The critical problem of Kirchhoff type elliptic equations in dimension four

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Abstract

We study the following Kirchhoff type elliptic problem,

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = \lambda u^q + \mu u^3, \ u>0 \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^4$ is a bounded domain with smooth boundary $\partial\Omega$. Moreover we assume $a, \lambda, \mu > 0$, $b \geq 0$ and $1 \leq q < 3$. In this paper, we prove the existence of solutions of (P). Our tools are the variational method and the concentration compactness argument for PS sequences.

Keywords: Kirchhoff, nonlocal, elliptic, critical, variational method

1. Introduction

We investigate a Kirchhoff type elliptic problem,

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = \lambda u^q + \mu u^3 \text{ in } \Omega, \\ u>0 \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^4$ is a bounded domain with smooth boundary $\partial\Omega$. We assume $a, \lambda, \mu > 0$, $b \geq 0$ and $1 \leq q < 3$. In this paper, we prove the existence of solutions of (P).

Our problem (P) describes the stationary state of the Kirchhoff type quasilinear hyperbolic equation such as

$$\frac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,t,u), \tag{P$_0$}$$

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where $M : \mathbb{R}^+ \to \mathbb{R}^+$ is some function. (P_0) appears in the theory of the nonlinear vibrations on physics [15]. The solvability of (P_0) is also discussed on mathematics [6][7][9][14][24] etc. We can refer to the survey [1].

In recent years, the analysis of the stationary problems of (P_0) has been extensively carrying out by many authors, see [2][3][4][10][11][16][17][18][20][21] [22][26][28][29] and so on. By them, several existence results are successfully obtained via the variational and topological methods even for the critical case. But most of them treat only three or less dimensional case except for [3], [10] and [18]. Here we emphasize that we would treat the 4-dimensional critical problem (P). In our case, a typical difficulty occurs in proving the existence of solutions because of the lack of the compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. Furthermore, in view of the corresponding energy, the interaction between the Kirchhoff type perturbation $\|u\|_{H_0^1(\Omega)}^4$ and the critical nonlinearity $\int_\Omega u^4 dx$ is crucial. In the followings, we can see the effect of such an interaction on the existence. To our best knowledge, this paper is the first one which essentially attacks the Brezis-Nirenberg problem for four dimensional Kirchhoff type equations.

1.1. Statement of results

Firstly we consider the cases $q = 1$. Let $S, \lambda_1 > 0$ be the usual Sobolev constant defined by

$$\inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega u^4 dx\right)^{1/2}},$$

and the principal eigenvalue of $-\Delta$ on Ω respectively. Our result is the following.

Theorem 1.1. *Let $q = 1$, $a > 0$ $b \geq 0$, $0 < \lambda < a\lambda_1$ and $\mu > 0$. Then (P) has a solution if and only if $bS^2 < \mu$.*

Remark 1.2. *Recall the result by Brezis-Nirenberg [8]. In [8], the case $a = 1$, $b = 0$ and $\mu = 1$ is considered. Theorem 1.1 gives an extension of their result to the Kirchhoff type problem for 4-dimensional case.*

As we shall see in Section 2, the proof of Theorem 1.1 is successfully straightforward. The problems lie in the case $1 < q < 3$. Certainly, we can confirm the existence if $a, \lambda, \mu > 0$ and $b = 0$ by [8]. Thus here we only deal with the case $b > 0$. In this case, the boundedness of the PS sequences is hard to prove. Hence inspired by [17], we consider the problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \nu(\lambda u^q + \mu u^3) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P_ν}$$

where $\nu \in (\delta, 1]$ for some $1/2 < \delta < 1$. By the aid of the result by Jeanjean [13], we prove the next theorems.

Theorem 1.3. *We suppose* $1 < q < 3$. *Let* $b, \mu > 0$ *satisfy* $bS^2 < \mu < 2bS^2$ *and take* $1/2 < \delta < 1$ *so that* $bS^2/\delta < \mu$. *Furthermore, assume one of the following (C1), (C2) and (C3) holds,*

(C1) $a > 0$ *and* $0 < \lambda < \lambda_0$ *where* $0 < \lambda_0 = \lambda_0(a, b, q, \mu) \leq \infty$ *is chosen sufficiently small if necessary.*

(C2) $\lambda > 0$ *and* $a > a_0$ *where* $a_0 = a_0(b, q, \lambda, \mu) \geq 0$ *is taken sufficiently large if necessary.*

(C3) $a > 0$, $\lambda > 0$ *and* $b_0 < b < \mu/S^2$ *where* $\mu/(2S^2) \leq b_0 = b_0(a, q, \lambda, \mu) < \mu/S^2$ *is selected sufficiently large if necessary.*

Then (P_ν) *poses a solution for almost every* $\nu \in (\delta, 1]$. *Furthermore we can find an increasing sequence* $(\nu_n) \subset (\delta, 1]$ *such that* $\nu_n \to 1$ *as* $n \to \infty$ *and* (P_ν) *with* $\nu = \nu_n$ *has a solution* u_n *and further, which shows one of the followings,*

(i) $u_n \to \infty$ *in* $H_0^1(\Omega)$ *as* $n \to \infty$,

(ii) u_n *is bounded in* $H_0^1(\Omega)$ *and consequently, (P) has a solution.*

Remark 1.4. *In Theorem 1.3, we give the condition* $bS^2 < \mu < 2bS^2$ *on* $b, \mu > 0$. *Comparing to that in Theorem 1.1, we can see an additional part such as* $\mu < 2bS^2$. *This condition is used to get the appropriate local compactness of our PS sequences. See the proof of Lemma 3.2. Note that it is also considered in the proof of Theorem 1.6 below.*

Remark 1.5. *Let* $1 < q < 3$ *and* b, μ *satisfy the hypothesis in Theorem 1.3 and put*

$$g(t) := \frac{a\mu}{2(\mu - bS^2)}t^2 - \frac{\lambda}{(q+1)S_{q+1}^{(q+1)/2}}t^{q+1} + \frac{\mu(2bS^2 - \mu)}{4S^2(\mu - bS^2)}t^4,$$

where

$$S_{q+1} := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{q+1} dx\right)^{\frac{2}{q+1}}}.$$

Firstly, fix $a > 0$. *Then it is enough if we choose* $0 < \lambda_0 = \lambda_0(a, b, q, \mu) < \infty$ *so small that if* $0 < \lambda < \lambda_0$, $g(t) \geq 0$ *for all* $t \geq 0$. *Next fix* $\lambda > 0$. *Then it is sufficient if we choose* $a_0 = a_0(b, q, \lambda, \mu) > 0$ *so large that if* $a > a_0$, $g(t) \geq 0$ *for all* $t \geq 0$. *Finally fix* $a > 0$ *and* $\lambda > 0$. *Then it is enough if we set* $\mu/(2S^2) < b_0 = b_0(a, q, \lambda, \mu) < \mu/S^2$ *so large that if* $b_0 < b < \mu/S^2$, $g(t) \geq 0$ *for all* $t \geq 0$. *We recall these constants* a_0, λ_0 *and* b_0 *in Lemma 3.2, Section 3.*

We can avoid the possibility of the assertion (i) in Theorem 1.3 if Ω is strictly star-shaped.

Theorem 1.6. *Assume* $a, b, \lambda, \mu > 0$ *satisfies the same hypotheses with that in Theorem 1.3. Furthermore we suppose* $\Omega \subset \mathbb{R}^3$ *is strictly star-shaped. Then (P) has a solution.*

Recently the Brezis-Nirenberg problem (cf.[8]) for the Kirchhoff type equations are observed in [2], [10], [20], [21] and [29]. By their works, a certain extension from the original Brezis-Nirenberg problem to the Kirchhoff type one is accomplished for 3-dimensional case. For larger dimensional case, only Figueiredo

[10] considers the case $N \geq 3$ and $\Omega \subset \mathbb{R}^N$. By his argument, we can prove that (P) with $1 < q < 3$, $a > 0$, $b \geq 0$ and $\mu > 0$ has a solution if $\lambda > 0$ is sufficiently large. But the result in [8] says that if $1 < q < 3$, $a > 0$, $b = 0$ and $\mu > 0$, (P) has a solution for all $\lambda > 0$. Hence we can naturally ask whether or not the existence result holds if $b > 0$ and $\lambda > 0$ is small or arbitrary. A positive answer to this question is obtained by Theorem 1.3 and 1.6. Lastly we note some questions on Theorem 1.3 and 1.6 which still remain unsolved. They are the followings, (1) whether or not we can choose $\lambda_0 = \infty$, $a_0 = 0$ and $b_0 = \mu/(2S^2)$, (2) whether or not the additional condition $\mu < 2bS^2$, which unexpectedly can be read as b should not be too small, is essential and further, (3) the clear answer for the general smooth bounded domain case. These are the left problems for our future.

