Cohomological non-rigidity of eight-dimensional complex projective towers

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Abstract. A complex projective tower or simply a $\mathbb{C}P$-tower is an iterated complex projective fibrations starting from a point. In this paper, we classify certain class of 8-dimensional $\mathbb{C}P$-towers up to diffeomorphism. As a consequence, we show that cohomological rigidity is not satisfied by the collection of 8-dimensional $\mathbb{C}P$-towers, i.e., there is a two distinct 8-dimensional $\mathbb{C}P$-towers which have the same cohomology rings.

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1. Introduction

Let $\mathcal{M}$ be a collection of diffeomorphism classes of smooth manifolds and $\mathbf{H}^*\mathcal{M}$ be the isomorphism classes of cohomology rings of manifolds in $\mathcal{M}$. Let $H^* : \mathcal{M} \to \mathbf{H}^*\mathcal{M}$ be the map defined by $M \in \mathcal{M} \mapsto H^*(M;\mathbb{Z})$. In general, H^* is not bijective. However, if we restrict the class of manifolds then this map sometimes becomes a bijection; e.g., if $\mathcal{M}$ is a collection of oriented 2-dimensional manifolds then it is well-known that the map H^* is bijective. We say such collection $\mathcal{M}$ is *cohomologically rigid* or $\mathcal{M}$ satisfies *cohomological rigidity*. The problem asking whether the map $H^* : \mathcal{M} \to \mathbf{H}^*\mathcal{M}$ is bijective or not is called a *cohomological rigidity problem*. In this paper, we study the cohomological rigidity problem for *complex projective towers* (or simply a $\mathbb{C}P$-*tower*) introduced in [**KuSu**].

A $\mathbb{C}P$-*tower* of height m is a sequence of complex projective fibrations

$$C_m \xrightarrow{\pi_m} C_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} C_1 \xrightarrow{\pi_1} C_0 = \{\text{a point}\} \tag{1.1}$$

where $C_i = P(\xi_{i-1})$ is the projectivization of a complex vector bundle ξ_{i-1} over C_{i-1}. We call each C_i the *ith stage* of the tower. If we forget the tower structure, then we call C_i an *(i-stage)* $\mathbb{C}P$-*manifolds*. In [**KuSu**], we show that the diffeomorphism types of 6-dimensional $\mathbb{C}P$-manifolds are

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determined by their cohomology rings, i.e., the collection of 6-dimensional $\mathbb{C}P$-manifolds $\mathcal{CPM}^6$ is cohomologically rigid. This is the generalization of the fact that the collection $\mathcal{GBM}^6$ of 6-dimensional generalized Bott manifolds is cohomologically rigid in [**CMS11**]. On the other hand, it is known that the collection $\mathcal{GBM}_2^{2n}$ of $2n$-dimensional 2-stage generalized Bott manifolds is also cohomologically rigid. The purpose of this paper is to show that the collection $\mathcal{CPM}_2^8$ of 8-dimensional 2-stage $\mathbb{C}P$-manifolds is not cohomologically rigid.

To state our main theorem, let us recall the theorem proved by Atiyah and Rees in [**AtRe**, (2.8) Theorem]. Let $\mathcal{VECT}_2(\mathbb{C}P^3)$ be the collection of vector bundle isomorphism classes of complex 2-dimensional vector bundles over $\mathbb{C}P^3$.

Theorem 1.1 (Atiyah-Rees). *There exists a bijective map $\phi : \mathcal{VECT}_2(\mathbb{C}P^3) \to \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ such that $\phi(\xi) = (\alpha(\xi), c_1(\xi), c_2(\xi))$, where $c_1(\xi)$ and $c_2(\xi)$ are the first and the second Chern classes of ξ, and $\alpha(\xi)$ is a mod 2 element which is 0 when $c_1(\xi)$ is odd.*

By Theorem 1.1, any element in $\mathcal{VECT}_2(\mathbb{C}P^3)$ can be denoted by $\eta_{(\alpha,c_1,c_2)}$, where $(\alpha, c_1, c_2) \in \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ such that $\alpha \equiv 0 \pmod 2$ when $c_1 \equiv 1 \pmod 2$. On the other hand, it can be seen easily that $P(\eta_{(\alpha,c_1,c_2)})$ is diffeomorphic to $P(\eta_{(0,1,c_2-(c_1^2-1)/4)})$ if $c_1 \equiv 1 \mod 2$, and is diffeomorphic to $P(\eta_{(\alpha,0,c_2-c_1^2/4)})$ if $c_1 \equiv 0 \mod 2$, see Lemma 3.2.

Let $N(u) := P(\eta_{(0,1,u)})$, and let $\mathcal{N} := \{N(u) \mid u \in \mathbb{Z}\}$. Similarly, let $M_\alpha(u) := P(\eta_{(\alpha,0,u)})$, and let $\mathcal{M} := \{M_\alpha(u) \mid \alpha \in \{0,1\}, u \in \mathbb{Z}\}$. We now state the main result of the paper (see Theorem 4.2 for (1) and see Theorem 5.2 for more precise statement of (2)).

Theorem 1.2. *For the classes $\mathcal{M}$ and $\mathcal{N}$, we have the following.*

(1) *The class $\mathcal{N}$ is cohomologically rigid. In fact, the following are equivalent:*
   (a) *$N(u)$ is diffeomorphic to $N(u')$;*
   (b) *$u = u'$;*
   (c) *$H^*(N(u);\mathbb{Z}) \cong H^*(N(u');\mathbb{Z})$ as graded rings.*
(2) *The class $\mathcal{M}$ is not cohomologically rigid. In fact, $H^*(M_0(u);\mathbb{Z}) \cong H^*(M_1(u);\mathbb{Z})$ as graded rings for all u, but if $\frac{u(u+1)}{12} \in \mathbb{Z}$ then $M_0(u)$ is not diffeomorphic, actually not homotopic, to $M_1(u)$.*

The second part of the theorem is proved in Proposition 5.4 by showing that $\pi_6(M_0(u)) \not\cong \pi_6(M_1(u))$ when $\frac{u(u+1)}{12} \in \mathbb{Z}$.

