IMPROVED RELLICH TYPE INEQUALITIES IN $\mathbb{R}^N$

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Abstract. We consider the second or higher-order Rellich inequalities on the whole space $\mathbb{R}^N$. In spite of the lack of Poincaré inequality on the whole space, we show that the higher-order Rellich inequalities with optimal constants can be improved, by adding explicit remainder terms to the inequalities.

1. Introduction

Let $N \ge 2, 1 \le p < N$, and let Ω be a bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$, or $\Omega = \mathbb{R}^N$. The classical Hardy inequality

$$\int_\Omega |\nabla u|^p dx \ge \left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \tag{1.1}$$

holds for all $u \in W_0^{1,p}(\Omega)$, or $u \in D^{1,p}(\mathbb{R}^N)$ when $\Omega = \mathbb{R}^N$. (1.1) gives an expression to the embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega; |x|^{-p} dx), \tag{1.2}$$

where $W_0^{1,p}(\Omega)$ (resp. $D^{1,p}(\mathbb{R}^N)$) is the completion of $C_0^\infty(\Omega)$ (resp. $C_0^\infty(\mathbb{R}^N)$) with respect to the norm $\|\nabla \cdot\|_{L^p(\Omega)}$ (resp. $\|\nabla \cdot\|_{L^p(\mathbb{R}^N)}$). It is known that for $1 < p < N$, the best constant $(\frac{N-p}{p})^p$ is never attained in $W_0^{1,p}(\Omega)$, or in $D^{1,p}(\mathbb{R}^N)$. Therefore, one can expect the existence of remainder terms on the right-hand side of the inequality (1.1). Indeed, there are many papers that deal with remainder terms for (1.1) when Ω is a smooth bounded domain (see [1], [8], [9], [12], [13], [20], to name a few). For example, Brezis and Vázquez [8] showed that the inequality

$$\int_\Omega |\nabla u|^2 dx \ge \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} dx + z_0^2 \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \int_\Omega |u|^2 dx \tag{1.3}$$

holds true for all $u \in W_0^{1,2}(\Omega)$ where $z_0 = 2.4048\cdots$ is the first zero of the Bessel function of the first kind. Chaudhuri and Ramaswamy [9] improved

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the Brezis-Vázquez's result by proving that for any $0 \le \beta < 2$ and $1 < q \le \frac{2(N-\beta)}{N-2}$, there exists a constant $C > 0$ depending on N, β, q and Ω such that

$$\int_\Omega |\nabla u|^2 \, dx \ge \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} \, dx + C \left(\int_\Omega \frac{|u|^q}{|x|^\beta} \, dx \right)^{2/q} \tag{1.4}$$

holds for all $u \in W_0^{1,2}(\Omega)$. This improved inequality (1.4) gives an expression to the embedding (1.2) and

$$W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega; |x|^{-\beta} dx). \tag{1.5}$$

When $\beta = 0$, (1.5) is the well-known Sobolev embedding.

On the other hand, when $\Omega = \mathbb{R}^N$, Ghoussoub and Moradifam [14] showed that there is no strictly positive $V \in C^1((0, +\infty))$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \ge \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} V(|x|)|u|^2 dx$$

holds for all $u \in W^{1,2}(\mathbb{R}^N)$. Therefore we cannot expect the same type of remainder terms as in (1.3) would exist in the whole space.

In spite of this fact, the authors of the paper recently showed the following result [22] : Let $2 \le p < N$ and $q > 2$. Set $\alpha = \alpha(p, q, N) = \frac{N}{2}(q-2) - \frac{pq}{2} + 2$. Then there exists $D = D(p, q, N) > 0$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \ge \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx + D \left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^\alpha dx}{\int_{\mathbb{R}^N} |u^\#|^p |x|^{2-p}} dx \right)^{\frac{2}{q-2}} \tag{1.6}$$

holds for all $u \in W^{1,p}(\mathbb{R}^N)$, $u \not\equiv 0$. Here $u^\#$ denotes the Schwartz symmetrization of a function u on $\mathbb{R}^N$:

$$u^\#(x) = u^\#(|x|) = \inf \left\{ \lambda > 0 \;\middle|\; \left| \{ x \in \mathbb{R}^N \mid |u(x)| > \lambda \} \right| \le |B_{|x|}(0)| \right\},$$

where $|A|$ denotes the measure of a set $A \subset \mathbb{R}^N$ (see e.g., [17]). We observe that (1.6) gives a new embedding

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{\frac{pq}{2}}(\mathbb{R}^N; |x|^\alpha dx) \quad \text{if } \alpha \le 0,$$

since when $\alpha \le 0$, we assure that $u^\#$ in (1.6) can be replaced by u and also the integral $\int_{\mathbb{R}^N} |u^\#|^p |x|^{2-p} dx$ is finite for any $u \in W^{1,p}(\mathbb{R}^N)$.