1.2. Setting

We put a notion of the weak solutions of (P). We call $u \in H_0^1(\Omega)$ is a weak solution of (P), if and only if u satisfies

$$\left(a + b\|u\|^2_{H_0^1(\Omega)}\right) \int_\Omega \nabla u \cdot \nabla h dx - \lambda \int_\Omega u_+^q h dx - \mu \int_\Omega u_+^3 h dx = 0$$

for all $h \in H_0^1(\Omega)$ where $\|\cdot\|_{H_0^1(\Omega)} := (\int_\Omega |\nabla u|^2 dx)^{1/2}$ and $u_+ := \max\{u, 0\}$. Applying the usual elliptic regularity theories and strong maximum principle, we can conclude that every weak solution of (P) belongs to $C^2(\overline{\Omega})$ and positive. Moreover we define the associated functional I on $H_0^1(\Omega)$ so that

$$I(u) := \frac{a}{2}\|u\|^2_{H_0^1(\Omega)} + \frac{b}{4}\|u\|^4_{H_0^1(\Omega)} - \frac{\lambda}{q+1}\int_\Omega u_+^{q+1} dx - \frac{\mu}{4}\int_\Omega u_+^4 dx \ (u \in H_0^1(\Omega)).$$

Then we can easily check that I is well-defined and belongs to $C^1(H_0^1(\Omega), \mathbb{R})$. Furthermore, every critical point of I is a weak solution of (P). Thus in the following sections we shall prove the existence of a nontrivial critical point of I. Similarly we can define the weak solutions of (P_ν) and the associated functional I_ν.

1.3. A description of PS sequences

In the present papers [10][20][21] etc., they investigate the compactness conditions of their PS sequences through Lions' second concentration compactness lemma [19]. In this paper, to understand the features of PS sequences for Kirchhoff type critical problems more clearly, we rather introduce a complete description of the PS sequences, following the argument in [25]. Here we define the Sobolev space $D^{1,2}(\mathbb{R}^4)$ as usual and write its norm as $\|\cdot\|_{D^{1,2}(\mathbb{R}^4)} := \left(\int_{\mathbb{R}^4} |\nabla \cdot|^2 dx\right)^{1/2}$.

Proposition 1.7. *Let $c \in \mathbb{R}$ and $(u_n) \subset H_0^1(\Omega) \subset D^{1,2}(\mathbb{R}^4)$ be a bounded $(PS)_c$ sequence for I, that is, $I(u_n) \to c$, $I'(u_n) \to 0$ in $H^{-1}(\Omega)$ and $\|u_n\|_{H_0^1(\Omega)}$ is bounded. Then (u_n) has a subsequence which strongly converges in $H_0^1(\Omega)$,* *or otherwise, there exist a nonnegative function $u_0 \in H_0^1(\Omega)$ which is a weak convergence of u_n, a number $k \in \mathbb{N}$ and further, for every $i \in \{1, 2, \cdots, k\}$, a sequence of values $(R_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}^+$, points $(x_n^i)_{n\in\mathbb{N}} \subset \overline{\Omega}$ and a nonnegative function $v_i \in D^{1,2}(\mathbb{R}^4)$ satisfying*

$$-\left\{a + b\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{k} \|v_j\|^2_{D^{1,2}(\mathbb{R}^4)}\right)\right\}\Delta u_0 = \lambda u_0^q + \mu u_0^3 \text{ in } \Omega, \tag{1}$$

$$-\left\{a + b\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{k} \|v_j\|^2_{D^{1,2}(\mathbb{R}^4)}\right)\right\}\Delta v_i = \mu v_i^3 \text{ in } \mathbb{R}^4, \tag{2}$$

such that up to subsequences, there hold $R_n^i \, dist(x_n^i, \partial\Omega) \to \infty$,

$$\left\|u_n - u_0 - \sum_{i=1}^{k} R_n^i v_i(R_n^i(\cdot - x_n^i))\right\|_{D^{1,2}(\mathbb{R}^4)} = o(1),$$

$$\|u_n\|^2_{H_0^1(\Omega)} = \|u_0\|^2_{H_0^1(\Omega)} + \sum_{i=1}^{k} \|v_i\|^2_{D^{1,2}(\mathbb{R}^4)} + o(1)$$

and

$$I(u_n) = \tilde{I}(u_0) + \sum_{i=1}^{k} \tilde{I}_\infty(v_i) + o(1)$$

where $o(1) \to 0$ as $n \to \infty$ and we define

$$\tilde{I}(u_0) := \left\{\frac{a}{2} + \frac{b}{4}\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{k} \|v_j\|^2_{D^{1,2}(\mathbb{R}^4)}\right)\right\}\|u_0\|^2_{H_0^1(\Omega)} - \frac{\lambda}{2}\int_\Omega u_0^2 dx - \frac{\mu}{4}\int_\Omega u_0^4 dx, \tag{3}$$

$$\tilde{I}_\infty(v_i) := \left\{\frac{a}{2} + \frac{b}{4}\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{k} \|v_j\|^2_{D^{1,2}(\mathbb{R}^4)}\right)\right\}\|v_i\|^2_{D^{1,2}(\mathbb{R}^4)} - \frac{\mu}{4}\int_{\mathbb{R}^4} v_i^4 dx. \tag{4}$$

Remark 1.8. *We note that (1), the equation for the weak convergence u_0 of u_n, depends on the nonlocal information of all bubbles,*

$$b\sum_{j=1}^{k} \|v_j\|^2_{D^{1,2}(\mathbb{R}^4)}.$$

This implies that, if $b > 0$ and (u_n) poses no subsequence which strongly converges in $H_0^1(\Omega)$, the weak convergence u_0 of u_n is never a critical point of I

because of the presence of bubbles, differently from the case $b = 0$. We also emphasize that in view of (3), the "energy" of the weak convergence u_0 has the cross term such as

$$\frac{b}{4}\left(\sum_{i=1}^{k}\|v_i\|^2_{D^{1,2}(\mathbb{R}^4)}\right)\|u_0\|^2_{H^1_0(\Omega)}.$$

Observe also that similar phenomena are confirmed in the limiting problem (2) for bubbles and the energies (4) of those. These are the features of the PS sequences of the Kirchhoff type critical problem. In the proof of Theorem 1.3, the careful analysis of such phenomena plays important role. In particular, see the proof of Lemma 3.2 in Section 3.

In Section 4, we argue with the details of this compactness result for a general dimensional problem.

1.4. Organization of this paper

This paper is organized as follows. In Section 2, we consider the case $q = 1$ and give the proof of Theorem 1.1. In Section 3, we treat the case $1 < q < 3$ and show the proof of Theorem 1.3 and 1.6. In addition, in Section 4, we give the global compactness result for the Kirchhoff type critical problem in the general dimension.

2. The case $q = 1$

In this section, we deal with the case $q = 1$ and prove Theorem 1.1. The conclusion for the case $b = 0$ is obtained by [8]. Hence we only consider the case $b > 0$. Let $a, b, \lambda, \mu > 0$ with $\lambda < a\lambda_1$. As we say in Section 1, we shall prove the existence of a nontrivial critical point of the functional

$$I(u) = \frac{a}{2}\|u\|^2_{H^1_0(\Omega)} + \frac{b}{4}\|u\|^4_{H^1_0(\Omega)} - \frac{\lambda}{2}\int_\Omega u_+^2\,dx - \frac{\mu}{4}\int_\Omega u_+^4\,dx.$$

Here we mainly treat the existence part of Theorem 1.1. For this, once we assume $bS^2 < \mu$, the proof is completely straightforward. To the first, we ensure the following local compactness result.

Lemma 2.1. *Let $a, b, \lambda, \mu > 0$ satisfy $\lambda < a\lambda_1$ and $bS^2 < \mu$. Then if $(u_n) \subset H^1_0(\Omega)$ is a $(PS)_c$ sequence for I with*

$$c < \frac{(aS)^2}{4(\mu - bS^2)},$$

then (u_n) strongly converges in $H^1_0(\Omega)$ up to subsequences.