The organization of this paper is as follows. In Section 2, as examples of $\mathbb{C}P$-towers, we explain when flag manifolds admit the structure of $\mathbb{C}P$-tower. In Section 3, we recall some basic facts from [**KuSu**]. In Section 4, we show that $\mathcal{N}$ satisfies the cohomological rigidity. In Section 5, we compute the 6-dimensional homotopy group of the elements in some class of $\mathcal{M}$ and show that $\mathcal{M}$ does not satisfies the cohomological rigidity.

2. Flag manifolds of type A and C

The $\mathbb{C}P$-towers contain many interesting classes of manifolds. In the previous paper [**KuSu**], we introduce that generalized Bott manifolds or the Milnor surface admits the structure of $\mathbb{C}P$-towers. We first introduce the other two examples of $\mathbb{C}P$-towers. Let $\mathcal{CPM}_m^{2n}$ be the collection of $2n$-dimensional m-stage $\mathbb{C}P$-manifolds up to diffeomorphism.

Example 2.1. The flag manifold $\mathcal{F}l(\mathbb{C}^{n+1}) = \{\{0\} \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}\}$, called type A, is well-known to be diffeomorphic to the homogeneous space $U(n+1)/T^{n+1} (\cong SU(n+1)/T^n)$. We will show that the flag manifold $U(n+1)/T^{n+1}$ is a $\mathbb{C}P$-tower with height n. Recall that if M is a smooth manifold with free K action and H is a subgroup of K, then we have a diffeomorphism $M/H \cong M \times_K (K/H)$. Also recall that $\mathbb{C}P^n \cong U(n+1)/(T^1 \times U(n))$. By using these facts, it is easy to check that there is the following $\mathbb{C}P$-tower structure of height n in $U(n+1)/T^{n+1}$:

$$
\begin{array}{c}
U(n+1)\times_{(T^1\times U(n))}(U(n)\times_{(T^1\times U(n-1))}(U(n-1)\times_{(T^1\times U(n-2))}\cdots(U(3)\times_{(T^1\times U(2))}\mathbb{C}P^1)\cdots)\\
\downarrow\\
\vdots\\
\downarrow\\
U(n+1)\times_{(T^1\times U(n))}(U(n)\times_{(T^1\times U(n-1))}\mathbb{C}P^{n-2})\\
\downarrow\\
U(n+1)\times_{(T^1\times U(n))}\mathbb{C}P^{n-1}\\
\downarrow\\
\mathbb{C}P^n,
\end{array}
$$

where the $U(k)$ action on $\mathbb{C}P^{k-1}$ in each stage is induced from the usual $U(k)$ action on $\mathbb{C}^k$. Hence, the flag manifold $U(n+1)/T^{n+1}$ of type A is an element of $\mathcal{CPM}_n^{n^2+n}$.

Example 2.2. The flag manifold of type C is defined by the homogeneous space $Sp(n)/T^n$. We claim that $Sp(n)/T^n$ is a $\mathbb{C}P$-tower with height n. It is well known that $Sp(n)/(T^1\times Sp(n-1))\cong S^{4n-1}/T^1\cong\mathbb{C}P^{2n-1}$, because $Sp(n)/Sp(n-1)\cong S^{4n-1}$. By using this fact and the method similar to that demonstrated in Example 2.1, it is easy to check that there is the following $\mathbb{C}P$-tower structure of height n in $Sp(n)/T^n$:

$$
\begin{array}{c}
Sp(n)\times_{(T^1\times Sp(n-1))}(Sp(n-1)\times_{(T^1\times Sp(n-2))}\cdots(Sp(2)\times_{(T^1\times Sp(1))}\mathbb{C}P^1)\cdots)\\
\downarrow\\
\vdots\\
\downarrow\\
Sp(n)\times_{(T^1\times Sp(n-1))}(Sp(n-1)\times_{(T^1\times Sp(n-2))}\mathbb{C}P^{2n-5})\\
\downarrow\\
Sp(n)\times_{(T^1\times Sp(n-1))}\mathbb{C}P^{2n-3}\\
\downarrow\\
\mathbb{C}P^{2n-1},
\end{array}
$$

where the $Sp(k)$-action on $\mathbb{C}P^{2k-1}$ in each stage is induced from the $Sp(k)$-action on $\mathbb{C}^{2k}(\cong\mathbb{H}^k)$ induced by the following representation to $U(2k)$:

$$
A+Bj\longrightarrow\begin{pmatrix} A & -B\\ \overline{B} & \overline{A}\end{pmatrix}.
$$

Here $A,\ B\in M(k;\mathbb{C})$ satisfy $A\overline{A}+B\overline{B}=I_k$ and $BA-AB=O$. Hence, the flag manifold $Sp(n)/T^n$ of type C is an element of $\mathcal{CPM}_n^{2n^2}$.

Remark 2.3. As is well-known, both of the flag manifolds $U(n+1)/T^{n+1}$ and $Sp(n)/T^n$ with $n\geq 2$ do not admit the structure of a *toric manifold* (see e.g. [**BuPa**]). On the other hand, $U(2)/T^2\cong Sp(1)/T^1\cong\mathbb{C}P^1$ is a toric manifold.

