A MEAN VALUE PROPERTY FOR POLYCALORIC FUNCTIONS

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Abstract. In this paper we prove a mean value property for polycaloric functions in one space dimensional case. The proof given here is a slight modification of that of the recent paper by Da Lio and Rodino [3] and seems more straightfoward.

1. Introduction

There are many papers that deal with a mean value property for polyharmonic functions (see [1, 2, 4, 6, 7] etc.). Especially, in 2011, G. Lysik ([7]) gave a simple and elegant proof of the following mean value property for polyharmonic functions and its inverse. Let $m \in \mathbb{N}$ and let U be a domain in $\mathbb{R}^N$. If $u \in C^{2m}(U)$ and $\Delta^m u = 0$, then for any ball $B_R(x) \subset U$ it holds

$$\frac{1}{|B_R(x)|}\int_{B_R(x)} u(y)dy = \sum_{k=0}^{m} \frac{\Delta^k u(x)}{4^k(\frac{N}{2}+1)_k k!}R^{2k} \tag{1.1}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \in \mathbb{N}$.

The main subject of this paper concerns the heat version of the result (1.1). First, we fix some terminologies. Let $U \subset \mathbb{R}^N$ be an open set and $U_T = U \times (0, T]$ denote a parabolic cylinder. We say that a function u defined on U_T is *caloric* if u is a solution of the linear heat equation $(\partial_t - \Delta_x)u(x,t) = 0$, $(x,t) \in U_T$, where $\Delta_x = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$. Also, in this paper, u is called *polycaloric* if u is a solution of the equation $(\partial_t - \Delta_x)^m u(x,t) = 0, (x,t) \in U_T$ for some $m \in \mathbb{N}$. For fixed $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, and $r > 0$, let

$$E(x,t;r) = \left\{ (y,s) \in \mathbb{R}^N \times \mathbb{R} \,\middle|\, s \le t, \Phi(x-y, t-s) \ge \frac{1}{r^N} \right\}$$

denote a heat ball with a top point (x,t), where

$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{N/2}} \exp\left(-\dfrac{|x|^2}{4t}\right) & (x \in \mathbb{R}^N, t > 0) \\ 0 & (x \in \mathbb{R}^N, t < 0) \end{cases}$$

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is the fundamental solution of the heat equation. Note that a heat ball is symmetric about y_i-axis ($i = 1, \cdots, N$) and

$$E(0,0;1) = \left\{ (y,s) \in \mathbb{R}^N \times \mathbb{R} \mid -\frac{1}{4\pi} \le s < 0, |y| \le \sqrt{2Ns\log(-4\pi s)} \right\}.$$

It is well known that caloric functions possess the mean value property. Namely, if u is caloric on U_T, then for each heat ball $E(x,t;r) \subset U_T$ it holds

$$u(x,t) = \frac{1}{4r^N} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds \tag{1.2}$$

(see [5]: p.p 53-54 Theorem 3, or [10]). There is also an inverse mean value property of caloric functions under certain conditions ([9]).

Heat version of the result (1.1) is also known. Namely, in 2006, F. Da Lio and L. Rodino [3] proved the following asymptotic expansion formula for the heat integral mean (1.2) as a power series with respect to the radius of the heat ball:

Let $u \in C^\infty(\mathbb{R}^{N+1})$ and $(x,t) \in \mathbb{R}^{N+1}$, then it holds

$$\begin{aligned} &\frac{1}{4r^N} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds \\ &= u(x,t) + \sum_{k=1}^{M} r^{2k} H_k u(x,t) + O\left(r^{2M+2}\right) \text{ as } r \to 0, \end{aligned} \tag{1.3}$$

where H_k is given by

$$H_k u = \beta_{k,N} \left(\partial_t - \frac{N}{2k+N} \Delta_x \right)^{k-1} (\partial_t - \Delta_x) u \tag{1.4}$$

and

$$\beta_{k,N} = (-1)^k \frac{N}{k!} \frac{1}{(2k+N)} \left(\frac{N}{2k+N} \right)^{\frac{N}{2}+1} \left(\frac{1}{4\pi} \right)^k.$$