Proof. Let $(u_n) \subset H^1_0(\Omega)$ be a $(\mathrm{PS})_c$ sequence for I with

$$c < \frac{(aS)^2}{4(\mu - bS^2)}.$$

We first claim that (u_n) is bounded in $H_0^1(\Omega)$. In fact, by the definition and the Poincare inequality, we have

$$\begin{aligned} c+1 &\geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle + \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &\geq \frac{a}{4}\left(1 - \frac{\lambda}{a\lambda_1}\right)\|u_n\|^2_{H_0^1(\Omega)} - \|u_n\|_{H_0^1(\Omega)} \end{aligned}$$

for large $n \in \mathbb{N}$. Since $\lambda < a\lambda_1$, this proves our claim. Now we suppose on the contrary that we can extract no subsequence from (u_n) which strongly converges in $H_0^1(\Omega)$. Then from Proposition 1.7, there exist a nonnegative weak convergence $u_0 \in H_0^1(\Omega)$ of u_n, a number $k \in \mathbb{N}$ and further, for every $i \in \{1, 2, \cdots, k\}$, a sequence of values $(R_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}^+$, points $(x_n^i)_{n\in\mathbb{N}} \subset \overline{\Omega}$ and a nonnegative function $v_i \in D^{1,2}(\mathbb{R}^4)$ satisfying (1) and (2) such that up to subsequences, there holds

$$I(u_n) = \tilde{I}(u_0) + \sum_{i=1}^{k} \tilde{I}_\infty(v_i) + o(1), \tag{5}$$

where $o(1) \to 0$ as $n \to \infty$ and we put $\tilde{I}(u_0)$ and $\tilde{I}_\infty(v_i)$ as in Proposition 1.7. Then we claim

$$\tilde{I}(u_0) \geq 0 \tag{6}$$

and

$$\tilde{I}_\infty(v_i) \geq \frac{(aS)^2}{4(\mu - bS^2)} \tag{7}$$

for all $i \in \{1, 2, \cdots, k\}$. First we prove (6). Noting (1), (3) and using the Poincare inequality, we have

$$\begin{aligned} \tilde{I}(u_0) &= \tilde{I}(u_0) - \frac{1}{4}\left\{(a+bA)\int_\Omega |\nabla u_0|^2 dx - \lambda \int_\Omega u_0^2 x - \mu \int_\Omega u_0^4 dx\right\} \\ &\geq \frac{a}{4}\left(1 - \frac{\lambda}{a\lambda_1}\right)\|u_0\|^2_{H_0^1(\Omega)}, \end{aligned}$$

where $A := \|u_0\|^2_{H_0^1(\Omega)} + \sum_{i=1}^k \|v_i\|^2_{D^{1,2}(\mathbb{R}^4)}$ for simplicity. Since $\lambda < a\lambda_1$, we conclude (6). Next we prove (7). From (2) and the Sobolev inequality, we get

$$\begin{aligned} 0 &= (a+bA)\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} - \mu \int_{\mathbb{R}^4} v_i^4 dx \\ &\geq a\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} + b\|v_i\|^4_{D^{1,2}(\mathbb{R}^N)} - \mu S^{-2}\|v_i\|^4_{D^{1,2}(\mathbb{R}^N)} \\ &\geq a\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} - S^{-2}(\mu - bS^2)\|v_i\|^4_{D^{1,2}(\mathbb{R}^N)}. \end{aligned}$$

Thus noting $bS^2 < \mu$, we obtain

$$\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} \geq \frac{aS^2}{\mu - bS^2}. \tag{8}$$

Moreover (2) and (4) imply

$$\begin{aligned}\tilde{I}_\infty(v_i) &= \tilde{I}_\infty(v_i) - \frac{1}{4}\left\{(a+bA)\int_{\mathbb{R}^4}|\nabla v_i|^2dx - \mu\int_{\mathbb{R}^4}v_i^4dx\right\}\\ &\geq \frac{a}{4}\|v_i\|^2_{D^{1,2}(\mathbb{R}^4)}.\end{aligned}$$

Using (8), we ensure (7). Finally, it follows from (5), (6) and (7),

$$\begin{aligned}c &= \lim_{n\to\infty} I(u_n)\\ &= \tilde{I}(u_0) + \sum_{i=1}^{k}\tilde{I}_\infty(v_i)\\ &\geq \frac{(aS)^2}{4(\mu - bS^2)},\end{aligned}$$

a contradiction. This finishes the proof. □

Here with no loss of generality we can assume $0 \in \Omega$. Owing to [8], we introduce the Talenti function [27] cut off appropriately,

$$u_\varepsilon(x) := \frac{\varepsilon\tau(x)}{\varepsilon^2 - |x|^2} \in H_0^1(\Omega)$$

where $\varepsilon > 0$ and $\tau \in C_0^\infty(\Omega)$ is an appropriate cut off function such that $0 \leq \tau \leq 1$ and $\tau(x) = 1$ on some neighborhood of $0 \in \Omega$. Then we put $v_\varepsilon := u_\varepsilon / \left(\int_\Omega u_\varepsilon^4 dx\right)^{1/4}$ and obtain

$$\begin{cases}\int_\Omega |\nabla v_\varepsilon|^2dx = S + O(\varepsilon^2),\\ \int_\Omega v_\varepsilon^4 dx = 1,\\ \int_\Omega v_\varepsilon^2 dx = \alpha_1\varepsilon^2|\log\varepsilon| + O(\varepsilon^2),\end{cases} \tag{9}$$

where $\alpha_1 > 0$ is some constant.

The next lemma will confirm a mountain pass level of I is below the desired energy level.

Lemma 2.2. *Let $a, b, \lambda, \mu > 0$ satisfy $\mu > bS^2$. Then there exists a constant $\varepsilon_1 > 0$ such that*

$$\sup_{t\geq 0} I(tv_\varepsilon) < \frac{(aS)^2}{4(\mu - bS^2)}$$

for all $\varepsilon \in (0, \varepsilon_1)$.

Proof. We consider v_ε defined as above. Noting $\mu > bS^2$ and (9) we estimate,

$$\begin{aligned}I(tv_\varepsilon) &\leq \frac{a\left(S - \lambda\alpha_1\varepsilon^2|\log\varepsilon| + O(\varepsilon^2)\right)}{2}t^2 - \frac{\left(\mu - bS^2 + O(\varepsilon^2)\right)}{4}t^4\\ &\leq \frac{(aS)^2 - 2\lambda a^2 S\alpha_1\varepsilon^2|\log\varepsilon| + O(\varepsilon^2)}{4(\mu - bS^2 + O(\varepsilon^2))}\\ &\leq \frac{(aS)^2}{4(\mu - bS^2)} - \frac{\lambda a^2 S\alpha_1}{2(\mu - bS^2)}\varepsilon^2|\log\varepsilon| + O(\varepsilon^2)\end{aligned}$$

for all $t \geq 0$. Thus there exists a constant $\varepsilon_1 > 0$ such that

$$\sup_{t\geq 0} I(tv_\varepsilon) < \frac{(aS)^2}{4(\mu - bS^2)}$$

for all $\varepsilon \in (0, \varepsilon_1)$. This concludes the proof. □

Remark 2.3. *Recall the argument by Brezis-Nirenberg [8]. In [8], they choose the Talenti function which attains the Sobolev constant*

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^4)\setminus\{0\}} \frac{\int_{\mathbb{R}^4} |\nabla u|^2 dx}{\left(\int_\Omega u^4 dx\right)^{\frac{1}{2}}},$$

and successfully show a mountain pass level below the desired energy level. Here, observe that (by trivial rescaling), the function can be regarded as the positive solution of the problem in whole space,

$$-\Delta U = U^3 \ \textit{in}\ \mathbb{R}^4,\ U(x) \to 0 \ \textit{as}\ |x| \to \infty,$$

which is given by

$$U_\varepsilon := \frac{8^{\frac{1}{2}}\varepsilon}{\varepsilon^2 + |\cdot - x_0|^2}$$

for some $\varepsilon > 0$ and $x_0 \in \mathbb{R}^4$. As a matter of the fact, when $\mu = 1$ (for simplicity) and $1 > bS^2$, the Talenti function multiplied by an appropriate constant,

$$W_\varepsilon := \left(\frac{a}{1 - bS^2}\right)^{\frac{1}{2}} U_\varepsilon,$$

is nothing but a solution of the Kirchhoff type equation in whole space,

$$-\left(a + b\|W\|^2_{D^{1,2}(\mathbb{R}^4)}\right)\Delta W = W^3 \ \textit{in}\ \mathbb{R}^4,\ W(x) \to 0 \ \textit{as}\ |x| \to \infty.$$

Moreover we can easily check that the energy of W_ε satisfies

$$\frac{a}{2}\|W_\varepsilon\|^2_{D^{1,2}(\mathbb{R}^4)} + \frac{b}{4}\|W_\varepsilon\|^4_{D^{1,2}(\mathbb{R}^4)} - \frac{1}{4}\int_{\mathbb{R}^4} W_\varepsilon^4 dx = \frac{(aS)^2}{4(1 - bS^2)}.$$

Thus similarly to [8], it is reasonable to choose the Talenti function to estimate the mountain pass level for our problem. Actually, we get the desired conclusion as in the previous lemma.

We now prove Theorem 1.1.

Proof of Theorem 1.1. Take $a, b, \lambda, \mu > 0$ with $\lambda < a\lambda_1$. First we assume $\mu > bS^2$. In this case, we apply the mountain pass lemma [5]. As usual, we shall ensure the mountain pass geometry of I, that is, $I(0) = 0$ and

(1) there exist constants $\alpha, \rho > 0$ such that $I(u) \geq \alpha$ for all $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1(\Omega)} = \rho$,

(2) there exists a function $e_0 \in H_0^1(\Omega)$ such that $\|e_0\|_{H_0^1(\Omega)} > \rho$ and $I(e_0) \le 0$.