Moreover, by computing the generators of flag manifolds of other types (B_n $(n\geq 3)$, D_n $(n\geq 4)$, G_2, F_4, E_6, E_7, E_8), they do not admit the structure of $\mathbb{C}P$-towers, see [**Bo**] (or [**FIM**] for classical types). Namely, we have the following proposition:

Proposition 2.4. Let M be a flag manifold denoted by G/T, where G is a compact simple Lie group and T is its maximal torus. If M admits the structure of a $\mathbb{C}P$-tower, then G must be a compact Lie group of type A or C.

The following problem also naturally arises (also see Remark 5.5).

Problem 2.5. *Let $H^*:\mathcal{CPM}\to\boldsymbol{H}^*\mathcal{CPM}$ be the map defined by taking the cohomology rings. Classify diffeomorphism types of all manifolds in the class $(H^*)^{-1}(H^*(U(n+1)/T^{n+1}))$ and $(H^*)^{-1}(H^*(Sp(n)/T^n))$.*

3. Some preliminaries

In this section, we recall some basic facts.

3.1. Preliminaries from [KuSu]. We first recall some basic facts from [**KuSu**, Section 2].

Let ξ be an n-dimensional complex vector bundle over a topological space X, and let $P(\xi)$ denote its projectivization. Then, the following formula holds (see [**KuSu**]):

$$H^*(P(\xi);\mathbb{Z}) \cong H^*(X;\mathbb{Z})[x]/\langle x^{n+1} + \sum_{i=1}^{n}(-1)^i c_i(\pi^*\xi)x^{n+1-i}\rangle \tag{3.1}$$

where $\pi^*\xi$ is the pull-back of ξ along $\pi : P(\xi) \to X$ and $c_i(\pi^*\xi)$ is the ith Chern class of $\pi^*\xi$. Here x can be viewed as the first Chern class of the canonical line bundle over $P(\xi)$, i.e., the complex 1-dimensional sub-bundle γ_ξ in $\pi^*\xi \to P(\xi)$ such that the restriction $\gamma_\xi|_{\pi^{-1}(a)}$ is the canonical line bundle over $\pi^{-1}(a) \cong \mathbb{C}P^{n-1}$ for all $a \in X$. Therefore $\deg x = 2$. Since it is well-known that the induced homomorphism $\pi^* : H^*(X;\mathbb{Z}) \to H^*(P(\xi);\mathbb{Z})$ is injective, we often abuse the notation $c_i(\pi^*\xi)$ by $c_i(\xi)$. The formula (3.1) is called the *Borel-Hirzebruch formula*.

In order to prove the main theorem, we often use the following two lemmas.

Lemma 3.1. *Let γ be any line bundle over M, and let $P(\xi)$ be the projectivization of a complex vector bundle ξ over M. Then, $P(\xi)$ is diffeomorphic to $P(\xi \otimes \gamma)$.*

Lemma 3.2. *Let γ be a complex line bundle, and let ξ be a 2-dimensional complex vector bundle over a manifold M. Then the Chern classes of the tensor product $\xi \otimes \gamma$ are as follows.*

$$c_1(\xi \otimes \gamma) = c_1(\xi) + 2c_1(\gamma);$$
$$c_2(\xi \otimes \gamma) = c_1(\gamma)^2 + c_1(\gamma)c_1(\xi) + c_2(\xi).$$

3.2. Atiyah-Rees's theorem. By Theorem 1.1, all of the complex 2-plane bundles over $\mathbb{C}P^3$ can be denoted by $\eta_{(\alpha,c_1,c_2)}$ for some $(\alpha, c_1, c_2) \in \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$. Using Lemma 3.1, its projectivization $P(\eta_{(\alpha,c_1,c_2)})$ is diffeomorphic to $P(\eta_{(\alpha,c_1,c_2)} \otimes \gamma)$ for any line bundle γ over $\mathbb{C}P^3$. Moreover, by Lemma 3.2 and the proof of Theorem 1.1 in [**AtRe**], we also have

$$\eta_{(\alpha,c_1,c_2)} \otimes \gamma \equiv \eta_{(\alpha,c_1+2c_1(\gamma),c_1(\gamma)^2+c_1(\gamma)c_1+c_2)}.$$

Therefore, we may assume $c_1 \in \{0, 1\}$. Consequently, in order to classify all $P(\eta_{(\alpha,c_1,c_2)})$ up to diffeomorphisms, it is enough to classify the following:

$$M_0(u) = P(\eta_{(0,0,u)});$$
$$M_1(u) = P(\eta_{(1,0,u)});$$
$$N(u) = P(\eta_{(0,1,u)}),$$

where $u \in \mathbb{Z}$. We denote the class of $M_0(u)$, $M_1(u)$ up to diffeomorphism by $\mathcal{M}$ and that of $N(u)$ by $\mathcal{N}$. Then, both classes $\mathcal{M}$ and $\mathcal{N}$ are the subclasses of $\mathcal{CPM}_2^8$ consisting of 8-dimensional 2-stage $\mathbb{C}P$-manifolds.

3.3. Intersection of two classes $\mathcal{M}$ and $\mathcal{N}$ are empty. Finally, in this section, we prove $\mathcal{M} \cap \mathcal{N} = \emptyset$ by comparing their cohomology rings. Namely, we prove the following lemma:

Lemma 3.3. *Two cohomology rings $H^*(M_\alpha(u))$ and $H^*(N(u'))$ are not isomorphic for any u, $u' \in \mathbb{Z}$.*

Proof. By the Borel-Hirzebruch formula (3.1), we have ring isomorphisms

$$H^*(M_\alpha(u)) \cong \mathbb{Z}[X, Y]/\langle X^4,\ uX^2 + Y^2\rangle, \quad \text{and}$$
$$H^*(N(u')) \cong \mathbb{Z}[x, y]/\langle x^4,\ u'x^2 + xy + y^2\rangle.$$

