SCALE INVARIANCE STRUCTURES OF THE CRITICAL AND THE SUBCRITICAL HARDY INEQUALITIES AND THEIR RELATIONSHIP

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Abstract. In this paper, first we establish an improved subcritical Hardy inequality in the whole space in spite of the lack of Poincaré inequality. This also enables us to improve the sharp version of the critical Hardy inequality on a ball. A key ingredient is a new transformation connecting the Hardy inequalities in critical and subcritical cases. By using the transformation, we reveal a relationship between the scale invariance structures of those Hardy inequalities.

1. Introduction

Let Ω be a smooth bounded domain with $0 \in \Omega$ in $\mathbb{R}^N$ ($N \geq 2$), or $\Omega = \mathbb{R}^N$, and let $1 \leq p < N$. The classical Hardy inequality

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \tag{1.1}$$

holds for all $u \in W_0^{1,p}(\Omega)$, or $u \in D^{1,p}(\mathbb{R}^N)$ when $\Omega = \mathbb{R}^N$. Here $D^{1,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\nabla \cdot\|_{L^p(\mathbb{R}^N)}$. It is known that for $1 < p$, the best constant $(\frac{N-p}{p})^p$ is never attained in $W_0^{1,p}(\Omega)$, or in $D^{1,p}(\mathbb{R}^N)$. Therefore, one can expect the existence of remainder terms on the right-hand side of the inequality (1.1). Indeed, there are many papers that deal with remainder terms for (1.1) when Ω is a smooth bounded domain (see [1], [4], [7], [9], [12], [19], to name a few). For example, Brezis and Vázquez [4] showed that the inequality

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} dx + z_0^2 \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \int_\Omega |u|^2 dx \tag{1.2}$$

holds true for all $u \in W_0^{1,2}(\Omega)$ where $z_0 = 2.4048\cdots$ is the first zero of the Bessel function of the first kind. After the usual symmetrization is employed, one of the key points of their proof is a clever transformation

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which estimates how far a function deviates from the "virtual" extremal of the Hardy inequality (1.1). The use of the Poincaré inequality on a ball is another key point. Furthermore, there are many applications of Hardy inequalities for parabolic equations with a singular potential (see [3], [11], [24], [21], [5]). Especially, Vázquez and Zuazua [24] applied the remainder term in (1.2) to study the large-time behavior of solutions to the linear heat equation with a singular potential.

On the other hand, when $\Omega = \mathbb{R}^N$, Ghoussoub and Moradifam [13] showed that there is no strictly positive $V \in C^1(\mathbb{R}^N \setminus \{0\})$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} V(|x|)|u|^2 dx$$

holds for all $u \in W^{1,2}(\mathbb{R}^N)$. Therefore we cannot expect the same type of remainder terms as in (1.2) would exist in the whole space. Instead, Cianchi and Ferone [8] provided the following "non-standard" remainder term: Let $p^* = \frac{Np}{N-p}$ be the critical Sobolev exponent, $v_a(x) = a|x|^{-\frac{N-p}{p}}$ for $x \in \mathbb{R}^N$, $a \in \mathbb{R}$, and define

$$d_p(u) = \inf_{a \in \mathbb{R}} \frac{\|u - v_a\|_{L^{p^*,\infty}(\mathbb{R}^N)}}{\|u\|_{L^{p^*,p}(\mathbb{R}^N)}} \quad (1 < p < N).$$

Here $L^{\rho,\sigma}(\mathbb{R}^N)$ $(0 < \rho \leq \infty, 1 \leq \sigma \leq \infty)$ is the Lorentz space with the norm

$$\|u\|_{L^{\rho,\sigma}(\mathbb{R}^N)} = \|s^{\frac{1}{\rho} - \frac{1}{\sigma}} u^*(\cdot)\|_{L^\sigma(0,\infty)},$$

where u^* denotes the (one-dimensional) decreasing rearrangement of u. Then in [8] it is shown that for any $1 < p < N$ there exists a constant $C = C(p, N)$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geq \left(\frac{N-p}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \left(1 + C d_p(u)^{2p^*} \right)$$

holds for every real-valued weakly differentiable function u in $\mathbb{R}^N$ decaying to zero at infinity with $|\nabla u| \in L^p(\mathbb{R}^N)$.

One of the aims of this paper is to obtain remainder terms of other forms for (1.1) when $\Omega = \mathbb{R}^N$. Note that the inequality (1.1) has the scale invariance under the scaling

$$u_\lambda(x) = \lambda^{-\frac{N-p}{p}} u\left(\frac{x}{\lambda} \right) \tag{1.3}$$

for $\lambda > 0$ when $\Omega = \mathbb{R}^N$. Therefore any possible remainder term to (1.1) should be invariant under the scaling (1.3) when $\Omega = \mathbb{R}^N$.

On the critical case $p = N$, the inequality (1.1) fails for every constant on the right-hand side and instead of (1.1) the inequality

$$\int_\Omega |\nabla u|^N dx \ge \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N(\log \frac{Re}{|x|})^N} d \tag{1.4}$$

holds for all $u \in W_0^{1,N}(\Omega)$, where $R = \sup_{x\in\Omega}|x|$. We call (1.4) as *the Hardy inequality in a limiting case*. It is also known that the constant $(\frac{N-1}{N})^N$ is optimal and never attained ([2], [22]). Adimurthi-Chaudhuri-Ramaswamy [1] have proved that, for any $R > 0$ and $k \in \mathbb{N}$, if we put $T(k) = e^{e^{\cdot^{e(\text{k-times})}}} R$, then the inequality

$$\int_{B_R^2(0)} |\nabla h|^2 dx \ge \frac{1}{4}\sum_{j=1}^{k} \int_{B_R^2(0)} \frac{|h|^2}{\left(|x| \prod_{i=1}^{j} \log^{(i)} \frac{T(k)}{|x|}\right)^2} dx \tag{1.5}$$

holds for all $h \in W_0^{1,2}(B_R^2(0))$, where $B_R^2(0) \subset \mathbb{R}^2$ denotes a 2-dimensiomal ball with radius R, $\log^{(k)}(\cdot)$ is defined inductively as $\log^{(1)}(\cdot) := \log(\cdot)$, $\log^{(k)}(\cdot) := \log\left(\log^{(k-1)}(\cdot)\right)$ for $k \ge 2$. However, it seems difficult to claim that we have obtained remainder terms for the inequality (1.4), due to the assumption $T(k) = e^{e^{\cdot^{e(\text{k-times})}}} R$; if we want to take $T(k) = Re$, we will not have any additional terms on the right-hand side of (1.5) other than the standard one $\frac{1}{4}\int_{B_R^2(0)} \frac{|h|^2}{\left(|x|\log\frac{Re}{|x|}\right)^2} dx$. Recently, the authors [22] showed that the following *Hardy inequality in a limiting case with a remainder term* holds true: For any $-1 < L < N-2$, let $q > 0$ be such that $\alpha(q, L) = \frac{N-1}{N}q + L + 2 \le N$. Then the inequality

$$\begin{aligned} \int_\Omega |\nabla u|^N dx \ge \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N(\log\frac{Re}{|x|})^N} dx \\ + \omega_N^{1-\frac{N}{q}} C(L, N, q)^{\frac{N}{q}} \left(\int_\Omega \frac{|u|^q}{|x|^N\left(\log\frac{Re}{|x|}\right)^{\alpha(q,L)}} dx\right)^{\frac{N}{q}} \end{aligned} \tag{1.6}$$

holds true for all $u \in W_0^{1,N}(\Omega)$, where ω_N is the area of the unit sphere in $\mathbb{R}^N$ and $C(L, N, q) = (L+1)^{\left(\frac{N-1}{N}q+1\right)}\Gamma\left(\frac{N-1}{N}q+1\right)^{-1}$.

Concerning the Hardy inequality in a limiting case, it is also known that the following *sharper version* of the inequality (1.4):

$$\int_\Omega |\nabla u|^N dx \ge \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N(\log\frac{R}{|x|})^N} dx \tag{1.7}$$

holds true for all $u \in W_0^{1,N}(\Omega)$, where $R = \sup_{x\in\Omega}|x|$; see [16], [18], [15], [23]. Note that the inequality (1.7) is not invariant under the scaling $u_\lambda(x) =$

$u(\lambda x)$ due to the term $(\log \frac{R}{|x|})^N$. However, when $\Omega = B_R^N(0)$ case, under the non-standard scaling

$$w^\lambda(x) = \lambda^{-\frac{N-1}{N}} w\left(\left(\frac{|x|}{R}\right)^{\lambda-1} x\right) \quad (\lambda > 0) \tag{1.8}$$

introduced by Cassani, Ruf, and Tarsi [6] (in the radial case), Ioku-Ishiwata [15], we check that the inequality (1.7) is invariant; see for example, [15] Proposition 3.2. Recently, Ioku and Ishiwata [15] showed that when Ω is a ball $B_R^N(0)$, the constant $(\frac{N-1}{N})^N$ in the inequality (1.7) is optimal and never attained in $W_0^{1,N}(B_R^N(0))$. One of key tools in their proof is the scale invariance structure of (1.7) mentioned above. Another aim of this paper is to obtain remainder terms in the shaper version of the critical Hardy inequality (1.7), at least when Ω is a ball. Possible remainder terms of (1.7) should also be invariant under the scaling (1.8). A key ingredient in this paper is the new transformation (2.3) (or (3.2)), which is connecting two scale invariant structures in the inequalities (1.1) and (1.7). By using this transformation, we obtain the relationship between remainder terms of the classical Hardy inequality (1.1) on $\mathbb{R}^m$ and the critical Hardy inequality (1.7) on $B_R^N(0)$.