Firstly, let us cofirm (1). To do this, take $\rho > 0$. Then for all $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1(\Omega)} = \rho$, we have by the Poincare inequality and the Sobolev inequality,

$$\begin{aligned} I(u) &\ge \frac{a}{2}\left(1 - \frac{\lambda}{a\lambda_1}\right)\|u\|^2_{H_0^1(\Omega)} - \frac{1}{4S^2}\left(\mu - bS^2\right)\|u\|^4_{H_0^1(\Omega)} \\ &= \frac{a}{2}\left(1 - \frac{\lambda}{a\lambda_1}\right)\rho^2 - \frac{1}{4S^2}\left(\mu - bS^2\right)\rho^4. \end{aligned}$$

Noting $\lambda < a\lambda_1$, we get (1). Next suppose $\varepsilon \in (0, \varepsilon_1)$ and $t \ge 0$ where $\varepsilon_1 > 0$ is taken from Lemma 2.2. Using (9) and the assumption $\mu > bS^2$, and further, taking $\varepsilon_1 > 0$ smaller if necessary, we obtain

$$\begin{aligned} I(tv_\varepsilon) &\le \frac{a\|v_\varepsilon\|^2_{H_0^1(\Omega)}}{2}t^2 - \frac{\left(\mu - b\|v_\varepsilon\|^4_{H_0^1(\Omega)}\right)}{4}t^4 \\ &= \frac{a(S + O(\varepsilon^2))}{2}t^2 - \frac{\left(\mu - bS^2 + O(\varepsilon^2)\right)}{4}t^4 \\ &\le aSt^2 - \frac{(\mu - bS^2)}{8}t^4, \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_1)$. We fix such a ε. Then it follows from the above inequality, $I(tv_\varepsilon) \to -\infty$ as $t \to \infty$. Thus choosing $t_0 > 0$ sufficiently large and putting $e_0 := t_0 v_\varepsilon$ we have a function $e_0 \in H_0^1(\Omega)$ satisfying (2). Now we define

$$\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega)) \mid \gamma(0) = 0,\ \gamma(1) = e_0\},$$

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I(u) > 0.$$

Then Lemma 2.2 implies

$$c < \frac{(aS)^2}{4(\mu - bS^2)}.$$

Thus from Lemma 2.1, I satisfies the $(\mathrm{PS})_c$ condition. Consequently the mountain pass theorem concludes the proof. Next, we suppose $\mu \le bS^2$ and $u > 0$ in Ω is a solution of (P). Then the Poincare inequality and the Sobolev inequality imply

$$\begin{aligned} 0 &= a\|u\|^2_{H_0^1(\Omega)} + b\|u\|^4_{H_0^1(\Omega)} - \lambda\int_\Omega u^2 dx - \mu\int_\Omega u^4 dx \\ &\ge a\left(1 - \frac{\lambda}{a\lambda_1}\right)\|u\|^2_{H_0^1(\Omega)} + \frac{(bS^2 - \mu)}{S^2}\|u\|^4_{H_0^1(\Omega)}. \end{aligned}$$

Since $0 < \lambda < a\lambda_1$ and $\mu \le bS^2$, we have $u = 0$, a contradiction. This completes the proof. $\square$

3. The case $1 < q < 3$

In this section, we consider the case $1 < q < 3$ and prove Theorem 1.3 and 1.6. To do this, we assume $a, b, \lambda, \mu > 0$ satisfy $bS^2 < \mu < 2bS^2$ and fix $1/2 < \delta < 1$ so that $bS^2/\delta < \mu$. Then for $\nu \in (\delta, 1]$, we consider the problem (P_ν). The associated functional is defined by

$$I_\nu(u) := \frac{a}{2}\|u\|^2_{H_0^1(\Omega)} + \frac{b}{4}\|u\|^4_{H_0^1(\Omega)} - \frac{\nu\lambda}{q+1}\int_\Omega u_+^{q+1}dx - \frac{\nu\mu}{4}\int_\Omega u_+^4 dx.$$

We prove the existence of a nontrivial critical point of I_ν. In this case, the boundedness of the PS sequences for I_ν is hard to get. To avoid this difficulty, we introduce the result by Jeanjean [13].

Theorem 3.1 (Jeanjean[13])**.** *Let X be a Banach space equipped with the norm $\|\cdot\|$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $(I_\nu)_{\nu\in J}$ of C^1-functionals on X of the form*

$$I_\nu(u) = A(u) - \nu B(u) \ (\nu \in J)$$

where $B(u) \geq 0$ for all $u \in X$ and such that $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$. We assume there are two points (e_1, e_2) in X such that setting

$$\Gamma = \{\gamma \in C([0,1], X), \gamma(0) = e_1, \gamma(1) = e_2\}$$

there holds, for all $\nu \in J$

$$c_\nu := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\nu(\gamma(t)) > \max\{I_\nu(e_1), I_\nu(e_2)\}.$$

Then, almost every $\nu \in J$, there is a sequence $(u_n) \subset X$ such that

(i) (u_n) is bounded, (ii)$I_\nu(u_n) \to c_\nu$, (iii) $I'_\nu(u_n) \to 0$ in the dual X^{-1} of X.

With the help of Theorem 3.1, we can get the bounded PS sequences for I_ν for almost all $\nu \in (\delta, 1]$. Here we prove the local compactness of those.

Lemma 3.2. *Let $b > 0$, $\mu > 0$ satisfy $bS^2 < \mu < 2bS^2$ and take $1/2 < \delta < 1$ so that $bS^2/\delta < \mu$. Furthermore choose constants $0 < \lambda_0 = \lambda_0(a, b, q, \mu) < \infty$, $a_0 = a_0(b, q, \lambda, \mu) > 0$ and $\mu/(2S^2) < b_0 = b_0(a, q, \lambda, \mu) < \mu/S^2$ as in Remark 1.5 and assume one of the following (C1), (C2) and (C3) holds,*

(C1) $a > 0$ and $0 < \lambda < \lambda_0$,
(C2) $\lambda > 0$ and $a > a_0$.
(C3) $a, \lambda > 0$ and $b_0 < b < \mu/S^2$

Then if (u_n) is a bounded (PS)$_c$ sequence for I_ν with $\nu \in (\delta, 1]$ and

$$c < \frac{(aS)^2}{4(\nu\mu - bS^2)},$$

then (u_n) strongly converges in $H_0^1(\Omega)$ up to subsequences.

Remark 3.3. *Here we use our condition $\mu < 2bS^2$ which is the different point from the case $q = 1$.*

Proof. We assume on the contrary that we can extract no subsequence from (u_n) which converges in $H_0^1(\Omega)$. Then similarly to Proposition 1.7, there exist a nonnegative weak convergence $u_0 \in H_0^1(\Omega)$ of u_n, a number $k \in \mathbb{N}$ and further, for every $i \in \{1, 2, \cdots, k\}$, a sequence of values $(R_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}^+$, points $(x_n^i)_{n\in\mathbb{N}} \subset \overline{\Omega}$ and a nonnegative function $v_i \in D^{1,2}(\mathbb{R}^4)$ satisfying

$$-\left\{a + b\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{k} \|v_j\|^2_{D^{1,2}(\mathbb{R}^4)}\right)\right\}\Delta v_i = \nu\mu v_i^3 \text{ in } \mathbb{R}^4, \tag{10}$$

such that up to subsequences,

$$I_\nu(u_n) = \tilde{I}_\nu(u_0) + \sum_{i=1}^{k} \tilde{I}_\nu^\infty(v_i) + o(1) \tag{11}$$

where $o(1) \to 0$ as $n \to \infty$ and we put

$$\begin{aligned}
\tilde{I}_\nu(u_0) :=& \left\{\frac{a}{2} + \frac{b}{4}\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{k} \|v_j\|^2_{D^{1,2}(\mathbb{R}^4)}\right)\right\}\|u_0\|^2_{H_0^1(\Omega)} \\
&- \frac{\nu\lambda}{q+1}\int_\Omega u_0^{q+1}dx - \frac{\nu\mu}{4}\int_\Omega u_0^4 dx, \\
\tilde{I}_\nu^\infty(v_i) :=& \left\{\frac{a}{2} + \frac{b}{4}\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{k} \|v_j\|^2_{D^{1,2}(\mathbb{R}^4)}\right)\right\}\|v_i\|^2_{D^{1,2}(\mathbb{R}^4)} \\
&- \frac{\nu\mu}{4}\int_{\mathbb{R}^4} v_i^4 dx.
\end{aligned} \tag{12}$$

Here we note that, since $1 < q < 3$, it is not obvious whether or not $\tilde{I}_\nu(u_0) \geq 0$, differently from the case $q = 1$ (see the proof of Lemma 2.1). To overcome this difficulty, we shall estimate the energy of our PS sequence more precisely, including the "cross terms" which we indicate in Subsection 1.3. Now we claim

$$\tilde{I}(v_i) \geq \frac{(aS)^2}{4(\nu\mu - bS^2)} + \frac{abS^2}{4(\nu\mu - bS^2)}\|u_0\|^2_{H_0^1(\Omega)} \tag{13}$$

for all $i \in \{1, 2, \cdots, k\}$. In fact, similarly to the proof of (7), using (10) and the Sobolev inequality, we have

$$\begin{aligned}
0 &= \left\{a + b\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{k} \|v_j\|^2_{D^{1,2}(\mathbb{R}^4)}\right)\right\}\|v_i\|^2_{D^{1,2}(\mathbb{R}^4)} - \nu\mu\int_{\mathbb{R}^4} v_i^4 dx \\
&\geq \|v_i\|^2_{D^{1,2}(\mathbb{R}^4)}\left\{\left(a + b\|u_0\|^2_{H_0^1(\Omega)}\right) - S^{-2}\left(\nu\mu - bS^2\right)\|v_i\|^2_{D^{1,2}(\mathbb{R}^4)}\right\},
\end{aligned}$$

for all $i \in \{1, 2, \cdots, k\}$. In this case, we estimate

$$\|v_i\|^2_{D^{1,2}(\mathbb{R}^4)} \geq \frac{\left(a + b\|u_0\|^2_{H^1_0(\Omega)}\right) S^2}{\nu\mu - bS^2}. \tag{14}$$