Assume that there is an isomorphism map $f : H^*(M_\alpha(u)) \to H^*(N(u'))$. Then we may put

$$f(X) = ax + by, \quad \text{and}$$
$$f(Y) = cx + dy,$$

for some $a,\ b,\ c,\ d \in \mathbb{Z}$ such that $ad - bc = \epsilon = \pm 1$. By taking the inverse of f, we also have

$$\begin{aligned} f^{-1}(x) &= d\epsilon X - b\epsilon Y, \quad \text{and} \\ f^{-1}(y) &= -c\epsilon X + a\epsilon Y. \end{aligned}$$

From the ring structures of $H^*(M_\alpha(u))$ and $H^*(N(u'))$, we have $f(uX^2 + Y^2) = 0$ and $f^{-1}(y^2 + xy + u'x^2) = 0$. Therefore we have the following equations:

$$u(a^2 - u'b^2) + (c^2 - u'd^2) = 0; \tag{3.2}$$

$$u(2ab - b^2) + (2cd - d^2) = 0; \tag{3.3}$$

$$c^2 - a^2u - cd + abu + u'd^2 - b^2uu' = 0; \tag{3.4}$$

$$-2ac + cb + ad - 2bdu' = 0. \tag{3.5}$$

Because $f^{-1}(x^4) = (dX - bY)^4 = 0$, we also have

$$bd(d^2 - ub^2) = 0.$$

Therefore $bd = 0$, or otherwise $d^2 = ub^2$. We first assume $bd = 0$. Then, there are two cases: $b = 0$ and $d = 0$. If $b = 0$, then $|a| = |d| = 1$. However, by using (3.3), we have $2cd = 1$. This gives a contradiction. If $d = 0$, then $|b| = |c| = 1$. By using (3.5), we have $c(-2a + b) = 0$, i.e., $b = 2a$ by $|c| = 1$. However, this contradicts to $|b| = 1$. Hence, $bd \neq 0$ and $d^2 = ub^2$, i.e., $|d| = \sqrt{|u|}|b|$. In this case, because $ad - bc = \epsilon = \pm 1$, we have $|b| = 1$ and $d^2 = u$. Let $b = \epsilon' = \pm 1$ and $d = \sqrt{u}\epsilon''$, where $\epsilon'' = \pm 1$. Then, it follows from $ad - bc = \epsilon$ that $c = -\epsilon\epsilon' + a\sqrt{u}\epsilon''\epsilon'$. Therefore, by using (3.2), we have the following equation:

$$\begin{aligned} & u(a^2 - u'b^2) + (c^2 - u'd^2) \\ = \ & u(a^2 - u') + (-\epsilon\epsilon' + a\sqrt{u}\epsilon''\epsilon')^2 - u'u \\ = \ & 2ua^2 - 2uu' + 1 - 2a\sqrt{u}\epsilon\epsilon'' = 0. \end{aligned}$$

However, this gives the equation $1 = 2(-ua^2 + uu' + a\sqrt{u}\epsilon\epsilon'')$, which is a contradiction. Hence, $H^*(M_\alpha(u)) \not\cong H^*(N(u'))$ for all $u,\ u' \in \mathbb{Z}$. □

Hence, we have the following corollary:

Corollary 3.4. There are no intersections between two classes $\mathcal{M}$ and $\mathcal{N}$.

4. Cohomological rigidity of $\mathcal{N}$

In this section, we shall prove the cohomological rigidity of the class $\mathcal{N}$. To show that, it is enough to prove the following lemma.

Lemma 4.1. *The following two statements are equivalent.*

(1) $H^*(N(u)) \cong H^*(N(u'))$
(2) $u = u' \in \mathbb{Z}$

Proof. Because (2) $\Rightarrow$ (1) is trivial, it is enough to show (1) $\Rightarrow$ (2). Assume there is an isomorphism $f : H^*(N(u)) \cong H^*(N(u'))$ where

$$\begin{aligned} H^*(N(u)) &\cong \mathbb{Z}[X, Y]/\langle X^4,\ uX^2 + xy + Y^2\rangle; \\ H^*(N(u')) &\cong \mathbb{Z}[x, y]/\langle x^4,\ u'x^2 + xy + y^2\rangle. \end{aligned}$$

Again, we use the same representation for f as in the proof of Lemma 3.3. Because $f(Y^2 + XY + uX^2) = 0$ and $f^{-1}(y^2 + xy + u'x^2) = 0$, we have that

$$c^2 - d^2u' = -ua^2 + b^2uu' - ac + bdu'; \tag{4.1}$$

$$2cd - d^2 = -2abu + b^2u - ad - bc + bd; \tag{4.2}$$

$$c^2 - a^2u = -u'd^2 + b^2uu' + cd - bau; \tag{4.3}$$

$$-2ac - a^2 = 2bdu' + b^2u' - ad - bc - ab. \tag{4.4}$$

Because $f(X^4) = 0$ and $f^{-1}(x^4) = 0$, there are the following two cases:

(1) $b = 0$;

(2) $b \neq 0$ and $4a^3-6a^2b+4ab^2(1-u')+b^3(2u'-1) = -4d^3-6d^2b-4db^2(1-u)+b^3(2u-1) = 0$.

If $b = 0$, then $|a| = |d| = 1$. Therefore, by (4.2), $2c = d - a$, i.e., $c = 0$ if $d = a$ or $c = -a$ if $d = -a$. Because $c^2 - u' = -u - ac$ by (4.1), we have that $u = u'$.