Before stating the main results we fix several notations: For $k \in \mathbb{N}$, $B_R^k(0)$ will denote a ball centered 0 with radius R and ω_k an area of the unit sphere in $\mathbb{R}^k$. $u^\#$ denotes a symmetric decreasing rearrangement (the Schwarz symmetrization) of a function u on $\mathbb{R}^k$:

$$u^\#(x) = u^\#(|x|) = \inf\{\lambda > 0 \mid \left|\{x \in \mathbb{R}^k \mid |u(x)| > \lambda\}\right| \le |B_{|x|}^k(0)|\}$$

where $|A|$ denotes the measure of a set $A \subset \mathbb{R}^k$.

Our main results are as follows:

Theorem 1. *(Improved subcritical Hardy inequality on the whole space) For given $m \ge 3, m \in \mathbb{N}$, $2 \le p < m$, and $q > 2$, set $\alpha = \alpha(p,q,m) = 2 - m + \frac{q(m-p)}{2}$. Then there exists $D = D(p,q,m) > 0$ such that the inequality*

$$\int_{\mathbb{R}^m} |\nabla u|^p dx \ge \left(\frac{m-p}{p}\right)^p \int_{\mathbb{R}^m} \frac{|u|^p}{|x|^p} dx + D\left(\frac{\int_{\mathbb{R}^m} |u^\#|^{\frac{pq}{2}} |x|^\alpha dx}{\int_{\mathbb{R}^m} |u^\#|^p |x|^{2-p}} dx\right)^{\frac{2}{q-2}} \tag{1.9}$$

holds for all $u \in W^{1,p}(\mathbb{R}^m)$, $u \not\equiv 0$.

If $\alpha(p,q,m) < 0$, then the inequality

$$\int_{\mathbb{R}^m} |\nabla u|^p dx \ge \left(\frac{m-p}{p}\right)^p \int_{\mathbb{R}^m} \frac{|u|^p}{|x|^p} dx + D\left(\frac{\int_{\mathbb{R}^m} |u|^{\frac{pq}{2}} |x|^\alpha dx}{\int_{\mathbb{R}^m} |u^\#|^p |x|^{2-p}} dx\right)^{\frac{2}{q-2}}$$

holds for all $u \in W^{1,p}(\mathbb{R}^m)$, $u \not\equiv 0$.

Furthermore, if $p = 2$, then the inequality

$$\int_{\mathbb{R}^m} |\nabla u|^2 dx \geq \left(\frac{m-2}{2}\right)^2 \int_{\mathbb{R}^m} \frac{|u|^2}{|x|^2} dx + D(2,q,m) \left(\frac{\int_{\mathbb{R}^m} |u|^q |x|^{\alpha(2,q,m)} dx}{\int_{\mathbb{R}^m} |u|^2 dx} \right)^{\frac{2}{q-2}} \tag{1.10}$$

holds for all radial function $u \in W^{1,p}(\mathbb{R}^m)$, $u \not\equiv 0$. (We need not assume that u is a radially decreasing function). The constant $D = D(p,q,m)$ is explicitly given as

$$D(p,q,m) = \frac{4\omega_m (p-1)}{\omega_2 p^2} \left(\frac{m-p}{p}\right)^{p-2} C(q,2,2,2)^{-\frac{2q}{q-2}},$$

where $C(q,2,2,2)$ is the positive constant in Proposition 4 in Appendix.

Remark 1. The remainder term of the inequality (1.9) is scale invariant under the scaling (1.3) on $\mathbb{R}^m$: $u_\lambda(x) = \lambda^{-\frac{m-p}{p}} u(y)$, $y = \frac{x}{\lambda}$, $x \in \mathbb{R}^m$. Indeed, for $a, b \in \mathbb{R}$, we have

$$\int_{\mathbb{R}^m} |u_\lambda(x)|^a |x|^b dx = \lambda^{-\left(\frac{m-p}{p}\right)a+b+m} \int_{\mathbb{R}^m} |u(y)|^a |y|^b dy. \tag{1.11}$$

Thus by taking $a = \frac{pq}{2}, b = \alpha(p,q,m)$, or $a = p, b = 2-p$ in (1.11), we have

$$\int_{\mathbb{R}^m} |(u_\lambda)^\#|^{\frac{pq}{2}} |x|^\alpha dx = \int_{\mathbb{R}^m} |(u^\#)_\lambda(x)|^{\frac{pq}{2}} |x|^\alpha dx = \lambda^2 \int_{\mathbb{R}^m} |u^\#(y)|^{\frac{pq}{2}} |y|^\alpha dy,$$
$$\int_{\mathbb{R}^N} |(u_\lambda)^\#|^p |x|^{2-p} dx = \int_{\mathbb{R}^N} |(u^\#)_\lambda(x)|^p |x|^{2-p} dx = \lambda^2 \int_{\mathbb{R}^m} |u^\#(y)|^p |y|^{2-p} dy$$

where the fact that $(u_\lambda)^\#(x) = (u^\#)_\lambda(x)$ comes from Proposition 6. Therefore the remainder term in the inequality (1.9) has the scale invariance.

Next theorem concerns the improvement of the sharper version of the Hardy inequality in a limiting case (1.7).

Theorem 2. *(Improved sharp critical Hardy inequality on balls) Let $N, m \in \mathbb{N}$ be $N \geq 2$, $m \geq 3$ and let $q > 2$. Take $a \in \mathbb{R}$ satisfying $a < \frac{m-2}{2}$, or $a > \frac{m}{2}$. Then the inequality*

$$\begin{aligned} &\int_{B_R^N(0)} |\nabla w|^N dx - \left(\frac{N-1}{N}\right)^N \int_{B_R^N(0)} \frac{|w|^N}{|x|^N (\log \frac{R}{|x|})^N} dx \\ &\geq K(a,q,m,N) \left(\frac{\int_{B_R^N(0)} |w|^{\frac{qN}{2}} |x|^{-N} (\log \frac{R}{|x|})^{-\beta} dx}{\int_{B_R^N(0)} |w|^N |x|^{-N} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx} \right)^{\frac{2}{q-2}} \end{aligned} \tag{1.12}$$

holds for all radial function $w \in C_0^1(B_R^N(0))$, $w \not\equiv 0$. *Here*

$$\beta = 1 + \frac{q(N-1)}{2} + \frac{2(N-1)}{m-2-2a},$$
$$\gamma = N + \frac{2(N-1)}{m-2-2a},$$
$$K = K(a,m,N,q) = \frac{2\omega_N D(2,q,m)}{\omega_m |m-2-2a|}\left(\frac{N-1}{N}\right)^{N-1},$$

and $D(2,q,m)$ *is the constant in Theorem 1.*

Remark 2. Again, the remainder term in (1.12) is scale invariant under the scaling (1.8). Indeed, for $\lambda > 0$, define

$$w^\lambda(r) = \lambda^{-\frac{N-1}{N}} w(R^{1-\lambda} r^\lambda), \quad r \in [0,R].$$

Then for any $c, d \in \mathbb{R}$, we see that

$$\int_0^R \frac{|w^\lambda(r)|^c}{r(\log \frac{R}{r})^d}\, dr = \lambda^{-\frac{N-1}{N}c+d-1} \int_0^R \frac{|w(s)|^c}{s(\log \frac{R}{s})^d}\, ds$$

holds by using the change of variables $s = R^{1-\lambda} r^\lambda$, $\frac{dr}{r} = \frac{1}{\lambda}\frac{ds}{s}$. Thus,

$$\int_0^R \frac{|w^\lambda(r)|^{\frac{qN}{2}}}{r(\log \frac{R}{r})^\beta} dr = \lambda^{-\frac{N-1}{N}\frac{qN}{2}+\beta-1} \int_0^R \frac{|w(s)|^{\frac{qN}{2}}}{s(\log \frac{R}{s})^\beta} ds,$$
$$\int_0^R \frac{|w^\lambda(r)|^N}{r(\log \frac{R}{r})^\gamma} dr = \lambda^{-\frac{N-1}{N}N+\gamma-1} \int_0^R \frac{|w(s)|^N}{s(\log \frac{R}{s})^\gamma} ds,$$

so the exponent of λ in the ratio of the above two integrals is $\beta - \gamma - \frac{N-1}{N}(\frac{qN}{2} - N) = 0$, by the definition of β and γ. Therefore the remainder term in the inequality (1.12) has the scale invariance.

This paper is organized as follows: Section 2 is devoted to proofs of main results, Theorem 1 and Theorem 2. In Section 3, we give some facts concerning the new transformation (2.3) introduced in the proof of Proposition 1. In Section 4, we discuss other improvements of the Hardy inequalities in both subcritical and critical cases. In Appendix, we collect useful facts for proofs.

Throughout the paper, if a radial function u is written as $u(x) = \tilde{u}(|x|)$ by some function $\tilde{u} = \tilde{u}(r)$, we write $u(x) = u(|x|)$ with admitting some ambiguity. We hope no confusion occurs by this abbreviation.