Consequently, (10), (12) and (14) imply

$$\begin{aligned}\tilde{I}^\infty_\nu(v_i) &\geq \frac{a}{4}\|v_i\|^2_{D^{1,2}(\mathbb{R}^4)} \\ &\geq \frac{(aS)^2}{4(\nu\mu - bS^2)} + \frac{abS^2}{4(\nu\mu - bS^2)}\|u_0\|^2_{H^1_0(\Omega)},\end{aligned}$$

for all $i \in \{1, 2, \cdots, k\}$. This is (13). Next using (14) and the Sobolev inequalities, we get for some $i \in \{1, 2, \cdots, k\}$,

$$\begin{aligned}\tilde{I}_\nu(u_0) \geq\ & \frac{a}{2}\|u_0\|^2_{H^1_0(\Omega)} + \frac{b}{4}\left(\|u_0\|^2_{H^1_0(\Omega)} + \|v_i\|^2_{D^{1,2}(\mathbb{R}^4)}\right)\|u_0\|^2_{H^1_0(\Omega)} \\ & - \frac{\nu\lambda}{q+1}\int_\Omega u_0^{q+1}dx - \frac{\nu\mu}{4}\int_\Omega u_0^4 dx \\ \geq\ & a\left(\frac{1}{2} + \frac{bS^2}{4\left(\nu\mu - bS^2\right)}\right)\|u_0\|^2_{H^1_0(\Omega)} - \frac{\nu\lambda}{(q+1)S_{q+1}^{(q+1)/2}}\|u_0\|^{q+1}_{H^1_0(\Omega)} \\ & + \frac{\mu\nu\left(2bS^2 - \nu\mu\right)}{4S^2\left(\nu\mu - bS^2\right)}\|u_0\|^4_{H^1_0(\Omega)}.\end{aligned} \tag{15}$$

Then it follows from (11), (13) and (15) that

$$\begin{aligned}c &= \lim_{n\to\infty} I_\nu(u_n) \\ &\geq \tilde{I}_\nu(u_0) + \tilde{I}^\infty_\nu(v_i) \\ &\geq \frac{(aS)^2}{4(\nu\mu - bS^2)} + \frac{a\mu}{2\left(\mu - bS^2\right)}\|u_0\|^2_{H^1_0(\Omega)} - \frac{\lambda}{(q+1)S_{q+1}^{(q+1)/2}}\|u_0\|^{q+1}_{H^1_0(\Omega)} \\ &\quad + \frac{\mu\left(2bS^2 - \mu\right)}{4S^2\left(\mu - bS^2\right)}\|u_0\|^4_{H^1_0(\Omega)},\end{aligned}$$

here for the last inequality, we use the fact $\nu \leq 1$. Observe that, the coefficient of $\|u_0\|^4_{H^1_0(\Omega)}$ in the right hand side of the last inequality is positive thanks to our assumption $bS^2 < \mu < 2bS^2$. Finally (C1), (C2) or (C3) shows

$$c \geq \frac{(aS)^2}{4\left(\nu\mu - bS^2\right)},$$

a contradiction. Thus (u_n) strongly converges in $H^1_0(\Omega)$ up to subsequences. This completes the proof. $\square$

As Section 2, we prove a mountain pass level of I_ν is below the desired energy level.

Lemma 3.4. *Let $a, b, \lambda, \mu > 0$ satisfy $bS^2 < \mu$ and take $1/2 < \delta < 1$ so that $bS^2/\delta < \mu$. We suppose $\nu \in (\delta, 1]$. Then there exists a constant $\varepsilon_2 > 0$ such that*

$$\sup_{t \geq 0} I_\nu(tv_\varepsilon) < \frac{(aS)^2}{4\,(\nu\mu - bS^2)}$$

for all $\varepsilon \in (0, \varepsilon_2)$, where v_ε is defined as previous section.

Proof. First observe that we have the estimate

$$\int_\Omega v_\varepsilon^{q+1} dx = \alpha_2 \varepsilon^{3-q},$$

where $\alpha_2 > 0$ is some constant. Here, using (9), we can easily check that there exists a constant $\varepsilon_2 > 0$ such that we can find constants $0 < \tau_0 < T_0$ such as

$$I_\nu(tv_\varepsilon) < \frac{(aS)^2}{8\,(\nu\mu - bS^2)}$$

for all $0 \leq t \leq \tau_0$ and all $t \geq T_0$ if $\varepsilon \in (0, \varepsilon_2)$. Noting this, we consider only $t \in (\tau_0, T_0)$. As $bS^2 < \nu\mu$, we have

$$\begin{aligned} I_\nu(tv_\varepsilon) &\leq \frac{aS}{2} t^2 - \frac{\left(\nu\mu - bS^2\right)}{4} t^4 - C\alpha_2 \varepsilon^{3-q} + O(\varepsilon^2) \\ &\leq \frac{(aS)^2}{4(\nu\mu - bS^2)} - C\alpha_2\varepsilon^{3-q} + O(\varepsilon^2), \end{aligned}$$

for some constant $C > 0$ which is independent of $\varepsilon \in (0, \varepsilon_2)$. Then since $1 < q < 3$, taking $\varepsilon_2 > 0$ smaller if necessary, we conclude

$$\sup_{t \geq 0} I_\nu(tv_\varepsilon) < \frac{(aS)^2}{4(\nu\mu - bS^2)},$$

for all $\varepsilon \in (0, \varepsilon_2)$. This finishes the proof. □

We prove Theorem 1.3.

The proof of Theorem 1.3. Let $b, \mu > 0$ satisfy $bS^2 < \mu < 2bS^2$. Choose $1/2 < \delta < 1$ so that $bS^2/\delta < \mu$ and suppose $\nu \in (\delta, 1]$. Furthermore assume one of (C1)-(C3) in Lemma 3.2 holds. To apply Theorem 3.1, we confirm the mountain pass geometry of I_ν which is determined independently of $\nu \in (\delta, 1]$. To do this, first assume $\rho > 0$ and take $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1(\Omega)} = \rho$. Then as $\nu \leq 1$, the Sobolev embeddings imply

$$\begin{aligned} I(u) &\geq \frac{a}{2}\|u\|^2_{H_0^1(\Omega)} - \frac{\lambda}{(q+1)S_{q+1}^{(q+1)/2}}\|u\|^{q+1}_{H_0^1(\Omega)} - \frac{\mu - bS^2}{4S^2}\|u\|^4_{H_0^1(\Omega)} \\ &\geq \frac{a}{2}\rho^2 - \frac{\lambda}{(q+1)S_{q+1}^{(q+1)/2}}\rho^{q+1} - \frac{\mu - bS^2}{4S^2}\rho^4. \end{aligned}$$

Since $1 < q < 3$ and the right hand side of the last inequality is independent of $\nu \in (\delta, 1]$, we conclude that

(a) there exist constants $\alpha, \rho > 0$ such that $I_\nu(u) \geq \alpha$ for all $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1(\Omega)} = \rho$ and all $\nu \in (\delta, 1]$.

Next noting $\delta < \nu \leq 1$ and (9), we get for all $t > 0$,

$$
\begin{aligned}
I_\nu(tv_\varepsilon) &\leq \frac{a\|v_\varepsilon\|_{H_0^1(\Omega)}^2}{2}t^2 + \frac{b\|v_\varepsilon\|_{H_0^1(\Omega)}^4}{4}t^4 - \frac{\delta\mu}{4}t^4 \\
&= \frac{a\left(S + O(\varepsilon^2)\right)}{2}t^2 - \frac{\left(\delta\mu - bS^2 + O(\varepsilon^2)\right)}{4}t^4.
\end{aligned}
$$