One of the key ideas in [3] is to introduce the differential operator H_k which is the k-th power of different heat operators whose diffusion coefficients depending on the iteration number k, though the exact meaning of H_k is less clear. Eventually, after some calculations we realize that

$$H_k u = \beta_{k,N} \sum_{l=0}^{k-1} \binom{k-1}{l} \left(\frac{2k}{2k+N} \right)^l \left(\frac{N}{2k+N} \right)^{k-1-l} (\partial_t - \Delta_x)^{k-l} (\partial_t)^l u,$$

so the formula (1.3) can be considered as the generalization of (1.1) to the polycaloric case.

In this paper, we prove the formula (1.3) in [3] by another method, when the space dimension $N = 1$. We do not need to introduce the weighted power H_k and, in the author's opinion, the method seems more straightforward.

In the following, we set $(x, t) = (0, 0)$ to simplify the description. Let $u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a smooth function (u need not be a caloric or polycaloric function). Set $E(r) = E(0, 0, r)$ and put

$$\phi(r) = \frac{1}{r^N} \iint_{E(r)} u(x,t) \frac{|x|^2}{t^2} dxdt = \iint_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2} dyds. \tag{1.5}$$

In the following, we will carry out the Maclaurin expansion of $\phi(r)$ with respect to $r \in \mathbb{R}$. By the argument in [5], we deduce

$$\phi^{(1)}(r) = -4Nr \iint_{E(1)} \left(\frac{\partial}{\partial t} - \Delta_x \right) u(ry, r^2 s) \psi(ry, r^2 s) dyds$$

where $\psi(y, s) = -\frac{N}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + N \log r$. Moreover, we get

$$\phi^{(2)}(r) = -4N \iint_{E(1)} \left(\frac{\partial}{\partial t} - \Delta_x \right) u(ry, r^2 s) \left[(N+1)\psi(ry, r^2 s) - N \right] dyds$$

by integration by parts. Therefore we obtain

$$\phi^{(1)}(0) = 0,$$

$$\phi^{(2)}(0) = \frac{2N^{\frac{N}{2}+2}}{\pi(N+2)^{\frac{N}{2}+2}} \left(\frac{\partial}{\partial t} - \Delta_x \right) u(0, 0),$$

since $|E(1)| = \dfrac{N^{\frac{N}{2}}}{2\pi(N+2)^{\frac{N}{2}+1}}$, and $\displaystyle\iint_{E(1)} \psi(ry, r^2 s) dyds = \frac{N^{\frac{N}{2}+1}}{2\pi(N+2)^{\frac{N}{2}+2}}$.

However, it seems difficult to calculate $\phi^{(n)}(0)$ by integration by parts when $n \geq 3$, since

$$\begin{aligned}\phi^{(3)}(r) = \frac{4N(N+1)}{r} & \left[\iint_{E(1)} (\partial_t - \Delta_x) u(ry, r^2 s) \left[N - (N+2)\psi(ry, r^2 s) \right] dyds \right] \\ & + 2r \left[\iint_{E(1)} (\partial_t - \Delta_x) \left[\partial_t u(ry, r^2 s) - \Delta_x u(ry, r^2 s) \psi(ry, r^2 s) \right] s\, dyds \right].\end{aligned}$$

Therefore we calculate $\phi^{(n)}(0)$ by the different method.

Theorem 1. *Let $N = 1$, $u \in C^\infty(U_T)$, $r > 0$ and $M \in \mathbb{N}$. Then we have*

$$\phi(r) = 4u(0,0) + \sum_{k=1}^{M} \frac{r^{2k}}{k!} \sum_{l=0}^{k-1} (\partial_t - \partial_x^2)^{k-l} (\partial_t)^l u(0,0) \times C_{l,k} + O(r^{2M+2}) \text{ as } r \to 0,$$

$$\text{where } C_{l,k} = \frac{(-1)^k 4}{(4\pi)^k (2k+1)^{k+\frac{3}{2}}} \binom{k-1}{l} (2k)^l.$$

Theorem 1 is the formula (1.3) in one space dimensional case.