2. Proofs of Main results

In this section, first we improve the classical Hardy inequality on the whole space. Brezis and Vázquez's well-known idea will be used in the proof. Next we prove Theorem 2 by using Theorem 1. In the course of proof, we use a new transformation which connects two Hardy inequalities.

Next simple lemma is used to prove Theorem 1.

Lemma 1 ([12] Lemma 1)**.** *Let $p \ge 2$ and ξ, η be real numbers such that $\xi \ge 0$ and $\xi - \eta \ge 0$. Then*

$$(\xi - \eta)^p + p\xi^{p-1}\eta - \xi^p \ge \max\{(p-1)\eta^2\xi^{p-2}, |\eta|^p\}.$$

Proof of Theorem 1. [Step 1] First, we show that the inequality (1.9) holds for nonnegative, radially symmetric and nonincreasing function $u = u(r) \in C_0^\infty(\mathbb{R}^m)$ where $r = |x|$. Following the idea of Brezis and Vázquez [4], we define the new function on $\mathbb{R}^2$: For $s \in [0, +\infty)$, define

$$v(s) = r^{\frac{m-p}{p}} u(r), \quad \text{where } s = r \in [0, \infty),$$

and put $v(y) = v(|y|)$ for $y \in \mathbb{R}^2$. Note that $v(0) = 0$ and also $v(+\infty) = 0$ since the support of u is compact. We claim that if $u \in W^{1,p}(\mathbb{R}^m)$, then $v \in L^p(\mathbb{R}^2)$. Indeed, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} |v(y)|^p dy = \omega_2 \int_0^\infty |v(s)|^p s ds \\
&= \omega_2 \int_0^\infty |u(r)|^p r^{m-p+1}\, dr = \frac{\omega_2}{\omega_m} \int_{\mathbb{R}^m} \frac{|u|^p}{|x|^{p-2}}\, dx \\
&\le \frac{\omega_2}{\omega_m} \left(\int_{\mathbb{R}^m} \frac{|u|^p}{|x|^p}\, dx \right)^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^m} |u|^p\, dx \right)^{\frac{2}{p}} \\
&\le \frac{\omega_2}{\omega_m} \left(\frac{p}{m-p} \right)^{p-2} \left(\int_{\mathbb{R}^m} |\nabla u|^p\, dx \right)^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^m} |u|^p\, dx \right)^{\frac{2}{p}} < \infty,
\end{aligned}$$

here we have used Hölder's inequality, Hardy's inequality, and the assumption that u and $|\nabla u|$ belong to $L^p(\mathbb{R}^m)$. Therefore we have checked $v \in L^p(\mathbb{R}^2)$. Now we observe by using $u'(r) \le 0$ that

$$\begin{aligned}
J &= \int_{\mathbb{R}^m} |\nabla u|^p dx - \left(\frac{m-p}{p} \right)^p \int_{\mathbb{R}^m} \frac{|u|^p}{|x|^p}\, dx \\
&= \omega_m \int_0^\infty (-u'(r))^p r^{m-1}\, dr - \left(\frac{m-p}{p} \right)^p \omega_m \int_0^\infty |u(r)|^p r^{m-p-1}\, dr \\
&= \omega_m \int_0^\infty \left(\frac{m-p}{p} s^{-\frac{m}{p}} v(s) - s^{-\frac{m-p}{p}} v'(s) \right)^p s^{m-1} ds \\
&- \left(\frac{m-p}{p} \right)^p \omega_m \int_0^\infty v^p(s) s^{-1}\, ds.
\end{aligned}$$

Here, applying Lemma 1 with the choice

$$\xi = \frac{m-p}{p} s^{-\frac{m}{p}} v(s) \quad \text{and} \quad \eta = s^{-\frac{m-p}{p}} v'(s),$$

and using the fact $v(0) = v(+\infty) = 0$, we get

$$\begin{aligned} J &\geq -\omega_m\, p \left(\frac{m-p}{p} \right)^{p-1} \int_0^\infty v(s)^{p-1} v'(s)\, ds \\ &+ \omega_m (p-1) \left(\frac{m-p}{p} \right)^{p-2} \int_0^\infty v^{p-2}(s)(v'(s))^2\, s\, ds \\ &= \omega_m (p-1) \left(\frac{m-p}{p} \right)^{p-2} \frac{4}{p^2} \int_0^\infty \left(\frac{d}{ds} \left(v^{\frac{p}{2}}(s) \right) \right)^2 s\, ds \\ &= \frac{\omega_m}{\omega_2} (p-1) \left(\frac{m-p}{p} \right)^{p-2} \frac{4}{p^2} \| \nabla (v^{\frac{p}{2}}) \|^2_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{2.1}$$

Now, we apply the Gagliardo-Nirenberg inequality (see Proposition 4 in Appendix) to $v^{\frac{p}{2}} \in L^2(\mathbb{R}^2)$:

$$\| v^{\frac{p}{2}} \|_{L^q(\mathbb{R}^2)} \leq C(q,2,2,2) \| v^{\frac{p}{2}} \|^{\frac{2}{q}}_{L^2(\mathbb{R}^2)} \| \nabla (v^{\frac{p}{2}}) \|^{\frac{q-2}{q}}_{L^2(\mathbb{R}^2)}.$$

Combining this to (2.1), we obtain

$$\begin{aligned} J &\geq \frac{\omega_m}{\omega_2} (p-1) \left(\frac{m-p}{p} \right)^{p-2} \frac{4}{p^2} C(q,2,2,2)^{-\frac{2q}{q-2}} \left(\frac{\int_{\mathbb{R}^2} v^{\frac{pq}{2}}(y) dy}{\int_{\mathbb{R}^2} v^p(y) dy} \right)^{\frac{2}{q-2}} \\ &= D \left(\frac{\int_{\mathbb{R}^m} |u|^{\frac{pq}{2}} |x|^\alpha dx}{\int_{\mathbb{R}^m} |u|^p |x|^{2-p} dx} \right)^{\frac{2}{q-2}}. \end{aligned}$$

Note that if $p = 2$, we can expand $\left(\frac{m-2}{2} s^{-\frac{m}{2}} v(s) - s^{-\frac{m-2}{2}} v'(s) \right)^2$ term without any sign information of $u'(r)$, thus getting the same conclusion.

[Step 2] Let $u \in W^{1,p}(\mathbb{R}^m)$ be given. By using density argument and symmetrization argument, we obtain

$$\begin{aligned} \int_{\mathbb{R}^m} |\nabla u|^p dx &\geq \int_{\mathbb{R}^m} |\nabla u^\#|^p dx \\ &\geq \left(\frac{m-p}{p} \right)^p \int_{\mathbb{R}^m} \frac{|u^\#|^p}{|x|^p} dx + D \left(\frac{\int_{\mathbb{R}^m} |u^\#|^{\frac{pq}{2}} |x|^\alpha dx}{\int_{\mathbb{R}^m} |u^\#|^p |x|^{2-p} dx} \right)^{\frac{2}{q-2}} \\ &\geq \left(\frac{m-p}{p} \right)^p \int_{\mathbb{R}^m} \frac{|u|^p}{|x|^p} dx + D \left(\frac{\int_{\mathbb{R}^m} |u^\#|^{\frac{pq}{2}} |x|^\alpha dx}{\int_{\mathbb{R}^m} |u^\#|^p |x|^{2-p} dx} \right)^{\frac{2}{q-2}}, \end{aligned}$$

where the first inequality comes from the Pólya-Szegö inequality, the second comes from Step 1, and the last comes from the Hardy-Littlewood inequality: $\int_{\mathbb{R}^m} f^\# g^\# \geq \int_{\mathbb{R}^m} fg$ for nonnegative measurable functions f and g. Hence (1.9) holds for all $u \in W^{1,p}(\mathbb{R}^m)$. If $\alpha < 0$, the numerator of the remainder term may be replaced by $\int_{\mathbb{R}^m} |u|^{\frac{pq}{2}} |x|^\alpha dx$, by another use of the Hardy-Littlewood inequality. Note that the denominator of the remainder term in (1.9) is bounded for any $u \in W^{1,p}(\mathbb{R}^m)$, since

$$\int_{\mathbb{R}^m} \frac{|u^\#|^p}{|x|^{p-2}}\, dx \leq \left(\int_{\mathbb{R}^m} \frac{|u^\#|^p}{|x|^p}\, dx \right)^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^m} |u^\#|^p\, dx \right)^{\frac{2}{p}}$$
$$\leq \left(\left(\frac{p}{m-p} \right)^p \int_{\mathbb{R}^m} |\nabla u^\#|^p\, dx \right)^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^m} |u|^p\, dx \right)^{\frac{2}{p}} < \infty.$$

The proof of Theorem 1 is now complete. □

Remark 3. We do not know the same type of remainder terms can be obtained when $1 < p < 2$ in Theorem 1. The point is a lack of nice pointwise estimate like Lemma 1 in this case. Also we do not know whether the conclusion holds true or not for $u \in D^{1,p}(\mathbb{R}^m)$ in Theorem 1.

Now, let us turn to the improvement of the sharper version of critical Hardy inequality (1.7) on balls. To prove Theorem 2, first we show a key proposition which reveals the relation between remainder terms of critical and subcritical Hardy inequalities in the radial case.