Now take $\varepsilon_2 > 0$ which is determined in Lemma 3.4. Then since $\delta\mu > bS^2$, taking $\varepsilon_2 > 0$ smaller if necessary, we have

$$
I_\nu(tv_\varepsilon) \leq aSt^2 - \frac{(\delta\mu - bS^2)}{8}t^4.
$$

for all $\varepsilon \in (0, \varepsilon_2)$. Then we fix such a ε and get $I_\nu(tv_\varepsilon) \to -\infty$ as $t \to \infty$ uniformly for $\nu \in (\delta, 1]$. Therefore there exists a constant $t_0 > 0$ such that if we put $e_0 := t_0 v_\varepsilon$, $\|e_0\|_{H_0^1(\Omega)} > \rho$ and $I_\nu(e_0) \leq 0$ for all $\nu \in (\delta, 1]$. Now we can define

$$
\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega)) \mid \gamma(0) = 0,\ \gamma(1) = e_0\},
$$

$$
c_\nu := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} I_\nu(u).
$$

Observe that $c_\nu > 0$ for all $\nu \in (\delta, 1]$ from (a). Consequently, utilizing Theorem 3.1, we have a bounded PS sequence of I_ν for almost every $\nu \in (\delta, 1]$. Furthermore by Lemma 3.2, 3.4 and the definition of c_ν, our bounded $(\mathrm{PS})_{c_\nu}$ sequence strongly converges to some nontrivial function in $H_0^1(\Omega)$ up to subsequences and thus, I_ν has a nontrivial critical point for almost every $\nu \in (\delta, 1]$. Then we can take an increasing sequence $(\nu_n) \subset (\delta, 1]$ such that $\nu_n \to 1$ as $n \to \infty$ and for every $n \in \mathbb{N}$, there exists a nontrivial critical point u_n of I_{ν_n} with critical value c_{ν_n}. Note that by the continuity, $c_{\nu_n} \to c_1$ as $n \to \infty$ (see Lemma 2.3 in [13]). Then

$$
\begin{aligned}
I_1(u_n) &= I_{\nu_n}(u_n) + (1 - \nu_n)\left(\frac{\lambda}{q+1}\int_\Omega (u_n)_+^{q+1}dx + \frac{\mu}{4}\int_\Omega (u_n)_+^4 dx\right) \\
&= c_1 + o(1)
\end{aligned}
$$

where $o(1) \to 0$ as $n \to \infty$. Similarly,

$$
I_1'(u_n) = I_{\nu_n}'(u_n) + o(1) = o(1)
$$

where $o(1) \to 0$ in $H^{-1}(\Omega)$ as $n \to \infty$. Here we assume $\|u_n\|_{H_0^1(\Omega)}$ is bounded. Then (u_n) is a bounded $(\mathrm{PS})_{c_1}$ sequence for I_1. Then Lemma 3.2, 3.4 and the definition of c_1 conclude the proof. □

We can get the boundedness of the sequence of solutions (u_n) above, if Ω is strictly star-shaped.

The proof of Theorem 1.6. Assume $\Omega \subset \mathbb{R}^4$ is strictly star-shaped and take a sequence of values $(\nu_n) \subset (\delta, 1]$ and corresponding solutions $(u_n) \subset H_0^1(\Omega)$ as in the proof of Theorem 1.3. Our aim is to prove (u_n) is bounded in $H_0^1(\Omega)$. We argue by the contradiction. Suppose $\|u_n\|_{H_0^1(\Omega)} \to \infty$ as $n \to \infty$. We put $w_n := u_n/\|u_n\|_{H_0^1(\Omega)}$. Then $w_n \geq 0$, $\|w_n\|_{H_0^1(\Omega)} = 1$ and consequently, there exists a nonnegative function $w_0 \in H_0^1(\Omega)$ such that $w_n \rightharpoonup w_0$ weakly in $H_0^1(\Omega)$ up to subsequences. Since w_n satisfies

$$\left(\frac{a}{\|u_n\|^2_{H_0^1(\Omega)}} + b\right) \int_\Omega \nabla w_n \cdot \nabla h dx = \frac{\nu_n \lambda}{\|u_n\|^{3-q}_{H_0^1(\Omega)}} \int_\Omega w_n^q h dx + \nu_n \mu \int_\Omega w_n^3 h dx \quad (16)$$

for all $h \in H_0^1(\Omega)$, taking $n \to \infty$, we get

$$b \int_\Omega \nabla w_0 \cdot \nabla h dx = \mu \int_\Omega w_0^3 h.$$

As Ω is strictly star-shaped, we have $w_0 = 0$ from the result by Pohozaev [23]. Furthermore, it follows from (16) and the argument in [25] that there exists a number $l \in \mathbb{N}$ and for every $i \in \{1, 2, \cdots, l\}$, a sequence of values $(R_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}^+$, points $(x_n^i)_{n\in\mathbb{N}} \subset \overline{\Omega}$ with $R_n^i \mathrm{dist}(x_n^i, \partial\Omega) \to \infty$ as $n \to \infty$, and a nonnegative function $v_i \in D^{1,2}(\mathbb{R}^4)$ satisfying

$$-b\Delta v_i = \mu v_i^3 \text{ in } \mathbb{R}^4,$$

such that up to subsequences,

$$1 = \|w_n\|^2_{H_0^1(\Omega)} = \sum_{i=1}^{l} \|v_i\|^2_{D^{1,2}(\mathbb{R}^4)} + o(1), \quad (17)$$

where $o(1) \to 0$ as $n \to \infty$. Since $\tilde{v}_i := (\mu/b)^{1/2} v \in D^{1,2}(\mathbb{R}^4)$ is a nonnegative solution of

$$-\Delta \tilde{v} = \tilde{v}^3 \text{ in } \mathbb{R}^4, \; w(x) \to 0 \text{ as } |x| \to \infty, \quad (18)$$

the uniqueness result from [12] implies that there exist a constant $\varepsilon_i > 0$ and a point $x_i \in \mathbb{R}^4$ such that

$$\tilde{v}_i(x) = \frac{8^{\frac{1}{2}} \varepsilon_i}{\varepsilon_i^2 + |x - x_i|^2}.$$

Therefore we have

$$\|v_i\|^2_{D^{1,2}(\mathbb{R}^4)} = \frac{b}{\mu} \|\tilde{v}_i\|^2_{D^{1,2}(\mathbb{R}^4)} = \frac{bS^2}{\mu},$$

for all $i \in \{1, 2, \cdots, l\}$. Then from (17), we get

$$1 = \frac{lbS^2}{\mu},$$

for $l \in \mathbb{N}$ which is impossible since $bS^2 < \mu < 2bS^2$. This is a contradiction. Thus (u_n) is bounded in $H_0^1(\Omega)$. Then Theorem 1.3 completes the proof. $\square$

4. A global compactness result

In this last section, we give the description of PS sequences for the Kirchhoff type critical problem and show the global compactness result. To this aim, we consider the problem

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2\right)\Delta u = \lambda u + |u|^{2^*-2}u \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{P$_N$}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ is a bounded domain with smooth boundary $\partial\Omega$ and further, we assume $a>0$, $b \geq 0$, $\lambda \in \mathbb{R}$ and $2^* = 2N/(N-2)$ is the critical exponent of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. The energy functional associated to (P$_N$) is given by

$$I(u) := \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} + \frac{b}{4}\|u\|^4_{H_0^1(\Omega)} - \frac{\lambda}{2}\int_\Omega u^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx,$$

where $\|\cdot\|_{H_0^1(\Omega)} := (\int_\Omega |\nabla u|^2 dx)^{1/2}$. Following the argument in [25], we firstly give the complete description of PS sequences for I. Here similarly to the previous sections, we introduce the Sobolev space $D^{1,2}(\mathbb{R}^N)$ as usual and put its norm as $\|\cdot\|_{D^{1,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla \cdot|^2 dx\right)^{1/2}$.

Theorem 4.1. *Let $(u_n) \subset H_0^1(\Omega) \subset D^{1,2}(\mathbb{R}^N)$ be a bounded PS sequence for I. Then (u_n) has a subsequence which converges strongly in $H_0^1(\Omega)$ or otherwise, there exist a function $u_0 \in H_0^1(\Omega)$ which is a weak convergence of u_n, a number $k \in \mathbb{N}$ and further, for every $i \in \{1,2,\cdots,k\}$, a sequence of values $(R_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}^+$, points $(x_n^i)_{n\in\mathbb{N}} \subset \overline{\Omega}$ and a function $v_i \in D^{1,2}(\mathbb{R}^N)$ which satisfy*

$$-\left\{a+b\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^k \|v_j\|^2_{D^{1,2}(\mathbb{R}^N)}\right)\right\}\Delta u_0 = \lambda u_0 + |u_0|^{2^*-2}u \textit{ in } \Omega,$$

$$-\left\{a+b\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^k \|v_j\|^2_{D^{1,2}(\mathbb{R}^N)}\right)\right\}\Delta v_i = |v_i|^{2^*-2}v_i \textit{ in } \mathbb{R}^N, \tag{19}$$

such that up to subsequences, $R_n^i \operatorname{dist}(x_n^i, \partial\Omega) \to \infty$ as $n \to \infty$,

$$\left\|u_n - u_0 - \sum_{i=1}^k (R_n^i)^{\frac{N-2}{2}} v_i(R_n^i(\cdot - x_n^i))\right\|_{D^{1,2}(\mathbb{R}^N)} = o(1),$$

$$\|u_n\|^2_{H_0^1(\Omega)} = \|u_0\|^2_{H_0^1(\Omega)} + \sum_{i=1}^k \|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} + o(1),$$

and

$$I(u_n) = \tilde{I}(u_0) + \sum_{i=1}^k \tilde{I}_\infty(v_i) + o(1),$$

where $o(1) \to 0$ as $n \to \infty$ and we put

$$\begin{aligned}\tilde{I}(u_0) &:= \frac{a}{2}\|u_0\|^2_{H^1_0(\Omega)} + \frac{b}{4}\left(\|u_0\|^2_{H^1_0(\Omega)} + \sum_{j=1}^{k}\|v_j\|^2_{D^{1,2}(\mathbb{R}^N)}\right)\|u_0\|^2_{H^1_0(\Omega)} \\ &\quad - \frac{\lambda}{2}\int_\Omega u_0^2 dx - \frac{1}{2^*}\int_\Omega |u_0|^{2^*} dx, \\ \tilde{I}_\infty(v_i) &:= \frac{a}{2}\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{b}{4}\left(\|u_0\|^2_{H^1_0(\Omega)} + \sum_{j=1}^{k}\|v_j\|^2_{D^{1,2}(\mathbb{R}^N)}\right)\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} \\ &\quad - \frac{1}{2^*}\int_{\mathbb{R}^N} |v_i|^{2^*} dx.\end{aligned} \tag{20}$$