Assume $b \neq 0$. By the equation $4a^3 - 6a^2b + 4ab^2(1 - u') + b^3(2u' - 1) = 0$, we have b is even. Substituting $a = A + \frac{b}{2}$ for some $A \in \mathbb{Z}$ to this equation (i.e., Tschirnhaus's transformation), we have the following equation:

$$\begin{aligned}
& 4(A+\frac{b}{2})^3 - 6(A+\frac{b}{2})^2b + 4(A+\frac{b}{2})b^2(1-u') + b^3(2u'-1) \\
= \; & 4(A^3 + 3A^2\frac{b}{2} + 3A\frac{b^2}{4} + \frac{b^3}{8}) - 6(A^2 + Ab + \frac{b^2}{4})b + 4(Ab^2 + \frac{b^3}{2})(1-u') + b^3(2u'-1) \\
= \; & 4A^3 + 6A^2b + 3Ab^2 + \frac{b^3}{2} - 6A^2b - 6Ab^2 - \frac{3b^3}{2} + 4Ab^2 + 2b^3 - 4Ab^2u' - 2b^3u' + 2b^3u' - b^3 \\
= \; & 4A^3 + Ab^2 - 4Ab^2u' \\
= \; & A(4A^2 + b^2 - b^2u') = 0
\end{aligned}$$

Therefore, there are the two cases: $A = 0$ or $A \neq 0$. We first assume $A \neq 0$. Then, by using the equation $4A^2 + b^2 - b^2u' = 0$, we have $u' \geq 1$. Now, there is the following commutative diagram:

$$\begin{array}{ccc}
H^2(N(u)) = \mathbb{Z}X \oplus \mathbb{Z}Y & \xrightarrow{X} & \mathbb{Z}X^2 \oplus \mathbb{Z}XY = H^4(N(u)) \\
\downarrow f & & \downarrow f \\
H^2(N(u')) = \mathbb{Z}x \oplus \mathbb{Z}y & \xrightarrow{ax+by} & \mathbb{Z}x^2 \oplus \mathbb{Z}xy = H^4(N(u'))
\end{array}$$

Because X and f are isomorphisms, so is $ax + by$ in the diagram. Using the indicated generators as bases, the determinant of the map $f \circ X : H^2(N(u)) \to H^4(N(u'))$ is equal to the determinant of the map $(ax + by) \circ f : H^2(N(u)) \to H^4(N(u'))$, which is equal to

$$a^2 - ab + b^2u' = \epsilon_1 = \pm 1. \tag{4.5}$$

Because $a \in \mathbb{Z}$, the discriminant of this equation satisfies

$$b^2 - 4(b^2u' - \epsilon_1) = b^2(1 - 4u') + 4\epsilon_1 \geq 0$$

Because $u' \geq 1$, we have that

$$0 < b^2 \leq \frac{4\epsilon_1}{4u' - 1} < 1.$$

This gives a contradiction to $b \in \mathbb{Z}$. Therefore, we have $A = 0$, i.e., $a = \frac{b}{2}$. Because $ad - bc = \epsilon(= \pm 1)$, we also have that $a = \epsilon' = \pm 1$, $b = 2\epsilon'$ and $d - 2c = \epsilon\epsilon'$. Hence, by (4.5), we have $-1 + 4u' = \epsilon_1$, i.e., $u' = 0$ and $\epsilon_1 = -1$. By applying a similar method to the one used to derive (4.5) for $f^{-1}(x)$, we have

$$d^2 + db + b^2u = \epsilon_2 = \pm 1. \tag{4.6}$$

Substituting (4.5) and (4.6) to (4.3) and (4.4), we have

$$\begin{aligned}
c^2 &= u\epsilon_1 - u'd^2 + cd = -u + cd; \\
-2ac &= \epsilon_1 + 2bdu' - ad - bc = -1 - (d + 2c)\epsilon'.
\end{aligned}$$

By using the second equation above, we also have $d = -\epsilon'$; therefore, by $d - 2c = \epsilon\epsilon'$, we have $c = \frac{-\epsilon' - \epsilon\epsilon'}{2} = 0$ or $-\epsilon'$. If $c = 0$, then $u = 0$ by the first equation above; if $c = -\epsilon'$ then we also have $u = 0$ by $d = -\epsilon'$ and the first equation above. This implies that $u = u' = 0$ for the case $b \neq 0$.

This establishes the statement. □

Therefore, by Theorem 1.1 and Lemma 4.1, we have the following theorem.

THEOREM 4.2. *The following three statements are equivalent.*

(1) *Two spaces $N(u)$ and $N(u')$ are diffeomorphic.*
(2) *Two cohomology rings $H^*(N(u))$ and $H^*(N(u'))$ are isomorphic.*
(3) $u = u' \in \mathbb{Z}$.

In particular, the class $\mathcal{N}$ is cohomologically rigid.

This establishes Theorem 1.2 (1).

5. Cohomological non-rigidity of $\mathcal{CPM}_2^8$

In this section, we prove that $\mathcal{M}$ is not cohomologically rigid. We first show the following fact about the cohomology rings of elements in $\mathcal{M}$.

Lemma 5.1. *The following two statements are equivalent.*
(1) $H^*(M_\alpha(u)) \cong H^*(M_{\alpha'}(u'))$ *where* $\alpha,\ \alpha' \in \{0,1\}$.
(2) $u = u' \in \mathbb{Z}$

Proof. Because (2) $\Rightarrow$ (1) is trivial, it is enough to show (1) $\Rightarrow$ (2). Assume there is an isomorphism $f : H^*(M_\alpha(u)) \cong H^*(M_{\alpha'}(u'))$ where

$$H^*(M_\alpha(u)) \cong \mathbb{Z}[X,Y]/\langle X^4,\ uX^2+Y^2\rangle;$$
$$H^*(M_{\alpha'}(u')) \cong \mathbb{Z}[x,y]/\langle x^4,\ u'x^2+y^2\rangle.$$