Proposition 1. *Let $m, N \in \mathbb{N}$ satisfy $m \geq 3$, $N \geq 2$. Let $a \in \mathbb{R}$, $a \neq \frac{m-2}{2}$ and $w = w(s) \in C_0^1([0,R))$, $w \geq 0$ be given. Then there exists a nonnegative function $u = u(r)$, $r \in [0,+\infty)$ with $|u'(r)|^2 r^{m-1} \in L^1(0,+\infty)$ such that*

$$\int_0^\infty |u'(r)|^2 r^{m-1} dr - \left(\frac{m-2}{2} \right)^2 \int_0^\infty |u|^2 r^{m-3} dr$$
$$\leq H \left(\int_0^R |\nabla w(s)|^N s^{N-1} ds - \left(\frac{N-1}{N} \right)^N \int_0^R \frac{|w(s)|^N}{s \left(\log \frac{R}{s} \right)^N} ds \right) \tag{2.2}$$

holds where

$$H = H(a,m,N) = \left(\frac{N}{N-1} \right)^{N-1} \frac{|m-2-2a|}{2}.$$

Equality holds if $N = 2$. Furthermore, if $a < \frac{m-2}{2}$ or $a > \frac{m}{2}$, then $u \in W^{1,2}([0,\infty), r^{m-1} dr)$.

Proof of Proposition 1. Let $a \in \mathbb{R}$, $a \neq \frac{m-2}{2}$. For a given nonnegative function $w = w(s) \in C_0^1([0, R))$, we define the function $u = u(r)$ as follows:

$$u(r) = r^{-a} w^{\frac{N}{2}}(s), \quad \text{where} \quad s = s(r) = R\exp\left(-r^b\right), \tag{2.3}$$

$$\left(\text{i.e., } r = r(s) = \left(\log \frac{R}{s}\right)^{\frac{1}{b}}\right), \quad s'(r) = (-b)r^{b-1}s(r),$$

where $b \in \mathbb{R}$, $b \neq 0$ will be chosen later. Note that $s^{-1}ds = (-b)r^{b-1}dr$ and

$$s(0) = R, s(+\infty) = 0, \quad \text{if } b > 0, \quad \text{and}$$
$$s(0) = 0, s(+\infty) = R, \quad \text{if } b < 0.$$

For the sake of simplicity, we put

$$A(w) = \int_0^R |w'(s)|^N s^{N-1} ds, \quad B(w) = \left(\frac{N-1}{N}\right)^N \int_0^R \frac{|w(s)|^N}{s\left(\log \frac{R}{s}\right)^N} ds,$$

$$C(u) = \int_0^\infty |u'(r)|^2 r^{m-1} dr, \quad D(u) = \left(\frac{m-2}{2}\right)^2 \int_0^\infty |u|^2 r^{m-3} dr.$$

Since

$$u'(r) = -ar^{-a-1}w^{\frac{N}{2}}(s) + \frac{N}{2} r^{-a} w^{\frac{N}{2}-1}(s) w'(s) s'(r),$$

where $s = s(r)$, we have

$$\begin{aligned}
C(u) &= \int_0^\infty \left(-ar^{-a-1}w^{\frac{N}{2}}(s(r)) + \frac{N}{2} r^{-a} w^{\frac{N-2}{2}}(s(r)) w'(s(r)) s'(r)\right)^2 r^{m-1}\, dr \\
&= a^2 \int_0^\infty w^N(s(r))\, r^{m-2a-3}\, dr - Na \int_0^\infty w^{N-1}(s(r))\, w'(s(r)) r^{m-2a-2} s'(r)\, dr \\
&+ \frac{N^2}{4} \int_0^\infty w^{N-2}(s(r))\, \left(w'(s(r))\right)^2\, r^{m-2a-1} (s'(r))^2\, dr \\
&= \frac{a^2}{-b} \int_{s(0)}^{s(\infty)} w^N(s)\, r(s)^{m-2a-2-b} s^{-1}\, ds - Na \int_{s(0)}^{s(\infty)} w^{N-1}(s) w'(s) r(s)^{m-2a-2} ds \\
&+ \frac{N^2}{4} \int_{s(0)}^{s(\infty)} w^{N-2}(s) (w'(s))^2 r(s)^{m-2a-1} s'(r(s))\, ds \\
&= \frac{a^2}{|b|} \int_0^R w^N(s) s^{-1} \left(\log \frac{R}{s}\right)^{\frac{m-2a-2-b}{b}} ds - a \int_{s(0)}^{s(\infty)} \frac{d}{ds}\left[w^N(s)\right] \left(\log \frac{R}{s}\right)^{\frac{m-2a-2}{b}} ds \\
&+ \frac{N^2}{4} |b| \int_0^R w^{N-2}(s) (w'(s))^2 s \left(\log \frac{R}{s}\right)^{\frac{m-2a-2+b}{b}} ds \\
&= I + II + III.
\end{aligned}$$

From now on, we assume $N \geq 3$. Then by the Hölder inequality and integrating by parts, we have

$$III \leq \frac{N^2}{4}|b|\left(\int_0^R (w'(s))^N s^{N-1} ds\right)^{\frac{2}{N}} \left(\int_0^R w^N(s)s^{-1}\left(\log \frac{R}{s}\right)^{\frac{N}{N-2}\cdot\frac{m-2a-2+b}{b}} ds\right)^{\frac{N-2}{N}}.$$

and

$$II = \frac{a(m-2a-2)}{b}\int_{s(0)}^{s(\infty)} w^N(s)s^{-1}\left(\log\frac{R}{s}\right)^{\frac{m-2a-2-b}{b}} ds - a\left[w^N(s)\left(\log\frac{R}{s}\right)^{\frac{m-2a-2}{b}}\right]_{s=s(0)}^{s=s(\infty)}.$$

On the other hand,

$$\begin{aligned} D(u) &= \left(\frac{m-2}{2}\right)^2 \int_0^\infty w^N(s(r))r^{m-2a-3}dr \\ &= \left(\frac{m-2}{2}\right)^2 \left(\frac{1}{-b}\right)\int_{s(0)}^{s(\infty)} w^N(s)r(s)^{m-2a-2-b}s^{-1}ds \\ &= \left(\frac{m-2}{2}\right)^2 \frac{1}{|b|}\int_0^R w^N(s)s^{-1}\left(\log\frac{R}{s}\right)^{\frac{m-2a-2-b}{b}} ds. \end{aligned}$$

Here, we take b to satisfy

$$\frac{m-2-2a-b}{b} = -N, \quad \text{i.e.,} \quad b = -\frac{m-2-2a}{N-1}. \tag{2.4}$$

Then we have

$$\left[w^N(s)\left(\log\frac{R}{s}\right)^{\frac{m-2a-2}{b}}\right]_{s=s(0)}^{s=s(\infty)} = \left[w^N(s)\left(\log\frac{R}{s}\right)^{1-N}\right]_{s=s(0)}^{s=s(\infty)} = 0$$

since $w \in C_0^\infty([0,R))$, so that

$$II = \frac{a(m-2-2a)}{|b|}\int_0^R w^N(s)s^{-1}\left(\log\frac{R}{s}\right)^{-N} ds.$$

By the above choice of b, we also have

$$\frac{N}{N-2}\cdot\frac{m-2-2a+b}{b} = -N.$$

Hence we observe that

(2.5)

$$
\begin{aligned}
C(u) &= I + II + III \\
&\leq \left(\frac{a^2}{|b|} + \frac{a(m-2a-2)}{|b|} \right) \int_0^R w^N(s) s^{-1} \left(\log \frac{R}{s} \right)^{-N} ds \\
&\quad + \frac{N^2}{4} |b| \left(\int_0^R (w'(s))^N s^{N-1} ds \right)^{\frac{2}{N}} \left(\int_0^R w^N(s) s^{-1} \left(\log \frac{R}{s} \right)^{-N} ds \right)^{\frac{N-2}{N}}
\end{aligned}
$$

(2.6)

$$
= \left(\frac{-a^2 + (m-2)a}{|b|} \right) \left(\frac{N}{N-1} \right)^N B(w) + \frac{N^2}{4} |b| \left(\frac{N-1}{N} \right)^2 \left(\frac{N}{N-1} \right)^N A(w)^{\frac{2}{N}} B(w)^{\frac{N-2}{N}},
$$

and

$$
D(u) = \left(\frac{m-2}{2} \right)^2 \frac{1}{|b|} \left(\frac{N}{N-1} \right)^N B(w). \tag{2.7}
$$

From (2.5), we observe that $|u'|^2 r^{m-1} \in L^1(0, \infty)$. Moreover, since

$$
\frac{-a^2 + (m-2)a}{|b|} - \left(\frac{m-2}{2} \right)^2 \frac{1}{|b|} = \frac{-1}{|b|} \left(a - \frac{m-2}{2} \right)^2 = \frac{N-1}{4} |m-2-2a|
$$

and

$$
\frac{(N-1)^2}{4} |b| \left(\frac{N}{N-1} \right)^N = \frac{1}{|b|} \left(a - \frac{m-2}{2} \right)^2 \left(\frac{N}{N-1} \right)^N = \frac{N}{2} \left(\frac{N}{N-1} \right)^{N-1} \frac{|m-2-2a|}{2}
$$

by the choice of b (2.4), we have

$$
\begin{aligned}
&C(u) - D(u) \\
&\leq \frac{N}{2} \left(\frac{N}{N-1} \right)^{N-1} \frac{|m-2-2a|}{2} A(w)^{\frac{2}{N}} B(w)^{\frac{N-2}{N}} - \frac{N-1}{4} |m-2-2a| \left(\frac{N}{N-1} \right)^N B(w) \\
&= H(a, m, N) \left(\frac{N}{2} \right) B(w)^{\frac{N-2}{N}} \left(A(w)^{\frac{2}{N}} - B(w)^{\frac{2}{N}} \right).
\end{aligned}
$$

Note that the sharp critical Hardy inequality (1.7) for w implies $A(w) \geq B(w)$. Since $f(A) - f(B) \leq f'(B)(A-B)$ holds for the function $f(x) = x^{\frac{2}{N}}$, we conclude that

$$
C(u) - D(u) \leq H(a, m, N) \left(A(w) - B(w) \right),
$$

which proves (2.2).

