partial\Omega} f_n(u_{n,\lambda_n})\psi_1 ds_x \\ &\geq \lambda_n f_n \left(\int_{\partial\Omega} \psi_1 u_{n,\lambda_n} ds_x \right) \geq \lambda^* f_n \left(\int_{\partial\Omega} \psi_1 u_{n,\lambda_n} ds_x \right). \end{aligned}$$

Put $a_n = \int_{\partial\Omega} \psi_1 u_{n,\lambda_n} ds_x$. Then we have

$$a_n \geq \left(\frac{\lambda^*}{\kappa_1} \right) f_n(a_n). \tag{2.7}$$

Assume the contrary that $f_n(a_n) = f'(n)(a_n - n) + f(n)$ for some $n \in \mathbb{N}$ sufficiently large. Then, since $a_n > n$ and $f(n) > \left(\frac{\kappa_1}{\lambda^*}\right) n$, $f'(n) > \left(\frac{\kappa_1}{\lambda^*}\right)$ for n sufficiently large by (1.2) and (1.3), we have, by (2.7),

$$\begin{aligned} &a_n \geq \left(\frac{\lambda^*}{\kappa_1} \right) f_n(a_n) = \left(\frac{\lambda^*}{\kappa_1} \right) \{ f'(n)(a_n - n) + f(n) \} \\ &> a_n - n + n = a_n, \end{aligned}$$

which is a contradiction. Thus we conclude there exists $n_0 \in \mathbb{N}$ such that $f_n(a_n) = f(a_n)$ for any $n \geq n_0$. Again, this and (2.7) implies $a_n \geq \left(\frac{\lambda^*}{\kappa_1}\right) f(a_n)$ for any $n \geq n_0$. Now, by the assumption f, we have $C > 0$ such that $f(s) \geq \frac{2\kappa_1}{\lambda^*} s - C$ holds for any $s > 0$. From this and the former estimate, we have $a_n \leq \left(\frac{\lambda^*}{\kappa_1}\right) C$ for $n \geq n_0$. This implies the claim (2.5).

Step 2. By (2.5), we have $\|u_{n,\lambda_n}\|_{L^1(\partial\Omega)} \leq C$ for some C independent of n. Also recall $\|\varphi_n\|_{L^1(\partial\Omega)} = 1$ for a solution φ_n of (2.3). Thus we can apply Lemma 3, which yields the existence of $w, \varphi \in L^1(\Omega)$, $\varphi \geq 0$ a.e. such that

$$\begin{aligned} &u_{n,\lambda_n} \rightharpoonup w, \quad \varphi_n \rightharpoonup \varphi \quad \text{weakly in } W^{1,q}(\Omega), \\ &u_{n,\lambda_n} \to w, \quad \varphi_n \to \varphi \quad \text{strongly in } L^p(\partial\Omega) \text{ and a.e. on } \partial\Omega \end{aligned} \tag{2.8}$$

for any $1 < q < \frac{N}{N-1}$ and $1 \leq p < \frac{N-1}{N-2}$. Since $\int_{\partial\Omega} \varphi ds_x = 1$, we see $\varphi \not\equiv 0$ on $\partial\Omega$.

In the following, we prove that $\lambda_n \downarrow \lambda^*$ as $n \to \infty$ and $w = u^*$. We will show that $w \in W^{1,q}(\Omega)$ is a weak supersolution in the sense of Theorem

2. Then the conclusion is obtained by Theorem 2. To prove that w is a weak supersolution, put $\overline{\lambda} = \inf_{n\in\mathbb{N}} \lambda_n$. Since $\lambda_n \geq \lambda^*$, we have $\overline{\lambda} \geq \lambda^*$. We observe that

$$\begin{aligned}\int_\Omega (-\Delta\zeta + \zeta)\, u_{n,\lambda_n} dx &= \lambda_n \int_{\partial\Omega} f_n(u_{n,\lambda_n})\zeta ds_x - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} u_{n,\lambda_n} ds_x \\ &\geq \overline{\lambda} \int_{\partial\Omega} f_n(u_{n,\lambda_n})\zeta ds_x - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} u_{n,\lambda_n} ds_x\end{aligned}$$

holds for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$. Using the fact that $u_{n,\lambda_n} \to w$ in $L^1(\Omega)$, $L^1(\partial\Omega)$ respectively and Fatou's lemma, we have

$$\begin{aligned}\int_\Omega (-\Delta\zeta + \zeta)\, w dx &\geq \overline{\lambda} \int_{\partial\Omega} f(w)\zeta ds_x - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} w ds_x \\ &\geq \lambda^* \int_{\partial\Omega} f(w)\zeta ds_x - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} w ds_x, \quad \forall\zeta \in C^2(\overline{\Omega}),\ \zeta \geq 0.\end{aligned}$$

This implies also $f(w) \in L^1(\partial\Omega)$ if we take $\zeta \equiv 1$. Thus, we conclude that w is a weak supersolution to $(1.1)_{\lambda^*}$