Remark 4.2. *In Theorem 4.1 if we additionally assume that $u_n \geq 0$, then u_0 and v_i are nonnegative. Furthermore if we consider the functional*

$$I_+(u) := \frac{a}{2}\|u\|^2_{H^1_0(\Omega)} + \frac{b}{4}\|u\|^4_{H^1_0(\Omega)} - \frac{\lambda}{2}\int_\Omega u_+^2 dx - \frac{1}{2^*}\int_\Omega u_+^{2^*} dx,$$

instead of I, we also have u_0 and v_i are nonnegative.

Remark 4.3. *If $v_i \in D^{1,2}(\mathbb{R}^N)$ is nonnegative, then we have $\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} = \|v_j\|^2_{D^{1,2}(\mathbb{R}^N)}$ for all $i, j \in \{1, 2, \cdots, k\}$ with $i \neq j$. In fact, since v_i satisfies (19), if we put $w_i := (a + bA)^{-(N-2)/4} v_i$ where $A := \|u_0\|^2_{H^1_0(\Omega)} + \sum_{i=1}^{k}\|v_j\|^2_{D^{1,2}(\mathbb{R}^N)}$, $w_i \in D^{1,2}(\mathbb{R}^N)$ is a nonnegative solution of*

$$-\Delta w = w^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N, \ w(x) \to 0 \text{ as } |x| \to \infty.$$

The uniqueness assertion of the above problem (see [12]) implies that there exist a constant $\varepsilon_i > 0$ and a point $x_i \in \mathbb{R}^N$ such that

$$w_i = \frac{(N(N-2)\varepsilon_i^2)^{\frac{N-2}{4}}}{(\varepsilon_i^2 + |\cdot - x_i|^2)^{\frac{N-2}{2}}}.$$

Thus we have

$$\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} = (a + bA)^{\frac{N-2}{2}} S^{\frac{N}{2}}. \tag{21}$$

Since the right hand side of the above equality is independent of $i \in \{1, 2, \cdots, k\}$, we confirm the claim.

Let us see the global compactness results for the cases $N = 3, 4$. We note that the local compactness result for (P_N) with $N = 3$ is found in [10], [20], [21], [29] etc., and that for the case $N = 4$ is treated in previous sections. Here, we assume $u_n \geq 0$ in Theorem 4.1. It follows from Remark 4.2, 4.3 and (21),

$$\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} = \left(a + b\|u_0\|^2_{H^1_0(\Omega)} + kb\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)}\right)^{\frac{N-2}{2}} S^{\frac{N}{2}}.$$

Consequently we deduce an equation for $\|v_i\|_{D^{1,2}(\mathbb{R}^N)}$,

$$\|v_i\|_{D^{1,2}(\mathbb{R}^N)}^{\frac{4}{N-2}} - kbS^{\frac{N}{N-2}}\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} - \left(a + b\|u_0\|^2_{H_0^1(\Omega)}\right) S^{\frac{N}{N-2}} = 0. \tag{22}$$

Firstly let $N = 3$. Then we have

$$\|v_i\|^2_{D^{1,2}(\mathbb{R}^3)} = \frac{1}{2}\left(kbS^3 + \sqrt{(kbS^3)^2 + 4\left(a + b\|u_0\|^2_{H_0^1(\Omega)}\right) S^3}\right).$$

Using (19), (20) and the above equality, we get

$$\begin{aligned}
\tilde{I}_\infty(v_i) &= \tilde{I}_\infty(v_i) - \frac{1}{6}\left(a\|v_i\|^2_{D^{1,2}(\mathbb{R}^3)} + bA\|v_i\|^2_{D^{1,2}(\mathbb{R}^3)} - \int_\Omega v_i^6 dx\right) \\
&= \left(\frac{a}{3} + \frac{b\|u_0\|^2_{H_0^1(\Omega)}}{12}\right)\left\{\frac{1}{2}\left(kbS^3 + \sqrt{(kbS^3)^2 + 4\left(a + b\|u_0\|^2_{H_0^1(\Omega)}\right) S^3}\right)\right\} \\
&\quad + \frac{kb}{12}\left\{\frac{1}{2}\left(kbS^3 + \sqrt{(kbS^3)^2 + 4\left(a + b\|u_0\|^2_{H_0^1(\Omega)}\right) S^3}\right)\right\}^2 \\
&=: c_3^*\left(a, b, k, \|u_0\|^2_{H_0^1(\Omega)}\right).
\end{aligned}$$

Observe that the energy of a bubble depends on a, b and further, the number of all bubbles and the nonlocal information of the weak convergence u_0. Consequently we conclude that if $(u_n) \subset H_0^1(\Omega)$ is a $(\mathrm{PS})_c$ sequence for I with

$$c \notin \left\{\tilde{I}(u_0) + kc_3^*\left(a, b, k, \|u_0\|^2_{H_0^1(\Omega)}\right)\right\}_{k\in\mathbb{N}},$$

then we can extract a subsequence from (u_n) which strongly converges in $H_0^1(\Omega)$. This is a global compactness result for the Kirchhoff type problem in dimension three. In particular, note that

$$\begin{aligned}
c_3^*\left(a, b, k, \|u_0\|^2_{H_0^1(\Omega)}\right) &\geq c_3^*\left(a, b, k, \|u_0\|^2_{H_0^1(\Omega)}\right)\Big|_{k=1,\ \|u_0\|^2_{H_0^1(\Omega)}=0} \\
&= \frac{1}{3}\left\{\frac{1}{2}\left(bS^3 + \sqrt{(bS^3)^2 + 4aS^3}\right)\right\} \\
&\quad + \frac{b}{12}\left\{\frac{1}{2}\left(bS^3 + \sqrt{(bS^3)^2 + 4aS^3}\right)\right\}^2.
\end{aligned}$$

Note also that if $\lambda < a\lambda_1$, $\tilde{I}(u_0) \geq 0$. Hence in this case, all $(\mathrm{PS})_c$ sequences of I with

$$\begin{aligned}
c < &\frac{1}{3}\left\{\frac{1}{2}\left(bS^3 + \sqrt{(bS^3)^2 + 4aS^3}\right)\right\} \\
&+ \frac{b}{12}\left\{\frac{1}{2}\left(bS^3 + \sqrt{(bS^3)^2 + 4aS^3}\right)\right\}^2,
\end{aligned}$$

strongly converges in $H_0^1(\Omega)$ up to subsequences. This is a local compactness assertion for the case $N = 3$, which is observed in [20], [21] and [29]. Next suppose $N = 4$. We use (22) again to get a necessary condition on $k \in \mathbb{N}$,

$$1 - kbS^2 > 0$$

and

$$\|v_i\|^2_{D^{1,2}(\mathbb{R}^4)} = \frac{\left(a + b\|u_0\|^2_{H_0^1(\Omega)}\right) S^2}{1 - kbS^2}.$$

Then noting (19), (20) and the above equality, we obtain

$$\begin{aligned}\tilde{I}_\infty(v_i) &= \tilde{I}_\infty(v_i) - \frac{1}{4}\left(a\|v_i\|^2_{D^{1,2}(\mathbb{R}^4)} + bA\|v_i\|^2_{D^{1,2}(\mathbb{R}^4)} - \int_{\mathbb{R}^4} v_i^4 dx\right) \\ &= \frac{a\left(a + b\|u_0\|^2_{H_0^1(\Omega)}\right) S^2}{4(1 - kbS^2)} \\ &=: c_4^*\left(a, b, k, \|u_0\|^2_{H_0^1(\Omega)}\right).\end{aligned}$$

Thus if $(u_n) \subset H_0^1(\Omega)$ is a PS sequence for I with

$$c \notin \left\{\tilde{I}(u_0) + kc_4^*\left(a, b, k, \|u_0\|^2_{H_0^1(\Omega)}\right)\right\}_{k\in\mathbb{N}},$$

then (u_n) has a subsequence which strongly converges in $H_0^1(\Omega)$. This is a global compactness result for the case $N = 4$. In particular, we can also check that if $\lambda < a\lambda_1$, $\tilde{I}(u_0) \geq 0$ and

$$\begin{aligned}c_4^*\left(a, b, k, \|u_0\|^2_{H_0^1(\Omega)}\right) &\geq c_4^*\left(a, b, k, \|u_0\|^2_{H_0^1(\Omega)}\right)\Big|_{k=1,\ \|u_0\|^2_{H_0^1(\Omega)}=0} \\ &= \frac{(aS)^2}{4(1 - bS^2)}.\end{aligned}$$

Thus in this cases, if (u_n) is a $(\mathrm{PS})_c$ sequence for I with

$$c < \frac{(aS)^2}{4(1 - bS^2)},$$

then (u_n) strongly converges in $H_0^1(\Omega)$ up to subsequences. This is a local compactness of the PS sequences in dimension four, which is observed in previous sections.