When $N = 2$ case, the argument is the same and much simpler, so we omit it.

Finally we show that $u^2 r^{m-1} \in L^1(0,\infty)$ if $a < \frac{m-2}{2}$, or $a > \frac{m}{2}$. Indeed, by (2.3), we calculate

$$\int_0^\infty u(r)^2 r^{m-1} dr = \int_0^\infty w^N(s(r)) r^{m-2a-1} dr$$

$$= \int_{s(0)}^{s(\infty)} w^N(s) \left(\log \frac{R}{s}\right)^{\frac{m-2a-1}{b}} \frac{ds}{s'(r(s))} = \left(\frac{-1}{b}\right) \int_{s(0)}^{s(\infty)} w^N(s) s^{-1} \left(\log \frac{R}{s}\right)^{\frac{m-2a-b}{b}} ds$$

(2.8)

$$= \frac{1}{|b|} \int_0^R w^N(s) s^{-1} \left(\log \frac{R}{s}\right)^{-\gamma} ds,$$

where $\gamma = -\frac{m-2a-b}{b} = N + \frac{2(N-1)}{m-2-2a}$. Thus we obtain

$$\int_0^\infty u(r)^2 r^{m-1} dr = \int_0^\infty w^N(s(r)) r^{m-2a-1} dr \leq \frac{1}{|b|} \|w\|^N_{L^\infty(0,R)} \int_0^{\mathrm{supp} w} \left(\log \frac{R}{s}\right)^{-\gamma} s^{-1} ds.$$

Consequently, $u^2 r^{m-1} \in L^1(0,\infty)$ if $\gamma > 1$. The last condition is equivalent to $a < \frac{m-2}{2}$, or $a > \frac{m}{2}$ by a simple observation. Thus we have proved Proposition 1. □

Now, we prove Theorem 2 by combining it to Theorem 1 via Proposition 1.

Proof of Theorem 2. Let $r = |x|$, $x \in \mathbb{R}^m$ and $s = |y|$, $y \in \mathbb{R}^N$. Given a nonnegative radially symmetric function $w = w(s) \in C_0^1(B_R^N(0))$, define $u = u(r)$ through (2.3). By using (2.2) in Proposition 1 and (1.9) in Theorem 1, we obtain

$$\int_{B_R^N(0)} |\nabla w|^N dy - \left(\frac{N-1}{N}\right)^N \int_{B_R^N(0)} \frac{|w|^N}{|y|^N \left(\log \frac{R}{|y|}\right)^N} dy$$

$$= \omega_N \left(\int_0^R |w'(s)|^N s^{N-1} ds - \left(\frac{N-1}{N}\right)^N \int_0^R \frac{|w|^N}{s\left(\log \frac{R}{s}\right)^N} ds \right)$$

$$\geq \omega_N H^{-1}(a,m,N) \left(\int_0^\infty |u'(r)|^2 r^{m-1} - \left(\frac{m-2}{2}\right)^2 \int_0^\infty |u|^2 r^{m-3} dr \right)$$

$$\geq \frac{\omega_N D(2,q,m)}{\omega_m H(a,m,N)} \left(\frac{\int_0^\infty |u|^q r^{1+\frac{q(m-2)}{2}} dr}{\int_0^\infty |u|^2 r^{m-1} dr} \right)^{\frac{2}{q-2}}.$$

By the transformation (2.3), we see

$$\begin{aligned}\int_0^\infty |u|^q r^{1+\frac{q(m-2)}{2}} dr &= \int_0^\infty |w(s(r))|^{\frac{qN}{2}} r^{1+\frac{q(m-2-2a)}{2}} dr \\ &= \int_{s(0)}^{s(\infty)} |w(s)|^{\frac{qN}{2}} \left(\log \frac{R}{s}\right)^{\frac{1}{b}+\frac{q(m-2-2a)}{2b}} \frac{ds}{s'(r(s))} \\ &= \frac{1}{|b|} \int_0^R |w(s)|^{\frac{qN}{2}} s^{-1} \left(\log \frac{R}{s}\right)^{\frac{2}{b}-1+\frac{q(m-2-2a)}{2b}} ds.\end{aligned}$$

Now, by the choice of b (2.4), we see

$$-\frac{2}{b} + 1 - \frac{q(m-2-2a)}{2b} = \frac{2(N-1)}{m-2-2a} + 1 + \frac{q(N-1)}{2} = \beta,$$

which implies

$$\int_0^\infty |u|^q r^{1+\frac{q(m-2)}{2}} dr = \frac{1}{|b|} \int_0^R |w(s)|^{\frac{qN}{2}} s^{-1} \left(\log \frac{R}{s}\right)^{-\beta} ds. \tag{2.9}$$

Thus we conclude (1.12) from (2.8) and (2.9), where

$$K(a,m,N,q) = \frac{\omega_N}{\omega_m} \frac{D(2,q,m)}{H(a,m,N)} = \frac{2\omega_N D(2,q,m)}{\omega_m |m-2-2a|} \left(\frac{N-1}{N}\right)^{N-1}.$$

The proof of Theorem 2 is now complete. □

3. Relation between the critical and the subcritical Hardy inequalities

We have proved Proposition 1 by exploiting a transformation (2.3). However, we do not know whether the equality in (2.2) holds true when $N \geq 3$. In this section, first by using another transformation (3.2), we prove the equality in Proposition 1. Next we show that the transformation (2.3) introduced in Proposition 1 and (3.2) both preserve the scale invariance structures of the classical Hardy inequality (1.1) in the whole space and the sharp Hardy inequality (1.7) on balls.

Proposition 2. *Let $m, N \in \mathbb{N}$ satisfy $N \geq 2$, $m \geq N+1$. Then for any nonnegative radially symmetric function $w \in C_0^1(B_R^N(0))$, there exists a nonnegative radially symmetric function $u \in C_0^1(\mathbb{R}^m)$ such that*

(3.1)

$$\begin{aligned}&\int_{\mathbb{R}^m} |\nabla u|^N dx - \left(\frac{m-N}{N}\right)^N \int_{\mathbb{R}^m} \frac{|u|^N}{|x|^N} dx \\ &= \frac{\omega_m}{\omega_N} \left(\frac{m-N}{N-1}\right)^{N-1} \left(\int_{B_R^N(0)} |\nabla w|^N dy - \left(\frac{N-1}{N}\right)^N \int_{B_R^N(0)} \frac{|w|^N}{|y|^N \left(\log \frac{R}{|y|}\right)^N} dy \right)\end{aligned}$$

holds true.

Proof. Let $r = |x|, x \in \mathbb{R}^m$ and $s = |y|, y \in \mathbb{R}^N$. For a given nonnegative radial function $w = w(y) \in C_0^1(B_R^N(0))$, we define a radial function $u = u(x) \in C_0^1(\mathbb{R}^m)$ as follows:

$$(3.2) \qquad u(r) = w(s(r)), \text{ where } s(r) = R\exp\left(-r^{-\frac{m-N}{N-1}}\right),$$
$$\left(\text{i.e., } r^{-\frac{m-N}{N-1}} = \log\frac{R}{s}\right), \; s'(r) = \frac{m-N}{N-1} r^{-\frac{m-N}{N-1}-1} s(r).$$

Note that $s'(r) > 0$ for any $r \in [0, +\infty)$ and $s(0) = 0, s(+\infty) = R$. Also $u \equiv 0$ near $r = \infty$ since $w(s) \equiv 0$ near $s = R$. Then we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^m} |\nabla u|^N dx - \left(\frac{m-N}{N}\right)^N \int_{\mathbb{R}^m} \frac{|u|^N}{|x|^N} dx \\
&= \omega_m \int_0^\infty |u'(r)|^N r^{m-1}\, dr - \left(\frac{m-N}{N}\right)^N \omega_m \int_0^\infty u^N(r) r^{m-N-1}\, dr \\
&= \omega_m \int_0^R |w'(s)s'(r(s))|^N r(s)^{m-1} \frac{ds}{s'(r(s))} - \left(\frac{m-N}{N}\right)^N \omega_m \int_0^R w^N(s) r(s)^{m-N-1} \frac{ds}{s'(r(s))} \\
&= \omega_m \left(\frac{m-N}{N-1}\right)^{N-1} \int_0^R |w'(s)|^N s^{N-1}\, ds - \left(\frac{m-N}{N}\right)^N \frac{N-1}{m-N} \omega_m \int_0^R \frac{w^N(s)}{s(\log\frac{R}{s})^N}\, ds \\
&= \frac{\omega_m}{\omega_N} \left(\frac{m-N}{N-1}\right)^{N-1} \left(\int_{B_R^N(0)} |\nabla w|^N dy - \left(\frac{N-1}{N}\right)^N \int_{B_R^N(0)} \frac{|w|^N}{|y|^N \left(\log\frac{R}{|y|}\right)^N} dy \right).
\end{aligned}$$

The proof of Proposition 2 is now complete. □

Remark 4. Up to now, we do not have an improved subcritical Hardy inequality (1.9) in $W^{1,p}(\mathbb{R}^m)$ without using $u^{\#}$ when $p > 2$; see Theorem 1. Therefore it seems difficult to improve the critical Hardy inequality in $B_R^N(0)$ when $N > 2$ by using Proposition 2.