2. Proof of the theorem 1

In this section, we prove Theorem 1. Set $v(r) = u(x,t) = u(ry, r^2 s)$ for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$. By differentiating $\phi(r)$ directly, we have

$$\phi^{(n)}(0) = \iint_{E(1)} v^{(n)}(0) \frac{|y|^2}{s^2} dyds. \tag{2.1}$$

In the following, we use standard notations of multi-indices; for $y = (y_1, \cdots, y_N) \in \mathbb{R}^N$ and a multi-index $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}_0^N$, we write $y^\alpha = y_1^{\alpha_1} \cdots y_N^{\alpha_N}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$. Next lemma concerns the evaluation of $v^{(n)}(0)$ and is valid for general dimension $N \in \mathbb{N}$.

Lemma 2 ($v^{(n)}(0)$)**.** *For $k \in \mathbb{N}_0$, we obtain*

$$v^{(2k-1)}(0) = 0, \tag{2.2}$$

$$v^{(2k)}(0) = \sum_{j=0}^{k} \sum_{|\beta|=k-j} (\partial_x^2)^\beta (\partial_t)^j u(0,0) \times A_{\beta,k}(y,s) \tag{2.3}$$

where

$$A_{\beta,k}(y,s) = \frac{(2k)!}{(2\beta)!\, j!} y^{2\beta} s^j.$$

Proof. Since $v(r)$ is a C^∞ function of r, for all $M \geq 1$ we have

$$v(r) = \sum_{n=0}^{2M+1} \frac{v^{(n)}(0)}{n!} r^n + O(r^{2M+2}) \text{ as } r \to 0. \tag{2.4}$$

On the other hand, since $v(r)$ is a composed function of $u(x,t)$ and $x = ry, t = r^2 s$, we have

$$\begin{aligned} v(r) &= \sum_{m=0}^{2M+1} \frac{1}{m!} \left((ry_1) \frac{\partial}{\partial x_1} + \cdots + (ry_N) \frac{\partial}{\partial x_N} + (r^2 s) \frac{\partial}{\partial t} \right)^m u(0,0) + O(r^{2M+2}) \\ &= \sum_{m=0}^{2M+1} \frac{1}{m!} \sum_{|\alpha|+j=m} \frac{m!}{\alpha_1! \cdots \alpha_N!\, j!} (ry)^\alpha (r^2 s)^j (\partial_x^\alpha \partial_t^j) u(0,0) + O(r^{2M+2}) \\ &= \sum_{m=0}^{2M+1} \sum_{|\alpha|+j=m} \frac{y^\alpha s^j}{\alpha!\, j!} (\partial_x^\alpha \partial_t^j) u(0,0) \times r^{|\alpha|+2j} + O(r^{2M+2}). \end{aligned} \tag{2.5}$$

By comparing the coefficients of r^n in the both expressions of (2.4) and (2.5), we obtain

$$\frac{v^{(n)}(0)}{n!} = \sum_{|\alpha|+2j=n} \frac{y^\alpha s^j}{\alpha!\, j!} (\partial_x^\alpha \partial_t^j) u(0,0).$$

Thus,

$$\phi^{(n)}(0) = \iint_{E(1)} v^{(n)}(0)\frac{|y|^2}{s^2}dyds$$
$$= \sum_{|\alpha|+2j=n} \frac{n!}{\alpha!\,j!}\,(\partial_x^\alpha \partial_t^j)u(0,0) \times \iint_{E(1)} y^\alpha s^j \frac{|y|^2}{s^2}dyds\,.$$