We may use the same representation for f as in the proof of Lemma 3.3. Note that $f(uX^2+Y^2)=0$ and $f^{-1}(u'x^2+y^2)=0$. By using the representation of f, we have the following equations:

$$ua^2 - uu'b^2 + c^2 - u'd^2 = 0; \tag{5.1}$$
$$uab + cd = 0; \tag{5.2}$$
$$u'd^2 - uu'b^2 + c^2 - a^2u = 0; \tag{5.3}$$
$$u'bd + ac = 0. \tag{5.4}$$

By (5.1) and (5.3), we have

$$c^2 = b^2uu'; \tag{5.5}$$
$$ua^2 = u'd^2. \tag{5.6}$$

Because $X^4 = 0$, we also have that

$$ab(a^2 - b^2u') = 0.$$

We first assume $ab \neq 0$. Then

$$a^2 = b^2u'$$

by this equation. Together with (5.5) and (5.6), we have that

$$c^2b^2 = b^4uu' = b^2a^2u = b^2d^2u' = a^2d^2.$$

This implies that

$$(ad - bc)(ad + bc) = \epsilon(ad + bc) = 0.$$

Hence, $ad = -bc$. However this gives a contradiction because $ad - bc = 2ad = \epsilon = \pm 1$. Consequently, we have $ab = 0$. Since $ad - bc = \epsilon$, if $a = 0$ then $|b| = |c| = 1$; therefore, we have $u = u' = \pm 1$ by (5.5); if $b = 0$ then $|a| = |d| = 1$; therefore, we have $u = u'$ by (5.6). This establishes the statement. $\square$

Lemma 5.1 says that cohomology rings of $\mathcal{M}$ are not affected by $\alpha \in \mathbb{Z}_2$. On the other hand, the goal of this section is to prove the following theorem, i.e., some topological types of $\mathcal{M}$ are affected by $\alpha \in \mathbb{Z}_2$.

Theorem 5.2. *Assume $u(u+1)/12 \in \mathbb{Z}$. The following three statements are equivalent.*
(1) *Two spaces $M_\alpha(u)$ and $M_\beta(u')$ are diffeomorphic.*
(2) $(\alpha, u) = (\beta, u') \in \mathbb{Z}_2 \times \mathbb{Z}$.
(3) *Two spaces $M_\alpha(u)$ and $M_\beta(u')$ are homotopy equivalent.*

In order to prove Theorem 5.2, we first compute the 6-dimensional homotopy group of $M_\alpha(u)$ in Proposition 5.4. Now $M_\alpha(u)$ can be defined by the following pull-back diagram:

$$\begin{tikzcd} M_\alpha(u) \arrow[r] \arrow[d] & EU(2)\times_{U(2)} \mathbb{C}P^1 \arrow[d] \\ \mathbb{C}P^3 \arrow[r, "\mu_{\alpha,u}"] & BU(2) \end{tikzcd}$$

Let $p : S^7 \to \mathbb{C}P^3$ be the canonical S^1-fibration and $P(\xi_{\alpha,u})$ be the pull-back of $M_\alpha(u)$ along p. Namely, we have the following diagram:

$$\begin{tikzcd} P(\xi_{\alpha,u}) \arrow[r] \arrow[d] & M_\alpha(u) \arrow[r] \arrow[d] & EU(2)\times_{U(2)} \mathbb{C}P^1 \arrow[d] \\ S^7 \arrow[r, "p"] & \mathbb{C}P^3 \arrow[r, "\mu_{\alpha,u}"] & BU(2) \end{tikzcd} \tag{5.7}$$

Then, we have the following lemma.

Lemma 5.3. *For* $* \geq 3$, $\pi_*(P(\xi_{\alpha,u})) \cong \pi_*(M_\alpha(u))$.

Proof. Because $P(\xi_{\alpha,u})$ is the pull-back of $M_\alpha(u)$, the homotopy exact sequences of $P(\xi_{\alpha,u})$ and $M_\alpha(u)$ satisfy the following commutative diagram:

$$\begin{tikzcd} \pi_{*+1}(S^7) \arrow[r] \arrow[d] & \pi_*(\mathbb{C}P^1) \arrow[r] \arrow[d] & \pi_*(P(\xi_{\alpha,u})) \arrow[r] \arrow[d] & \pi_*(S^7) \arrow[r] \arrow[d] & \pi_{*-1}(\mathbb{C}P^1) \arrow[d] \\ \pi_{*+1}(\mathbb{C}P^3) \arrow[r] & \pi_*(\mathbb{C}P^1) \arrow[r] & \pi_*(M_\alpha(u)) \arrow[r] & \pi_*(\mathbb{C}P^3) \arrow[r] & \pi_{*-1}(\mathbb{C}P^1) \end{tikzcd}$$

From the homotopy exact sequence of the fibration $S^1 \to S^7 \to \mathbb{C}P^3$, we have $\pi_*(S^7) \cong \pi_*(\mathbb{C}P^3)$ for $* \geq 3$. Therefore, by using the 5 lemma, we have the statement. □

Now we may prove the following proposition.

Proposition 5.4. Assume $u(u+1)/12 \in \mathbb{Z}$. The following two isomorphisms hold.