We find the new transformations (2.3) or (3.2) which connect each remainder term in Hardy inequalities (1.7) or (1.1) to the other. Here we show that these transformations also preserve the scale invariance structures of both Hardy inequalities.

Theorem 3. *Let $m \geq 3$ and $N \geq 2$ be integers and let $1 < p < m$. For functions $u = u(r)$, $r \in [0, +\infty)$ and $w = w(s)$, $s \in [0, R)$, define the scaled functions*

$$u_\lambda(r) = \lambda^{-\frac{m-p}{p}} u\left(\frac{r}{\lambda}\right),$$
$$w^\lambda(s) = \lambda^{-\frac{N-1}{N}} w(s^\lambda R^{1-\lambda}).$$

Then we have the following.

(i) *Let $a \in \mathbb{R}$, $a \neq \frac{m-2}{2}$. The transformations*

$$\tilde{\mathcal{M}} : C_0^1([0,R),\mathbb{R}_+) \ni w = w(s) \mapsto u = u(r) = r^{-a} w^{\frac{N}{2}}(s(r)),$$

$$\text{where } s = s(r) = R\exp\left(-r^b\right),\ b = -\frac{m-2-2a}{N-1}, \quad \text{and}$$

$$\tilde{\mathcal{M}}^{-1} : C_0^1([0,+\infty),\mathbb{R}_+) \ni u = u(r) \mapsto w = w(s) = r(s)^{\frac{2a}{N}} u^{\frac{2}{N}}(r(s)),$$

$$\text{where } r = r(s) = \left(\log \frac{R}{s}\right)^{\frac{1}{b}},$$

satisfy

$$\tilde{\mathcal{M}}(w^\lambda)(r) = r^{-a}\left(w^\lambda\right)^{\frac{N}{2}}(s(r)) = u_\mu(r) \quad \text{where } \mu = \lambda^{-\frac{1}{b}},$$

$$\tilde{\mathcal{M}}^{-1}(u_\lambda)(s) = r(s)^{\frac{2a}{N}}\left(u_\lambda\right)^{\frac{2}{N}}(r(s)) = w^\nu(s) \quad \text{where } \nu = \lambda^{-b}.$$

(ii) *The transformations*

$$\mathcal{M} : C_0^1([0,R),\mathbb{R}_+) \ni w = w(s) \mapsto u = u(r) = w(s(r)),$$

$$\text{where } s = s(r) = R\exp\left(-r^{-\frac{m-N}{N-1}}\right), \quad \text{and}$$

$$\mathcal{M}^{-1} : C_0^1([0,+\infty),\mathbb{R}_+) \ni u = u(r) \mapsto w = w(s) = u(r(s)),$$

$$\text{where } r = r(s) = \left(\log \frac{R}{s}\right)^{-\frac{N-1}{m-N}},$$

satisfy

$$\mathcal{M}(w^\lambda)(r) = w^\lambda(s(r)) = u_\mu(r), \quad \text{where } \mu = \lambda^{\frac{N-1}{m-N}},$$

$$\mathcal{M}^{-1}(u_\lambda)(s) = u_\lambda(r(s)) = w^\nu(s), \quad \text{where } \nu = \lambda^{\frac{m-N}{N-1}}.$$

Proof. We prove only (i). By direct calculation,

$$s^\lambda R^{1-\lambda} = \left(R\exp\left(-r^b\right)\right)^\lambda R^{1-\lambda} = R\exp\left(-\lambda r^b\right) = R\exp\left(-\left(\frac{r}{\lambda^{-\frac{1}{b}}}\right)^b\right) = s\left(\frac{r}{\mu}\right),$$

where $\mu = \lambda^{-1/b}$. Therefore we obtain

$$\begin{aligned}\left(w^\lambda\right)^{\frac{N}{2}}(s) &= \lambda^{-\frac{N-1}{N}\cdot\frac{N}{2}} w^{\frac{N}{2}}(s^\lambda R^{1-\lambda}) = \lambda^{-\frac{N-1}{2}} w^{\frac{N}{2}}\left(s\left(\frac{r}{\mu}\right)\right) \\ &= \lambda^{-\frac{N-1}{2}}\left(\frac{r}{\mu}\right)^a u\left(\frac{r}{\mu}\right) = r^a\left(\mu^{-b}\right)^{-\frac{N-1}{2}}\mu^{-a} u\left(\frac{r}{\mu}\right).\end{aligned}$$

Now, since $b\left(\frac{N-1}{2}\right) - a = \frac{N-1}{2}\cdot\frac{2a+2-m}{N-1} - a = -\frac{m-2}{2}$, we have

$$\left(w^\lambda\right)^{\frac{N}{2}}(s) = r^a \mu^{-\frac{m-2}{2}} u\left(\frac{r}{\mu}\right) = r^a u_\mu(r).$$

Thus we conclude $\tilde{\mathcal{M}}(w^\lambda)(r) = u_\mu(r)$. The proof of $\tilde{\mathcal{M}}^{-1}(u_\lambda)(s) = w^\nu(s)$, $\nu = \lambda^{-b}$, is similar. □

On the other hand, there is a relation between the subcritical Hardy inequality (1.1) in $B_1^m(0)$ and the Hardy inequality in a limiting case (1.4) in $B_R^N(0)$ as follows.

Proposition 3. *Let $m, N \in \mathbb{N}$ satisfy $m \geq 3, N \geq 2, m \geq N+1$. Then for any nonnegative radially symmetric function $w \in C_0^1(B_R^N(0))$, there exists a nonnegative radially symmetric function $u \in C_0^1(B_1^m(0))$ such that*

$$\int_{B_1^m(0)} |\nabla u|^N \, dx - \left(\frac{m-N}{N}\right)^N \int_{B_1^m(0)} \frac{|u|^N}{|x|^N} \, dx$$

(3.3)

$$= \frac{\omega_m}{\omega_N} \left(\frac{m-N}{N-1}\right)^{N-1} \left(\int_{B_R^N(0)} |\nabla w|^N \, dy - \left(\frac{N-1}{N}\right)^N \int_{B_R^N(0)} \frac{|w|^N}{|y|^N \left(\log \frac{Re}{|y|}\right)^N} \, dy \right)$$

Proof of Proposition 3. Let $r = |x|, x \in \mathbb{R}^m$ and $s = |y|, y \in \mathbb{R}^N$. For a given nonnegative radial function $w = w(y) \in C_0^1(B_R^N(0))$, we define a radial function $u = u(x) \in C_0^1(B_1^m(0))$ as follows:

$$w(s) = u(r), \text{ where } s = s(r) = R \exp\left(1 - r^{-\frac{m-N}{N-1}}\right), \tag{3.4}$$
$$\left(\text{i.e. } r^{-\frac{m-N}{N-1}} = \log \frac{Re}{s}\right), \; s'(r) = \frac{m-N}{N-1} r^{-\frac{m-N}{N-1}-1} s(r).$$

We go through the rest of the proof by using the same argument as Proposition 2. □

4. Other improved Hardy inequalities

In this section, we prove miscellaneous improved Hardy inequalities, in both subcritical and critical cases. In Theorem 1, we have used the Gagliardo-Nirenberg inequality as a substitute for the Poincaré inequality, which is usually used to improve the Hardy inequality on bounded domains. In the next theorem, we will employ *the logarithmic Sobolev inequality* on the whole space.

Theorem 4. *Let $m \geq 3$ and $2 \leq p < m$. Then the inequality*

$$\int_{\mathbb{R}^m} |\nabla u|^p dx - \left(\frac{m-p}{p}\right)^p \int_{\mathbb{R}^m} \frac{|u|^p}{|x|^p} \, dx$$

$$\geq C E(u^\#) \exp\left(1 + E(u^\#)^{-1} \int_{\mathbb{R}^m} \frac{|u^\#|^p}{|x|^{p-2}} \log\left(\frac{\omega_m |x|^{m-p} |u^\#|^p}{\omega_2 E(u^\#)}\right) dx\right) \tag{4.1}$$

holds for all $u \in W^{1,p}(\mathbb{R}^m)$, where $u^\#$ is the symmetric decreasing rearrangement of u, $C = C(p,m) = \left(\frac{m-p}{p}\right)^{p-2} \frac{4\pi(p-1)}{p^2}$, and $E(u^\#) = \int_{\mathbb{R}^m} |u^\#|^p |x|^{2-p} dx$.