Since $E(1)$ is symmetric about y_i-axis($i = 1, \cdots, N$), $\iint_{E(1)} y^\alpha s^j \frac{|y|^2}{s^2}dyds$ vanishes when at least one α_i of $\alpha = (\alpha_1, \cdots, \alpha_N)$ is odd (i.e. when n is odd because $|\alpha| + 2j = n$). This proves (2.2). Next, we consider the case $\alpha = 2\beta$ for some $\beta \in \mathbb{N}_0^N$ and let $n = 2k\,(k \in \mathbb{N})$. Then we obtain

$$v^{(2k)}(0) = \sum_{2|\beta|+2j=2k} (\partial_x^2)^\beta (\partial_t)^j u(0,0) \times \frac{(2k)!}{(2\beta)!\,j!} y^{2\beta} s^j$$
$$= \sum_{j=0}^{k} \sum_{|\beta|=k-j} (\partial_x^2)^\beta (\partial_t)^j u(0,0) \times \frac{(2k)!}{(2\beta)!\,j!} y^{2\beta} s^j,$$

which implies (2.3).

□

Lemma 3 (Factorization)**.** *Let $N = 1$. Then*

(2.6) $$v^{(2k)}(0) = \sum_{l=0}^{k} (\partial_t - \partial_x^2)^{k-l} (\partial_t)^l u(0,0) \times B_{l,k}(y,s)$$

where

(2.7) $$B_{l,k}(y,s) = (-1)^{k+l} \sum_{m=0}^{l} \binom{k-l+m}{m} \times A_{k-l+m,k}(y,s)$$

for $0 \le l \le k$.

Proof. By the assumption $N = 1$ and (2.3), it is enough to prove that

(2.8) $$\sum_{j=0}^{k} (\partial_x^2)^{k-j} (\partial_t)^j u(0,0) \times A_{k-j,k} = \sum_{l=0}^{k} (\partial_t - \partial_x^2)^{k-l} (\partial_t)^l u(0,0) \times B_{l,k}.$$

We prove (2.8) by comparing the coefficients of $(\partial_x^2)^{k-j}(\partial_t)^j u(0,0)$ in both sides.

Since

$$\sum_{l=0}^{k} (\partial_t - \partial_x^2)^{k-l} (\partial_t)^l u(0,0) B_{l,k}$$
$$= (\partial_t - \partial_x^2)^k u(0,0) B_{0,k} + (\partial_t - \partial_x^2)^{k-1} (\partial_t) u(0,0) B_{1,k} + \cdots + (\partial_t)^k u(0,0) B_{k,k},$$

the coefficient of $(\partial_x^2)^{k-j}(\partial_t)^j u(0,0)$ on the right hand side of (2.8) is given by

$$\begin{aligned}
&(-1)^{k-j}\left[\binom{k}{k-j}B_{0,k}+\binom{k-1}{k-j}B_{1,k}+\binom{k-2}{k-j}B_{2,k}+\cdots\right.\\
&\quad\left.+\binom{k-j+1}{k-j}B_{j-1,k}+\binom{k-j}{k-j}B_{j,k}\right]\\
&=(-1)^{k-j}\sum_{l=0}^{j}\binom{k-l}{k-j}B_{l,k}.
\end{aligned}$$

Inserting the definition of $B_{l,k}$ in (2.7) into this expression, we assure that the coefficient of $(\partial_x^2)^{k-j}(\partial_t)^j u(0,0)$ on the right hand side of (2.8) is given by

$$(-1)^{k-j}\sum_{l=0}^{j}\binom{k-l}{k-j}(-1)^{k+l}\sum_{m=0}^{l}\binom{k-l+m}{m}\times A_{k-l+m,k}. \tag{2.9}$$