(1) $\pi_6(P(\xi_{\alpha,u})) \cong \pi_6(M_\alpha(u)) \cong \mathbb{Z}_{12}$ if $\alpha \equiv u(u+1)/12 \pmod 2$
(2) $\pi_6(P(\xi_{\beta,u})) \cong \pi_6(M_\beta(u)) \cong \mathbb{Z}_6$ if $\beta \not\equiv u(u+1)/12 \pmod 2$

Proof. We first claim the 1st statement. If $u(u+1)/12 \in \mathbb{Z}$ and $\alpha \equiv u(u+1)/12 \pmod 2$, then it follows from [**AtRe**] that $\xi_{\alpha,u}$ is induced from the rank 2 complex vector bundle over $\mathbb{C}P^4$. Namely, there is the following commutative diagram:

$$\begin{tikzcd} \xi_{\alpha,u} \arrow[r] \arrow[d] & \eta_{(\alpha,0,u)} \arrow[r] \arrow[d] & \widetilde{\mu}_{\alpha,u} \arrow[r] \arrow[d] & EU(2)\times_{U(2)} \mathbb{C}^2 \arrow[d] \\ S^7 \arrow[r, "p"] & \mathbb{C}P^3 \arrow[r] & \mathbb{C}P^4 \arrow[r] & BU(2) \end{tikzcd} \tag{5.8}$$

On the other hand, we have that $\pi_7(\mathbb{C}P^4) \cong \pi_7(S^9) = \{0\}$, by using the homotopy exact sequence for the fibration $S^1 \to S^9 \to \mathbb{C}P^4$. This implies that $\xi_{\alpha,u}$ is the trivial $\mathbb{C}^2$-bundle over S^7. Therefore,

$$P(\xi_{\alpha,u}) = S^7 \times \mathbb{C}P^1$$

when $u(u+1)/12 \in \mathbb{Z}$ and $\alpha \equiv u(u+1)/12 \pmod 2$. Hence, we also have that

$$\pi_6(M_\alpha(u)) \cong \pi_6(S^7 \times \mathbb{C}P^1) \cong \pi_6(\mathbb{C}P^1) \cong \mathbb{Z}_{12}.$$

Next we claim the 2nd statement. Let $\mu_{\alpha,u} : \mathbb{C}P^3 \to BU(2)$ be a continuous map which induces the above $\eta_{(\alpha,0,u)}$, and β be the element in $\mathbb{Z}_2$ which is not equal to α. Let $x \in \mathbb{C}P^3$ and $s = \mu_{\alpha,u}(x) \in BU(2)$ be base points. Take a disk neighborhood around $x \in \mathbb{C}P^3$ and pinch its boundary to a point, i.e., the boundary of $D^6 \subset \mathbb{C}P^3$ pinches to a point, then we obtain the surjective map

$$\rho : \mathbb{C}P^3 \to \mathbb{C}P^3 \vee S^6,$$

where $\mathbb{C}P^3 \vee S^6$ may be regarded as the wedge sum with respect to the base points $x \in \mathbb{C}P^3$ and $y \in S^6$. Due to theorem of Atiyah-Rees [**AtRe**], we have $\eta_{(\beta,0,u)} \not\equiv \eta_{(\alpha,0,u)}$. This implies that the vector bundle $\eta_{(\beta,0,u)}$ is induced from the following continuous map:

$$\mu_{\beta,u} : \mathbb{C}P^3 \xrightarrow{\rho} \mathbb{C}P^3 \vee S^6 \xrightarrow{\nu_\alpha} BU(2) \tag{5.9}$$

where $\nu_\alpha = \mu_{\alpha,u} \vee \kappa$ for the generator $\kappa \in \pi_6(BU(2), s) \cong \mathbb{Z}_2$.[1] Hence, we have the following commutative diagram.

$$\begin{tikzcd}
P(\xi_{\beta,u}) \arrow[r] \arrow[d] & M_\beta(u) \arrow[r] \arrow[d] & EU(2) \times_{U(2)} \mathbb{C}P^1 \arrow[d] \\
S^7 \arrow[r, "p"] \arrow[dr] & \mathbb{C}P^3 \arrow[r, "\mu_{\beta,u}"] \arrow[d, "\rho"] & BU(2) \\
& \mathbb{C}P^3 \vee S^6 \arrow[ur, "\nu_\alpha"] &
\end{tikzcd} \tag{5.10}$$

From the $\mathbb{C}P^1$-fibrations $\mathbb{C}P^1 \to P(\xi_{\beta,u}) \to S^7$ and $\mathbb{C}P^1 \to EU(2) \times_{U(2)} \mathbb{C}P^1 \cong BT^2 \to BU(2)$ in the above diagram (5.10), there is the following commutative diagram.

$$\begin{tikzcd}
\pi_7(S^7) \cong \mathbb{Z} \arrow[r] \arrow[d] & \pi_6(\mathbb{C}P^1) \arrow[r] \arrow[d, "\cong"] & \pi_6(P(\xi_{\beta,u})) \arrow[r] \arrow[d] & \pi_6(S^7) = \{0\} \arrow[d] \\
\pi_7(BU(2)) \cong \mathbb{Z}_{12} \arrow[r, "\cong"] & \pi_6(\mathbb{C}P^1) \arrow[r] & \pi_6(BT^2) = \{0\} \arrow[r] & \pi_6(BU(2)) \cong \mathbb{Z}_2
\end{tikzcd}$$

This diagram shows that the following exact sequence:

$$\mathbb{Z} \cong \pi_7(S^7) \to \pi_7(BU(2))(\cong \mathbb{Z}_{12}) \to \pi_6(P(\xi_{\beta,u})) \to \{0\}. \tag{5.11}$$

In this diagram, the left homomorphism is induced from $\widetilde{\mu} := \mu_{\beta,u} \circ p \colon S^7 \to BU(2)$, say $\widetilde{\mu}_\# : \mathbb{Z} \to \mathbb{Z}_{12}$. We claim $\widetilde{\mu}_\#(1) = [6]_{12} \in \mathbb{Z}_{12}$. Because the diagram (5.10) is commutative, we may regard that $\widetilde{\mu} := \mu_{\beta,u} \circ p \colon S^7 \to BU(2)$ can be defined by passing through the map $\nu_\alpha : \mathbb{C}P^3 \vee S^6 \to BU(2)$, i.e., $\widetilde{\mu} = \nu_\alpha \circ \rho \circ p$. Because $\nu_\alpha = \mu_{\alpha,u} \vee \kappa$, we also have

$$\widetilde{\mu} = (\mu_{\alpha,u} \vee \kappa) \circ \rho \circ p = (\mu_{\alpha,u} \circ \rho \circ p) \vee (\kappa \circ \rho \circ p).$$