Proof. Proof is similar to that of Theorem 1. First we show the inequality (4.1) for any nonnegative, radially symmetric and nonincreasing function $u = u(r) \in C_0^\infty(\mathbb{R}^m)$ where $r = |x|$. Put $v = v(y) = |y|^{\frac{m-p}{p}} u(|y|)$, $y \in \mathbb{R}^2$, as in the proof of Theorem 1 and recall that $v^{\frac{p}{2}} \in W^{1,2}(\mathbb{R}^2)$ when $u \in W^{1,p}(\mathbb{R}^m)$. We apply the logarithmic Sobolev inequality on $\mathbb{R}^2$ (see Proposition 5 in Appendix) to $f(y) = \|v\|_{L^p(\mathbb{R}^2)}^{-\frac{p}{2}} v^{\frac{p}{2}}(y)$:

$$\pi \|v\|_{L^p(\mathbb{R}^2)}^p \exp\left(1 + \int_{\mathbb{R}^2} \frac{v^p(y)}{\|v\|_{L^p(\mathbb{R}^2)}^p} \log\left(\frac{v^p(y)}{\|v\|_{L^p(\mathbb{R}^2)}^p}\right) dy\right) \le \int_{\mathbb{R}^2} |\nabla(v^{p/2})|^2 \, dy.$$

By this inequality, we estimate (2.1) from below. Therefore we obtain

$$\begin{aligned}
J &= \int_{\mathbb{R}^m} |\nabla u|^p dx - \left(\frac{m-p}{p}\right)^p \int_{\mathbb{R}^m} \frac{|u|^p}{|x|^p} \, dx \\
&\ge C(p,m) \frac{\omega_m}{\omega_2} \|v\|_{L^p(\mathbb{R}^2}^p \exp\left(1 + \int_{\mathbb{R}^2} \frac{v^p(y)}{\|v\|_{L^p(\mathbb{R}^2)}^p} \log\left(\frac{v^p(y)}{\|v\|_{L^p(\mathbb{R}^2)}^p}\right) dy\right) \\
&= C(p,m) E(u) \exp\left(1 + \frac{\omega_m}{E(u)} \int_0^\infty r^{m-p} u^p(r) \log\left(\frac{\omega_m r^{m-p} u^p(r)}{\omega_2 E(u)}\right) r \, dr\right) \\
&= C(p,m) E(u) \exp\left(1 + E(u)^{-1} \int_{\mathbb{R}^m} \frac{|u|^p}{|x|^{p-2}} \log\left(\frac{\omega_m |x|^{m-p} |u|^p}{\omega_2 E(u)}\right) dx\right)
\end{aligned}$$

where $E(u) = \frac{\omega_m}{\omega_2} \|v\|_{L^p(\mathbb{R}^2)}^p = \int_{\mathbb{R}^m} |u|^p |x|^{2-p} dx$. Hence the inequality (4.1) holds for any nonnegative, radially symmetric and nonincreasing function $u = u(r) \in C_0^\infty(\mathbb{R}^m)$. The rest of the proof is done by the usual density and the symmetrization argument. □

Remark 5. The inequality (4.1) has an invariance under the scaling $u_\lambda(x) = \lambda^{-\frac{m-p}{p}} u(y)$ where $y = \frac{x}{\lambda}$, $(\lambda > 0, x \in \mathbb{R}^m)$. Indeed, by virtue of Proposition 6, we have $(u^\#)_\lambda = (u_\lambda)^\#$ so that

$$E((u^\#)_\lambda) = \int_{\mathbb{R}^m} |(u^\#)_\lambda(x)|^p |x|^{2-p} dx = \lambda^{-m+p} \int_{\mathbb{R}^m} |u^\#(y)|^p |\lambda y|^{2-p} \lambda^m dy = \lambda^2 E(u^\#).$$

Thus

$$
\begin{aligned}
&CE((u^{\#})_\lambda)\exp\left(1+E((u^{\#})_\lambda)^{-1}\int_{\mathbb{R}^m}\frac{|(u^{\#})_\lambda(x)|^p}{|x|^{p-2}}\log\left(\frac{\omega_m|x|^{m-p}|(u^{\#})_\lambda(x)|^p}{\omega_2 E((u^{\#})_\lambda)}\right)dx\right)\\
&=\lambda^2 CE(u^{\#})\exp\left(1+E(u^{\#})^{-1}\lambda^{-2+p-m}\int_{\mathbb{R}^m}\frac{|u^{\#}(y)|^p}{\lambda^{p-2}|y|^{p-2}}\log\left(\frac{\omega_m|y|^{m-p}|u^{\#}(y)|^p}{\omega_2\lambda^2 E(u^{\#})}\right)\lambda^m\,dy\right)\\
&=\lambda^2 CE(u^{\#})\exp\left(1+E(u^{\#})^{-1}\int_{\mathbb{R}^m}\frac{|u^{\#}(y)|^p}{|y|^{p-2}}\left(\log\lambda^{-2}+\log\left(\frac{\omega_m|y|^{m-p}|u^{\#}(y)|^p}{\omega_2 E(u^{\#})}\right)\right)dy\right)\\
&=CE(u^{\#})\exp\left(1+E(u^{\#})^{-1}\int_{\mathbb{R}^m}\frac{|u^{\#}(y)|^p}{|y|^{p-2}}\log\left(\frac{\omega_m|y|^{m-p}|u^{\#}(y)|^p}{\omega_2 E(u^{\#})}\right)dy\right),
\end{aligned}
$$

so the inequality (4.1) also enjoys a scale invariance.

Next we will show another type of improvement of the sharp Hardy inequality in a limiting case (1.7).

Theorem 5. *The inequality*

$$
\begin{aligned}
&\int_{B_R^2(0)}|\nabla w|^2dx-\frac{1}{4}\int_{B_R^2(0)}\frac{|w|^2}{|x|^2\left(\log\frac{R}{|x|}\right)^2}dx\\
&\geq\frac{4}{R^2\omega_2}\sup_{\lambda>0}\ \lambda^{-4}\left|\int_{B_R^2(0)}w(x)\frac{\lambda-\log\frac{R}{|x|}}{|x|(\log\frac{R}{|x|})^{\frac{1}{2}}}\left(\frac{R}{|x|}\right)^{1-\frac{1}{\lambda}}dx\right|^2
\end{aligned}
\tag{4.2}
$$

holds for all radial function $w\in W_0^{1,2}(B_R^2(0))$.

Proof. We show the inequality (4.2) holds for any radial function $w = w(x)\in C_0^\infty(B_R^2(0))$. We define the new function $v=v(x)$ on $B_R^2(0)$ as

$$
v(r)=\left(\log\frac{R}{r}\right)^{-\frac{1}{2}}w(r),\ \ r=|x|, x\in B_R^2(0).
$$

By direct calculation, we obtain

$$
\begin{aligned}
I &= \int_{B_R^2(0)} |\nabla w|^2 dx - \frac{1}{4}\int_{B_R^2(0)} \frac{|w|^2}{|x|^2(\log \frac{R}{|x|})^2} dx \\
&= \omega_2 \int_0^R (w'(r))^2 r\, dr - \frac{\omega_2}{4}\int_0^R \frac{|w(r)|^2}{r^2(\log \frac{R}{r})^2} r\, dr \\
&= \omega_2 \int_0^R \left(-\frac{1}{2}\left(\log \frac{R}{r}\right)^{-\frac{1}{2}} \frac{v(r)}{r} + \left(\log \frac{R}{r}\right)^{\frac{1}{2}} v'(r) \right)^2 r\, dr - \frac{\omega_2}{4}\int_0^R \frac{|v(r)|^2}{r \log \frac{R}{r}}\, dr \\
&= -\omega_2 \int_0^R v(r)v'(r)\, dr + \omega_2 \int_0^R |v'(r)|^2 r \log \frac{R}{r}\, dr \\
&= -\frac{\omega_2}{2}\int_0^R \left(v^2(r)\right)'\, dr + \frac{R^2\omega_2}{4}\left(\int_0^R |v'(r)|^2 \frac{4}{R^2} r \log \frac{R}{r}\, dr \right) \\
&= \frac{R^2\omega_2}{4}\left(\int_0^R |v'(r)|^2 \frac{4}{R^2} r \log \frac{R}{r}\, dr \right),
\end{aligned}
$$

where we have used $v(0) = v(R) = 0$ for the last equality. Since $\frac{4}{R^2}\int_0^R r \log \frac{R}{r}\, dr = 1$ and $t \mapsto t^2$ is convex, we apply Jensen's inequality to the above expression of I. Hence we obtain

$$
I \geq \frac{R^2\omega_2}{4}\left(\int_0^R |v'(r)| \frac{4}{R^2} r \log \frac{R}{r}\, dr \right)^2 \geq \frac{4\omega_2}{R^2}\left| \int_0^R v'(r) r \log \frac{R}{r} dr \right|^2 .
$$

By using the boundary condition $v(0) = v(R) = 0$, we have

$$
\begin{aligned}
&\int_0^R v'(r) r \log \frac{R}{r}\, dr = -\int_0^R v(r) \left(\log \frac{R}{r} - 1 \right) dr \\
&= \int_0^R w(r) \frac{1 - \log \frac{R}{r}}{(\log \frac{R}{r})^{\frac{1}{2}}}\, dr = \frac{1}{\omega_2}\int_{B_R^2(0)} w(x) \frac{1 - \log \frac{R}{|x|}}{|x|(\log \frac{R}{|x|})^{\frac{1}{2}}}\, dx.
\end{aligned}
$$