Since

$$\begin{aligned}
&\sum_{l=0}^{j}\binom{k-l}{k-j}(-1)^{k+l}\sum_{m=0}^{l}\binom{k-l+m}{m}A_{k-l+m,k}\\
&=\binom{k}{k-j}(-1)^{k}\binom{k}{0}A_{k,k}\\
&+\binom{k-1}{k-j}(-1)^{k+1}\left[\binom{k-1}{0}A_{k-1,k}+\binom{k}{1}A_{k,k}\right]\\
&+\binom{k-2}{k-j}(-1)^{k+2}\left[\binom{k-2}{0}A_{k-2,k}+\binom{k-1}{1}A_{k-1,k}+\binom{k}{2}A_{k,k}\right]\\
&+\cdots\\
&+\binom{k-j}{k-j}(-1)^{k+j}\left[\binom{k-j}{0}A_{k-j,k}+\cdots+\binom{k-1}{j-1}A_{k-1,k}+\binom{k}{j}A_{k,k}\right],
\end{aligned}$$

coefficients of $A_{k-i,k}$ for all $0\le i\le j-1$ in (2.9) is given by

$$\begin{aligned}
&(-1)^{k-j}(-1)^{k+i}\sum_{n=0}^{j-i}(-1)^n\binom{k-i-n}{k-j}\binom{k-i}{n}\\
&=(-1)^{i-j}\sum_{n=0}^{j-i}(-1)^n\binom{k-i}{k-j}\binom{j-i}{n}=0,
\end{aligned}$$

where the last equality comes from $\sum_{n=0}^{p}(-1)^n\binom{p}{n}=(-1+1)^p=0$.

Then we prove that

(2.10)
$$\sum_{l=0}^{j}\begin{pmatrix} k-l \\ k-j \end{pmatrix}(-1)^{k+l}\sum_{m=0}^{l}\begin{pmatrix} k-l+m \\ m \end{pmatrix}A_{k-l+m,k} = \begin{pmatrix} k-j \\ k-j \end{pmatrix}(-1)^{k+j}A_{k-j,k}.$$

Therefore, by (2.9) and (2.10), the coefficient of $(\partial_x^2)^{k-j}(\partial_t)^j u(0,0)$ on the right hand side of (2.8) is $A_{k-j,k}$. We have thus proved Lemma 3. □

From (2.1) and (2.6), we deduce

$$\phi^{(2k)}(0) = \sum_{l=0}^{k}(\partial_t - \partial_x^2)^{k-l}(\partial_t)^l u(0,0) \times \iint_{E(1)} B_{l,k}(y,s)dyds \tag{2.11}$$

Note that, on the right hand side of (2.11), the heat operator $(\partial_t - \partial_x^2)$ acts on u except for $l = k$.

Lemma 4. *We put*

$$\tilde{C}_{l,k} = \iint_{E(1)} B_{l,k}(y,s)\frac{y^2}{s^2}dyds.$$

Then we get

$$\tilde{C}_{l,k} = \frac{(2k)!(-1)^k 4}{k!(4\pi)^k(2k+1)^{k+\frac{3}{2}}}\begin{pmatrix} k-1 \\ l \end{pmatrix}(2k)^l \tag{2.12}$$

for $0 \le l \le k-1$ *and* $\tilde{C}_{k,k} = 0$.

Proof. We prove Lemma 4 by simple calculations. First, by the definition of $B_{l,k}$ in (2.7)

$$B_{l,k} = (-1)^{k+l}\sum_{m=0}^{l}\begin{pmatrix} k-l+m \\ m \end{pmatrix}\frac{(2k)!}{(2k-2l+2m)!(l-m)!}y^{2k-2l+2m}s^{l-m}$$

for $0 \le l \le k$, we have

$$\tilde{C}_{l,k} = (-1)^{k+l}\sum_{m=0}^{l}\begin{pmatrix} k-l+m \\ m \end{pmatrix}\frac{(2k)!}{(2k-2l+2m)!(l-m)!}\iint_{E(1)} y^{2k-2l+2m+2}s^{l-m-2}dyds.$$