By the argument when we proved the 1st statement, we see that $\mu_{\alpha,u} \circ \rho \circ p$ induces the trivial bundle over S^7, i.e., $\mu_{\alpha,u} \circ \rho \circ p$ is homotopic to the trivial map. This also implies that there is the following decomposition up to homotopy:

$$\widetilde{\mu} : S^7 \xrightarrow{p} \mathbb{C}P^3 \xrightarrow{\rho} \mathbb{C}P^3 \vee S^6 \xrightarrow{\pi} S^6 \xrightarrow{\kappa} BU(2),$$

where π is the collapsing map of $\mathbb{C}P^3$ to a point. Therefore, we have the following decomposition for the induced map

$$\widetilde{\mu}_\# : \pi_7(S^7) \xrightarrow{\Psi_\#} \pi_7(S^6) \cong \mathbb{Z}_2 \xrightarrow{\kappa_\#} \pi_7(BU(2)) \cong \mathbb{Z}_{12},$$

where the 1st map is induced from the surjective map $\Psi = \pi \circ \rho \circ p$. Because Ψ is non trivial map, $\Psi_\#(1) = [1]_2$ (the generator of $\pi_7(S^6) \cong \mathbb{Z}_2$). Moreover, because $\kappa \in \pi_6(BU(2)) \cong \mathbb{Z}_2$ is the generator, i.e., non-trivial map, we have $\kappa_\#([1]_2) = [6]_{12} \in \mathbb{Z}_{12}$. This shows that $\widetilde{\mu}_\#(1) = [6]_{12}$; therefore, $\widetilde{\mu}_\#(\pi_7(S^7)) = \{[0]_{12},\ [6]_{12}\} \subset \mathbb{Z}_{12}$.

Consequently, by the exact sequence (5.11), we have that

$$\pi_6(P(\xi_{\beta,u})) \cong \pi_7(BU(2))/\widetilde{\mu}_\#(\pi_7(S^7)) \cong \mathbb{Z}_{12}/\{[0]_{12}, [6]_{12}\} \cong \mathbb{Z}_6.$$

By Lemma 5.3, we have the statement. □

[1] This construction induces the free $\pi_6(BU(2)) \cong \pi_5(U(2)) \cong \mathbb{Z}_2$ action on $\widetilde{KSp}(\mathbb{C}P^3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ (see [**AtRe**]).

Remark 5.5. For example, the relation $u(u+1)/12 \in \mathbb{Z}$ is true for the case when $u = 0$ and $u = 3$. In these cases, by using Proposition 5.4, we have

$$\pi_6(M_\alpha(0)) \cong \begin{cases} \mathbb{Z}_{12} & \text{for } \alpha \equiv 0 \\ \mathbb{Z}_6 & \text{for } \alpha \equiv 1 \end{cases}$$

and

$$\pi_6(M_\alpha(3)) \cong \begin{cases} \mathbb{Z}_{6} & \text{for } \alpha \equiv 0 \\ \mathbb{Z}_{12} & \text{for } \alpha \equiv 1 \end{cases}$$

On the other hand, the case when $u = 1$ does not satisfy the relation $u(u+1)/12 \in \mathbb{Z}$. It follows from the cohomology ring of the flag manifold of type C (see e.g. [**Bo**] or [**FIM**]) that the flag manifold $Sp(2)/T^2$ is one of this case, i.e., $M_0(1)$ or $M_1(1)$. However, by using the homotopy exact sequence for the fibration $T^2 \to Sp(2) \to Sp(2)/T^2$ and the computation in [**MiTo**], we have that

$$\pi_6(Sp(2)/T^2) \cong \pi_6(Sp(2)) = 0.$$

Therefore, Proposition 5.4 is not true for the case when $u(u+1)/12 \notin \mathbb{Z}$.

Let us prove Theorem 5.2

Proof of Theorem 5.2. By using Theorem 1.1, (2) $\Rightarrow$ (1) is trivial. The statement (1) $\Rightarrow$ (3) is also trivial. We claim (3) $\Rightarrow$ (2). Assume $M_\alpha(u)$ and $M_\beta(u')$ are homotopy equivalent. Then, $H^*(M_\alpha(u)) \cong H^*(M_\beta(u'))$. Therefore, it follows from Lemma 5.1 that $u = u'$. Moreover, in this case, $\pi_6(M_\alpha(u)) \cong \pi_6(M_\beta(u))$. If $\alpha \not\equiv \beta \mod 2$, then this gives a contradiction to Proposition 5.2. Hence, $\alpha \equiv \beta \mod 2$. We have (3) $\Rightarrow$ (2). This establishes Theorem 5.2. □

In summary, by Lemma 5.1 and Theorem 5.2, we have the following corollary:

Corollary 5.6. The set of 8-dimensional $\mathbb{C}P$-manifolds does not satisfy the cohomological rigidity.

This establishes Theorem 1.2 (2).

Note that if we restrict the class of 8-dimensional $\mathbb{C}P$-manifolds to the 8-dimensional generalized Bott manifolds with height 2, then cohomological rigidity holds by [**CMS10**]. On the other hand, the following question seems to be natural to ask for the class of $\mathbb{C}P$-manifolds $\mathcal{CPM}$ instead of the cohomological rigidity problem.

Problem 5.7. *Is the class of $\mathbb{C}P$-manifolds $\mathcal{CPM}$ (up to diffeomorphism) determined by their homotopy types? More precisely, are $M_1, M_2 \in \mathcal{CPM}$ diffeomorphic if they have the same homotopy types?*

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