Combining this to the above estimate of I from below, we obtain

$$
I \geq \frac{4}{R^2\omega_2}\left| \int_{B_R^2(0)} w(x) \frac{1 - \log \frac{R}{|x|}}{|x|(\log \frac{R}{|x|})^{\frac{1}{2}}} dx \right|^2 .
$$

Now, we know that I is invariant under the scale transformation $w \mapsto w^\lambda(r) = \lambda^{-\frac{1}{2}} w(R^{1-\lambda} r^\lambda)$. Therefore if we put

$$
J(w) = \int_{B_R^2(0)} w(x) \frac{1 - \log \frac{R}{|x|}}{|x|(\log \frac{R}{|x|})^{\frac{1}{2}}} dx = \omega_2 \int_0^R w(r) \frac{1 - \log \frac{R}{r}}{(\log \frac{R}{r})^{\frac{1}{2}}} dr,
$$

we obtain $I \geq \frac{4}{R^2\omega_2}|J(w^\lambda)|^2$ for any $\lambda > 0$. On the other hand, we see

$$\begin{aligned}
J(w^\lambda) &= \omega_2 \int_0^R w^\lambda(r) \frac{1 - \log \frac{R}{r}}{(\log \frac{R}{r})^{\frac{1}{2}}} dr \\
&= \omega_2 \lambda^{-\frac{1}{2}} \int_0^R w(r^\lambda R^{1-\lambda}) \frac{1 - \log \frac{R}{r}}{(\log \frac{R}{r})^{\frac{1}{2}}} dr \\
&= \omega_2 \lambda^{-\frac{1}{2}} \int_0^R w(s) \frac{1 - \log \left(\frac{R}{s}\right)^{\frac{1}{\lambda}}}{\left\{\log \left(\frac{R}{s}\right)^{\frac{1}{\lambda}}\right\}^{\frac{1}{2}}} \frac{1}{\lambda} \left(\frac{R}{s}\right)^{1-\frac{1}{\lambda}} ds \\
&= \omega_2 \lambda^{-2} \int_0^R w(s) \frac{\lambda - \log \frac{R}{s}}{\left(\log \frac{R}{s}\right)^{\frac{1}{2}}} \left(\frac{R}{s}\right)^{1-\frac{1}{\lambda}} ds \\
&= \lambda^{-2} \int_{B_R^2(0)} w(x) \frac{\lambda - \log \frac{R}{|x|}}{|x|(\log \frac{R}{|x|})^{\frac{1}{2}}} \left(\frac{R}{|x|}\right)^{1-\frac{1}{\lambda}} dx,
\end{aligned}$$

where we have used a change of variable $s = r^\lambda R^{1-\lambda}$, $dr = \frac{1}{\lambda}\left(\frac{R}{s}\right)^{\frac{\lambda-1}{\lambda}} ds$. This and $I \geq \frac{4}{R^2\omega_2} \sup_{\lambda>0} |J(w^\lambda)|^2$ imply (4.2). General case follows from density argument. □

Remark 6. The remainder term in (4.2) is invariant under the transformation (1.8). Indeed, put

$$\begin{aligned}
R(w) &= \sup_{\lambda>0} \lambda^{-4} \left| \int_{B_R^2(0)} w(x) \frac{\lambda - \log \frac{R}{|x|}}{|x|(\log \frac{R}{|x|})^{\frac{1}{2}}} \left(\frac{R}{|x|}\right)^{1-\frac{1}{\lambda}} dx \right|^2 \\
&= \sup_{\lambda>0} |J(w^\lambda)|^2.
\end{aligned}$$

Then, by the definition of (1.8), we easily check that $(w^\lambda)^\mu = w^{\lambda\mu}$ for any $\lambda, \mu > 0$. Thus we see

$$\begin{aligned}
R(w^\mu) &= \sup_{\lambda>0} |J((w^\mu)^\lambda)|^2 = \sup_{\lambda>0} |J(w^{\lambda\mu})|^2 \\
&= \sup_{\lambda\mu>0} |J(w^{\lambda\mu})|^2 = \sup_{\nu>0} |J(w^\nu)|^2 = R(w).
\end{aligned}$$

On the other hand, J itself is not invariant under the scaling (1.8). For example, take $w_\alpha(x) = 1 - \left(\frac{|x|}{R}\right)^\alpha \in W_0^{1,2}(B_R^2(0))$ for $\alpha > 0$. Elementary computations show that

$$J(w_\alpha) = R\left(\Gamma\left(\frac{1}{2}\right) - \Gamma\left(\frac{3}{2}\right) - (1+\alpha)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) + (1+\alpha)^{-\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)\right),$$

$$J((w_\alpha)^\lambda) = \lambda^{-\frac{1}{2}}R\left(\Gamma\left(\frac{1}{2}\right) - \Gamma\left(\frac{3}{2}\right) - (1+\alpha\lambda)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) + (1+\alpha\lambda)^{-\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)\right)$$

for any $\lambda > 0$. In this case, we observe that $J((w_\alpha)^\lambda) \to 0$ as $\lambda \to 0$ or $\lambda \to \infty$, which assures that $\sup_{\lambda>0} \ |J((w_\alpha)^\lambda)|^2 < \infty$. However, it may be difficult to show the fact $\sup_{\lambda>0} \ |J(w^\lambda)|^2 < \infty$ without appealing to (4.2) for each $w \in W_0^{1,2}(B_R^2(0))$.

5. Appendix

In this Appendix, we collect several facts which are useful throughout the paper.

Proposition 4. *(Gagliardo-Nirenberg inequality,* [10], [20]*) Let $N \in \mathbb{N}$, $1 \leq p, q, r \leq \infty$ and $0 \leq \sigma \leq 1$ satisfy*

$$\frac{1}{p} = \sigma\left(\frac{1}{r} - \frac{1}{N}\right) + (1-\sigma)\frac{1}{q}. \tag{5.1}$$

Moreover, assume $p \neq \infty$ or $r \neq N$ if $N \geq 2$. Then there exists a constant $C = C(p,q,r,N)$ such that

$$\|u\|_{L^p(\mathbb{R}^N)} \leq C\|u\|_{L^q(\mathbb{R}^N)}^{1-\sigma}\|\nabla u\|_{L^r(\mathbb{R}^N)}^{\sigma}.$$

for all $u \in C_0^1(\mathbb{R}^N)$.

Proposition 5. *(The logarithmic-Sobolev inequality in $2D$,* [14], [17]*) The following inequality*

$$\int_{\mathbb{R}^2} f^2(x)\log f^2(x)\,dx \leq \log\left(\frac{1}{\pi e}\int_{\mathbb{R}^2} |\nabla f|^2\,dx\right)$$

holds for all $f \in W^{1,2}(\mathbb{R}^2)$ with $\|f\|_{L^2(\mathbb{R}^2)} = 1$.

Proposition 6. *Put $r = |x|$, $x \in \mathbb{R}^N$ and let*

$$u^\#(r) = \inf\{\tau > 0 \mid \mu_u(\tau) \leq |B_r^N(0)|\}$$

be the symmetric decreasing rearrangement of a function u, where μ_u is a distribution function of u: $\mu_u(\tau) = \left|\{x \in \mathbb{R}^N \mid |u(x)| > \tau\}\right|$, $\tau \geq 0$. Define $u_\lambda(x) = \lambda^{-\frac{N-p}{p}} u\left(\frac{x}{\lambda}\right)$ for $\lambda > 0$. Then the equality

$$(u_\lambda)^\#(r) = (u^\#)_\lambda(r) \quad (^\forall r > 0) \tag{5.2}$$

holds.

Proof of Proposition 6. The distribution function of u_λ can be written as

$$\begin{aligned}
\mu_{u_\lambda}(\tau) &= \left|\{x \in \mathbb{R}^N \mid |u_\lambda(x)| > \tau\}\right| \\
&= \left|\left\{x \in \mathbb{R}^N \;\middle|\; \lambda^{-\frac{N-p}{p}} \left|u\left(\frac{x}{\lambda}\right)\right| > \tau\right\}\right| \\
&= |\{\lambda y \in \mathbb{R}^N \mid |u(y)| > \lambda^{\frac{N-p}{p}}\tau\}| \\
&= \lambda^N |\{y \in \mathbb{R}^N \mid |u(y)| > \lambda^{\frac{N-p}{p}}\tau\}| \\
&= \lambda^N \mu_u(\lambda^{\frac{N-p}{p}}\tau). \qquad (5.3)
\end{aligned}$$

Hence by the definition of $(u_\lambda)^\#$ and (5.3), we obtain

$$\begin{aligned}
(u_\lambda)^\#(r) &= \inf\{\tau > 0 \mid \mu_{u_\lambda}(\tau) \le |B_r|\} \\
&= \inf\{\tau > 0 \mid \lambda^N \mu_u(\lambda^{\frac{N-p}{p}}\tau) \le |B_r|\} \\
&= \inf\{\lambda^{-\frac{N-p}{p}}\tilde{\tau} > 0 \mid \mu_u(\tilde{\tau}) \le \lambda^{-N}|B_r|\} \\
&= \lambda^{-\frac{N-p}{p}} \inf\{\tilde{\tau} > 0 \mid \mu_u(\tilde{\tau}) \le |B_{\frac{r}{\lambda}}|\} \\
&= \lambda^{-\frac{N-p}{p}} u^\#\left(\frac{r}{\lambda}\right) = (u^\#)_\lambda(r).
\end{aligned}$$

The proof of Proposition 6 is now complete. □

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