Remark 4.4. *In the larger dimensional case, that is, when $N \geq 5$, the behaviors of PS sequences are drastically different from the cases $N = 3, 4$. For example, the energies of bubbles may be negative. Thus it seems to be difficult to get the clear compactness condition. This suggests that a certain difficulty will occur in dealing with the larger dimensional critical problems. But at this point, in view of our main aim of this paper, we rather stop here and proceed to the proof of Theorem 4.1.*

To prove Theorem 4.1, we introduce the following lemma.

Lemma 4.5. *Let $A \geq 0$ be a constant and $(w_n^0) \subset H_0^1(\Omega)$ be a sequence such that*

$$w_n^0 \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega)$$

and

$$J'(w_n^0) \to 0 \text{ in } H^{-1}(\Omega),$$
$$\tilde{I}(w_n^0) \to \beta \in \mathbb{R},$$

where $J, \tilde{I} \in C^1(H_0^1(\Omega), \mathbb{R})$ are defined by

$$J(u) := \frac{a+bA}{2}\|u\|_{H_0^1(\Omega)}^2 - \frac{\lambda}{2}\int_\Omega u^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx,$$

$$\tilde{I}(u) := \frac{a}{2}\|u\|_{H_0^1(\Omega)}^2 + \frac{bA}{4}\|u\|_{H_0^1(\Omega)}^2 - \frac{\lambda}{2}\int_\Omega u^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx.$$

Then if we can choose no subsequence from (w_n^0) which strongly converges to 0 in $H_0^1(\Omega)$, there exist a sequence of values $(R_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$, points $(x_n)_{n\in\mathbb{N}} \subset \overline{\Omega}$, a function $v_0 \in D^{1,2}(\mathbb{R}^N)$ which satisfy

$$-(a+bA)\Delta v_0 = |v_0|^{2^*-2}v_0 \text{ in } \mathbb{R}^N$$

and further, a sequence of functions $(w_n) \subset H_0^1(\Omega)$ such that $R_n \text{dist}(x_n, \partial\Omega) \to \infty$ as $n \to \infty$,

$$w_n \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega),$$
$$w_n = w_n^0 - (R_n)^{\frac{N-2}{2}} v_0(R_n(\cdot - x_n)) + o(1) \text{ in } D^{1,2}(\mathbb{R}^N),$$
$$\|w_n\|_{H_0^1(\Omega)}^2 = \|w_n^0\|_{H_0^1(\Omega)}^2 - \|v_0\|_{D^{1,2}(\mathbb{R}^N)}^2 + o(1)$$

and

$$J'(w_n) = o(1) \text{ in } H^{-1}(\Omega),$$
$$\tilde{I}(w_n) = \beta - \tilde{I}_\infty(v_0) + o(1)$$

up to subsequences.

Proof. Similar to that in [25]. □

Proof of Theorem 4.1. Let $(u_n) \subset H_0^1(\Omega)$ be a bounded $(\mathrm{PS})_c$ sequence for I. Then there exists a constant $A \geq 0$ and a function $u_0 \in H_0^1(\Omega)$ such that $\|u_n\|_{H_0^1(\Omega)}^2 \to A$ and $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ up to subsequences. We put $w_n^0 := u_n - u_0$ and then have, $w_n^0 \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. If we can extract a subsequence so that $w_n^0 \to 0$ strongly in $H_0^1(\Omega)$, the proof is finished. If not, using the Vitali's convergence theorem, we get

$$J'(w_n^0) = I'(u_n) - J'(u_0) + o(1) = o(1) \text{ in } H^{-1}(\Omega)$$

here we use the facts that $A = \|u_n\|^2_{H^1_0(\Omega)} + o(1)$ as $n \to \infty$ and u_0 is a critical point of J. Furthermore, noting the weak convergence, similarly we have

$$\tilde{I}(w_n^0) = I(u_n) - \tilde{I}(u_0) + o(1) = c - \tilde{I}(u_0) + o(1)$$

where $o(1) \to 0$ as $n \to \infty$. Thus from Lemma 4.5, there exist a sequence of values $(R_n^1)_{n\in\mathbb{N}}$, points $(x_n^1)_{n\in\mathbb{N}} \subset \overline{\Omega}$, a function $v_1 \in D^{1,2}(\mathbb{R}^N)$ which satisfy

$$-(a + bA)\Delta v_1 = |v_1|^{2^*-2} v_1 \text{ in } \mathbb{R}^N$$

and further, a sequence of functions $(w_n^1) \subset H_0^1(\Omega)$, such that $R_n^1 \mathrm{dist}(x_n^1, \partial\Omega) \to \infty$,

$$w_n^1 \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega),$$
$$w_n^1 = w_n^0 - (R_n^1)^{\frac{N-2}{2}} v_1(R_n^1(\cdot - x_n^1)) + o(1) \text{ in } D^{1,2}(\mathbb{R}^N),$$
$$\|w_n^1\|^2_{H^1_0(\Omega)} = \|w_n^0\|^2_{H^1_0(\Omega)} - \|v_1\|^2_{D^{1,2}(\mathbb{R}^N)} + o(1)$$

and

$$J'(w_n^1) = o(1) \text{ in } H^{-1}(\Omega),$$
$$\tilde{I}(w_n^1) = \tilde{I}(w_n^0) - \tilde{I}_\infty(v_1) + o(1)$$

up to subsequences. If we can select a subsequence from (w_n) which strongly converges to 0 in $H_0^1(\Omega)$, the proof is finished. If not, we repeat the same argument with the above one. Finally we reach a number $k \in \mathbb{N}$ such that for every $i \in \{1, 2, \cdots, k\}$, there exist a sequence of values $(R_n^i)_{n\in\mathbb{N}} \subset \mathbb{R}^N$, points $(x_n^i)_{n\in\mathbb{N}} \subset \overline{\Omega}$, a function $v_i \in D^{1,2}(\mathbb{R}^N)$ which satisfy

$$-(a + bA)\Delta v_i = |v_i|^{2^*-2} v_i \text{ in } \mathbb{R}^N$$

and further, a sequence of functions $(w_n^i) \subset H_0^1(\Omega)$, such that $R_n^i \mathrm{dist}(x_n^i, \partial\Omega) \to \infty$ and

$$w_n^k = u_n - u_0 - \sum_{i=1}^{k} (R_n^i)^{\frac{N-2}{2}} v_i(R_n^i(\cdot - x_n^i)) + o(1) \text{ in } D^{1,2}(\mathbb{R}^N),$$

$$\|w_n^k\|^2_{H^1_0(\Omega)} = \|u_n\|^2_{H^1_0(\Omega)} - \|u_0\|^2_{H^1_0(\Omega)} - \sum_{i=1}^{k} \|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} + o(1),$$

$$\tilde{I}(w_n^k) = I(u_n) - \tilde{I}(u_0) - \sum_{i=1}^{k} \tilde{I}_\infty(v_i) + o(1),$$

and further,

$$w_n^k \to 0 \text{ in } H_0^1(\Omega),$$

up to subsequences. If not, we can choose a number $k' \in \mathbb{N}$ with

$$k' > \frac{A}{a^{\frac{N-2}{2}} S^{\frac{N}{2}}} \tag{23}$$

such that

$$\|u_n\|^2_{H^1_0(\Omega)} = \|w_n^{k'}\|^2_{H^1_0(\Omega)} + \|u_0\|^2_{H^1_0(\Omega)} + \sum_{i=1}^{k'} \|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} + o(1), \tag{24}$$

instead of $k \in \mathbb{N}$ above. Here from (19) and the Sobolev inequality, we have for every $i \in \{1, 2, \cdots, k'\}$,

$$\begin{aligned} 0 &\geq a\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} |v_i|^{2^*} dx \\ &\geq a\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} - S^{-\frac{2^*}{2}} \|v_i\|^{2^*}_{D^{1,2}(\mathbb{R}^N)}. \end{aligned}$$

This inequality implies

$$\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} \geq a^{\frac{N-2}{2}} S^{\frac{N}{2}}. \tag{25}$$

Using (24), (25) and (23), we conclude that

$$\begin{aligned} A &= \lim_{n\to\infty} \|u_n\|^2_{H^1_0(\Omega)} \\ &\geq k' a^{\frac{N-2}{2}} S^{\frac{N}{2}} \\ &> A, \end{aligned}$$

a contradiction. This proves our claim. The proof is done. □

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