Step 3. Let φ_n, φ be as in Step 2. We claim that

$$\lambda_n f_n'(u_{n,\lambda_n})\varphi_n \to \lambda^* f'(u^*)\varphi \quad \text{strongly in } L^1(\partial\Omega) \tag{2.9}$$

as $n \to \infty$. For the proof, we invoke Vitali's Convergence Theorem. First, by (2.8), we see

$$\lambda_n f_n'(u_{n,\lambda_n}(x))\varphi_n(x) \to \lambda^* f'(u^*(x))\varphi(x) \quad \text{a.e. } x \in \partial\Omega$$

for a subsequence. Next, we prove the uniformly absolute continuous property of the sequence $\{\lambda_n f_n'(u_{n,\lambda_n})\varphi_n\}_{n\in\mathbb{N}}$. For that purpose, let $A \subset \partial\Omega$ and $\varepsilon > 0$ be given arbitrary. Since f_n is convex, we have

$$f_n\left(\frac{\chi_A(x)}{\varepsilon}\right) \geq f_n(u_{n,\lambda_n}(x)) + f_n'(u_{n,\lambda_n}(x))\left(\frac{\chi_A(x)}{\varepsilon} - u_{n,\lambda_n}(x)\right) \tag{2.10}$$

a.e. $x \in \partial\Omega$, here χ_A is the characteristic function of A. By (2.3) and (2.4), it holds that

$$\begin{aligned}\lambda_n \int_{\partial\Omega} f_n(u_{n,\lambda_n})\varphi_n ds_x &= \lambda_n \int_{\partial\Omega} f_n'(u_{n,\lambda_n}) u_{n,\lambda_n}\varphi_n ds_x + \mu \int_{\partial\Omega} u_{n,\lambda_n}\varphi_n ds_x \\ &\geq \lambda_n \int_{\partial\Omega} f_n'(u_{n,\lambda_n}) u_{n,\lambda_n}\varphi_n ds_x.\end{aligned} \tag{2.11}$$

Also easy consideration shows that

$$\left\{ f_n\left(\frac{\chi_A(x)}{\varepsilon}\right) - f(0)\right\} \varphi_n(x) \le f\left(\frac{1}{\varepsilon}\right)\varphi_n(x)\chi_A(x) \quad \text{a.e. on } \partial\Omega. \tag{2.12}$$

Thus by (2.10), (2.11) and (2.12), we have

$$\begin{aligned}
\int_{\partial\Omega} f_n'(u_{n,\lambda_n})\frac{\chi_A}{\varepsilon}\varphi_n ds_x &\le \int_{\partial\Omega} f_n\left(\frac{\chi_A}{\varepsilon}\right)\varphi_n ds_x + \int_{\partial\Omega} f_n'(u_{n,\lambda_n})u_{n,\lambda_n}\varphi_n ds_x - \int_{\partial\Omega} f_n(u_{n,\lambda_n})\varphi_n ds_x \\
&\le \int_{\partial\Omega} f_n\left(\frac{\chi_A}{\varepsilon}\right)\varphi_n ds_x \\
&= \int_{\partial\Omega}\left\{ f_n\left(\frac{\chi_A}{\varepsilon}\right) - f(0)\right\}\varphi_n ds_x + \int_{\partial\Omega} f(0)\varphi_n ds_x \\
&\le \int_{\partial\Omega} f\left(\frac{1}{\varepsilon}\right)\varphi_n\chi_A ds_x + f(0) \\
&\le f\left(\frac{1}{\varepsilon}\right)|A|^{\frac{1}{p'}}\|\varphi_n\|_{L^p(\partial\Omega)} + f(0) \\
&\le Cf\left(\frac{1}{\varepsilon}\right)|A|^{\frac{1}{p'}} + f(0) \qquad (2.13)
\end{aligned}$$

for any $1 \le p < \frac{N-1}{N-2}$, where $|A|$ denotes the $(N-1)$ dimensional measure of $A \subset \partial\Omega$ and $p' = \frac{p}{p-1}$. In (2.13) we have used $\|\varphi_n\|_{L^p(\partial\Omega)} \le C$ for some $C > 0$ independent of n by (2.8). Define

$$\delta(\varepsilon) = \left(\frac{f(0)}{f(\frac{1}{\varepsilon})C}\right)^{p'}.$$

Then for any $\varepsilon > 0$, we obtain $\int_A f_n'(u_{n,\lambda_n})\varphi_n ds_x \le 2f(0)\varepsilon$ if $A \subset \partial\Omega$ satisfies that $|A| < \delta(\varepsilon)$ by (2.13). This implies the uniform absolutely continuity of the sequence $\{\lambda_n f_n'(u_{n,\lambda_n})\varphi_n\}_{n\in\mathbb{N}}$. Also for any $\varepsilon > 0$, if we take $E \subset \partial\Omega$ such that $|\partial\Omega \setminus E| < \delta(\varepsilon)$ where $\delta(\varepsilon)$ is as above, we obtain that $\int_{\partial\Omega\setminus E}\lambda_n f_n'(u_{n,\lambda_n})\varphi_n ds_x \le C\varepsilon$. This implies the uniform integrability of $\{\lambda_n f_n'(u_{n,\lambda_n})\varphi_n\}_{n\in\mathbb{N}}$. Therefore, Vitali's Convergence Theorem assures the claim (2.9).

By (2.9), we pass to the limit $n \to \infty$ in the weak formulation of (2.3):

$$\int_\Omega (-\Delta\zeta + \zeta)\varphi_n dx = \int_{\partial\Omega}(\lambda_n f_n'(u_{n,\lambda_n}) + \mu)\varphi_n\zeta - \frac{\partial\zeta}{\partial\nu}\varphi_n ds_x, \quad \forall\zeta \in C^2(\overline{\Omega}),$$

and conclude that φ is a weak solution of

$$-\Delta\varphi + \varphi = 0 \quad \text{in } \Omega, \quad \frac{\partial\varphi}{\partial\nu} = \lambda^* f'(u^*)\varphi + \mu\varphi \quad \text{on } \partial\Omega.$$

Recall $\varphi \in W^{1,q}(\Omega)$ for any $1 \leq q < \frac{N}{N-1}$. The proof of Theorem 1 is finished. □

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