Direct calculation shows that

$$\iint_{E(1)} y^{2k-2l+2m+2} s^{l-m-2} dyds = \int_{s=-1/4\pi}^{s=0} s^{l-m-2} \int_{|y|\leq \sqrt{2s\log(-4\pi s)}} y^{2k-2l+2m+2} dyds$$
$$= \frac{2}{(2k-2l+2m+3)} \int_{-1/4\pi}^{0} s^{l-m-2} \{2s \log(-4\pi s)\}^{k-l+m+\frac{3}{2}} ds$$
$$= \frac{(-1)^{l-m} 2^{k-l+m+\frac{3}{2}}}{(k-l+m+\frac{3}{2})(4\pi)^{k+\frac{1}{2}}} \int_0^\infty t^{k-l+m+\frac{3}{2}} \exp\left(-\left(k+\frac{1}{2}\right)t\right) dt$$
$$= \frac{(-1)^{l-m} 4^{k-l+m} 2^3 \Gamma(k-l+m+\frac{3}{2})}{(4\pi)^k \sqrt{\pi}(2k+1)^{k-l+m+\frac{5}{2}}}$$

where $\Gamma(\cdot)$ is the Gamma function. Thus, we get

$$\tilde{C}_{l,k} = \frac{(-1)^k (2k)! 4^{k-l} 8}{(4\pi)^k \sqrt{\pi}(2k+1)^{k-l+\frac{5}{2}}(k-l)!} \sum_{m=0}^{l} \frac{(-1)^m (k-l+m)! 4^m \Gamma(k-l+m+\frac{3}{2})}{m!(2k-2l+2m)!(l-m)!(2k+1)^m}.$$
$$= \frac{(-1)^k (2k)! 4}{k!(4\pi)^k (2k+1)^{k-l+\frac{5}{2}}} \begin{pmatrix} k \\ l \end{pmatrix} \sum_{m=0}^{l} (-1)^m \begin{pmatrix} l \\ m \end{pmatrix} \frac{2k-2l+2m+1}{(2k+1)^m},$$

where the last equality comes from the fact $\Gamma(s+1) = s\Gamma(s)$.

Since we have the following equation

$$(2k+1)^l \sum_{m=0}^{l} (-1)^m \begin{pmatrix} l \\ m \end{pmatrix} \frac{2k-2l+2m+1}{(2k+1)^m}$$
$$= (2k+1) \sum_{m=0}^{l} \begin{pmatrix} l \\ m \end{pmatrix} (-1)^m (2k+1)^{l-m} - 2 \sum_{m=0}^{l-1} (-1)^m \begin{pmatrix} l \\ m \end{pmatrix} (l-m)(2k+1)^{l-m}$$
$$= (2k+1)(2k)^l - 2l(2k+1) \sum_{m=0}^{l-1} (-1)^m \begin{pmatrix} l-1 \\ m \end{pmatrix} (2k+1)^{l-m-1}$$
$$= 2(k-l)(2k)^{l-1}(2k+1)$$

Therefore we obtain $\tilde{C}_{k,k} = 0$ and (2.12). □

From all Lemmas, we obtain

$$\phi^{(2k)}(0) = \sum_{l=0}^{k-1} (\partial_t - \partial_x^2)^{k-l} (\partial_t)^l u(0,0) \times \tilde{C}_{l,k} \quad (k = 1, 2, \ldots),$$

which proves Theorem 1.

3. A mean value property for polycaloric functions

In this section, first we recall the well-known regularity property of (poly-) caloric functions.

Proposition 5 (caloric function is smooth)**.** *If $u : U_T \to \mathbb{R}$ is caloric, then $u \in C^\infty(U_T)$.*

Proof. See [5]: p.p 59-61 Theorem 8. □

Proposition 6 (polycaloric function is smooth)**.** *If $u : U_T \to \mathbb{R}$ is polycaloric, then $u \in C^\infty(U_T)$.*

Proof. Assume that there exists $m \in \mathbb{N}$ such that $(\partial_t - \Delta_x)^m u = 0$ in U_T. Then we find caloric functions $u_0, u_1, \cdots, u_{m-1} : U_T \to \mathbb{R}$ such that

$$u(x,t) = u_0(x,t) + tu_1(x,t) + \cdots + t^{m-1}u_{m-1}(x,t) \tag{3.1}$$

holds true, by proposition 1 in [8]. Indeed, for $j = 1, 2, \cdots, m$, we may choose

$$u_{m-j}(x,t) = \frac{1}{(m-j)!}\sum_{k=0}^{j-1}\frac{(-t)^k}{k!}(\partial_t - \Delta_x)^{m-j+k}u(x,t).$$

Therefore $u_0, u_1, \cdots, u_{m-1}$ are caloric and satisfy the equation (3.1). By proposition 5 and (3.1), we obtain $u \in C^\infty(U_T)$. □

By proposition 5 and proposition 6, we obtain several corollaries which are proved by Da Lio and Rodino [3] as follows. We do not need the additional assumption that u is smooth, after assuming that u is caloric or polycaloric.

corollary 7 (A mean value property for analytic functions. [3] Proposition 2.2)**.** *Let $N = 1$ and $u \in C^\infty(U_T)$. Assume that $(\partial_t - \Delta_x)u(x,t)$ is an analytic function in U_T. Then $\phi(r)$ given in (1.5) is an analytic function of $r \in \mathbb{R}$ in a neighborhood of $r = 0$, and it holds*

$$\begin{aligned}
&\frac{1}{4r}\iint_{E(x,t;r)} u(y,s)\frac{(x-y)^2}{(t-s)^2}dyds \\
&= u(x,t) + \sum_{k=1}^{\infty}\frac{r^{2k}}{k!}\sum_{l=0}^{k-1}(\partial_t - \partial_x^2)^{k-l}(\partial_t)^l u(x,t) \times C_{l,k} \\
&\textit{where } C_{l,k} = \frac{(-1)^k 4}{(4\pi)^k(2k+1)^{k+\frac{3}{2}}}\begin{pmatrix} k-1 \\ l \end{pmatrix}(2k)^l.
\end{aligned}$$

Remark 8. If u is caloric on U_T, then $u \in C^\infty(U_T)$ and $(\partial_t - \Delta_x)u(x,t)$ is obviously analytic in U_T and for each heat ball $E(x,t;r) \subset U_T$ the following equation holds:

$$\frac{1}{4r}\iint_{E(x,t;r)} u(y,s)\frac{(x-y)^2}{(t-s)^2}dyds = u(x,t).$$

corollary 9 (A mean value property for polycaloric functions)**.** *Let $N = 1$ and $(\partial_t - \Delta_x)u(x,t)$ be an analytic function in U_T. If u is polycaloric on U_T (i.e.$(\partial_s - \partial_y^2)^m u(y,s) = 0$, $(y,s) \in U_T$, $m \in \mathbb{N}$), then for each heat ball $E(x,t;r) \subset U_T$ the following equality holds:*

$$\begin{aligned}
&\frac{1}{4r}\iint_{E(x,t;r)} u(y,s)\frac{(x-y)^2}{(t-s)^2}dyds \\
&= u(x,t) + \sum_{k=1}^{m-1}\frac{r^{2k}}{k!}\sum_{l=0}^{k-1}(\partial_t - \partial_x^2)^{k-l}(\partial_t)^l u(x,t)\times C_{l,k} \\
&+ \sum_{k=m}^{\infty}\frac{r^{2k}}{k!}\sum_{l=k-m+1}^{k-1}(\partial_t - \partial_x^2)^{k-l}(\partial_t)^l u(x,t)\times C_{l,k}, \\
&\textit{where}\, C_{l,k} = \frac{(-1)^k 4}{(4\pi)^k(2k+1)^{k+\frac{3}{2}}}\begin{pmatrix} k-1 \\ l \end{pmatrix}(2k)^l.
\end{aligned}$$

Proof. This is a direct consequence of Theorem 1 and Proposition 6. ◻

corollary 10 ([3]Corollary 2.1)**.** *Let $N = 1$. Suppose that there exist $n_1 \geq 0$ and $n_2 \geq 1$ such that*

$$(\partial_t - \partial_x^2)(\partial_t)^{n_1}u = 0 \,\textit{and}\, (\partial_t - \partial_x^2)^{n_2}u = 0 \,\textit{in}\, U_T.$$

Then for all $r > 0$ we have

$$\begin{aligned}
&\frac{1}{4r}\iint_{E(x,t;r)} u(y,s)\frac{(x-y)^2}{(t-s)^2}dyds \\
&= u(x,t) + \sum_{k=1}^{M}\frac{r^{2k}}{k!}\sum_{l=0}^{k-1}(\partial_t - \partial_x^2)^{k-l}(\partial_t)^l u(x,t)\times C_{l,k},
\end{aligned} \tag{3.2}$$

with $M = n_1 + n_2 - 1$(when $n_1 = 0$ or $n_2 = 1$ the sum in the right-hand side of (3.2) does not appear).

Proof. Note that we get $u \in C^\infty(U_T)$, since u is polycaloric in U_T. See the proof of corollary 2.1 in [3]. ◻

We finally give a mean value property for the higher order heat equation $\partial_t u + (-1)^m\Delta^m u = 0\,(m \in \mathbb{N})$ for general dimension. In the proof, we use proposition 2.2 and a result in the proof of proposition 2.1 in [3].

Proposition 11 (A mean value property for the higher order heat equation)**.** *Let $u \in C^\infty(U_T)$ and $(\partial_t - \Delta_x)u(x,t)$ be an analytic function in U_T. Assume* *that u is a solution of the higher order heat equation $\partial_t u + (-1)^m \Delta^m u = 0$. Then for each heat ball $E(x,t;r) \subset U_T$ the following equality holds:*

$$\frac{1}{4r^N}\iint_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}dyds = u(x,t) + \sum_{k=1}^{\infty} r^{2k} H_k u(x,t), \tag{3.3}$$

where H_k is given by

$$H_k u = \begin{cases} \dfrac{\rho_{k,N}}{k!}\displaystyle\sum_{h=0}^{k}(-1)^{k-h}\binom{k}{h}(N+2h)\left(\frac{N}{2k+N}\right)^h \Delta^{mk+(1-m)h}u, & (m : odd) \\ \dfrac{\rho_{k,N}}{k!}\displaystyle\sum_{h=0}^{k}\binom{k}{h}(N+2h)\left(\frac{N}{2k+N}\right)^h \Delta^{mk+(1-m)h}u, & (m : even) \end{cases}$$

$$where \quad \rho_{k,N} = \frac{1}{2k+N}\left(\frac{N}{2k+N}\right)^{\frac{N}{2}+1}\left(\frac{1}{4\pi}\right)^k.$$

Proof. Let $p \in \mathbb{N}$. Note that u satisfies

$$\partial_t^p u = \begin{cases} \Delta^{pm} u, & (m : \text{odd}) \\ (-1)^p \Delta^{pm} u, & (m : \text{even}) \end{cases} \tag{3.4}$$

since u is a smooth solution of the higher order heat equation $\partial_t u + (-1)^m \Delta^m u = 0$. On the other hand, (3.3) holds by proposition 2.2 in [3], and according to a result in [3] (p,268, line 2 and 9), H_k is given by

$$H_k u = \frac{\rho_{k,N}}{k!}\sum_{h=0}^{k}(-1)^{k-h}\binom{k}{h}(N+2h)\left(\frac{N}{2k+N}\right)^h \Delta^h(\partial_t)^{k-h}u. \tag{3.5}$$

Finally, combining (3.4) and (3.5), we get the proposition 11. □

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