In this paper, we focus on the higher-order case. A higher-order generalization of (1.1) was first proved by Rellich [21]: it holds

$$\int_\Omega |\Delta u|^2 dx \ge \left(\frac{N(N-4)}{4}\right)^2 \int_\Omega \frac{|u|^2}{|x|^4} dx$$

for all $u \in W_0^{2,2}(\Omega)$, where Ω is a domain in $\mathbb{R}^N$, $N \geq 5$. More generally, let $k, m \in \mathbb{N}$ and $k < kp < N$. Define

$$|u|_{k,p}^p = \begin{cases} \int_\Omega |\Delta^m u|^p \, dx & \text{if } k = 2m, \\ \int_\Omega |\nabla(\Delta^m u)|^p \, dx & \text{if } k = 2m+1, \end{cases} \quad \text{and}$$

$$C_{k,p} = \begin{cases} p^{-2m} \prod_{j=1}^m (N - 2pj) \{N(p-1) + 2p(j-1)\} & \text{if } k = 2m, \\ \frac{N-p}{p} C_{2m,p} & \text{if } k = 2m+1. \end{cases}$$

We put $C_{0,p} = 1$, $C_{1,p} = \frac{N-p}{p}$ for the convenience of description. Then the inequality

$$|u|_{k,p}^p \geq C_{k,p}^p \int_\Omega \frac{|u|^p}{|x|^{kp}} dx \tag{1.7}$$

holds for all $u \in W_0^{k,p}(\Omega)$. It is also known that $C_{k,p}^p$ is optimal (see [10], [18], Proposition 7 in Appendix) and never attained by functions in $W_0^{k,p}(\Omega)$. Furthermore, Gazzola-Grunau-Mitidieri [13] provided the following inequality on a smooth bounded domain: there exist positive constants $A, B > 0$ such that the inequality

$$|u|_{2,2}^2 \geq C_{2,2}^2 \int_\Omega \frac{|u|^2}{|x|^4} dx + A \int_\Omega \frac{|u|^2}{|x|^2} dx + B \int_\Omega |u|^2 dx$$

holds for all $u \in W_0^{2,2}(\Omega)$, where $N \geq 5$. In addition to this, there are many papers that deal with various types of Rellich inequalities with remainder terms on bounded domains (see [2], [3], [4], [5], [6], [7], [11], [15], [19], [24], [25] etc.).

However, when $\Omega = \mathbb{R}^N$ case, it seems difficult to get a remainder term for the inequality (1.7) even in the case $k = p = 2$, due to the lack of appropriate Poincaré inequality on the whole space. Main aim of this paper is to obtain remainder terms for the inequality (1.7) when $\Omega = \mathbb{R}^N$. Note that the inequalities (1.1) and (1.7) have the scale invariance under the scaling

$$u_\lambda(x) = \lambda^{-\frac{N-kp}{p}} u\left(\frac{x}{\lambda}\right) \tag{1.8}$$

for $\lambda > 0$ when $\Omega = \mathbb{R}^N$. Therefore the possible remainder term to (1.7) should be invariant under the scaling (1.8) when $\Omega = \mathbb{R}^N$. In the following, ω_N will denote an area of the unit sphere in $\mathbb{R}^N$ and $\| \cdot \|_r = \| \cdot \|_{L^r(\mathbb{R}^N)}$.

Our main results are as follows:

Theorem 1. *(Radial case) Let $k \geq 2$ be an integer, $k < kp < N$ and $q > 2$. Set $\alpha_k = \frac{N}{2}(q-2) - \frac{kpq}{2} + 2$. Then there exists $E_k = E_k(p, q, N) > 0$ such that* *the inequality*

$$|u|_{k,p}^p \geq C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\,dx + E_k \left(\frac{\int_{\mathbb{R}^N} |u|^{\frac{pq}{2}} |x|^{\alpha_k}\,dx}{\int_{\mathbb{R}^N} |u|^p |x|^{2-kp}\,dx} \right)^{\frac{2}{q-2}} \tag{1.9}$$

holds for all radial function $u \in W_0^{k,p}(\mathbb{R}^N)$, $u \neq 0$. The constants E_k are explicitly given as

$$E_k = \frac{4\omega_N(p-1)}{\omega_2 p} C_{k-2,p} C_{k,p}^{p-1} C(q)^{-\frac{2q}{q-2}},$$

where $C(q)$ is the positive constant in the Gagliardo-Nirenberg inequality (2.5) below.

In the non-radial case, we obtain only partial results for $k = 2, 3$.

Theorem 2. *(Non-radial case) For $k = 2$ or $k = 3$, let $k < kp < N$ and $q > 2$. Set $\alpha_k = \frac{N}{2}(q-2) - \frac{kpq}{2} + 2$ and $r = \frac{Np}{N+2p}$ (i.e. $\frac{1}{p} = \frac{1}{r} - \frac{2}{N}$). Then there exists $F_k = F_k(p, q, N) > 0$ such that the inequality*

$$|u|_{k,p}^p \geq C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}}\,dx + F_k \left(\frac{\int_{\mathbb{R}^N} |u^{\#}|^{\frac{pq}{2}} |x|^{\alpha_k}\,dx}{|u|_{k,p}^{p-1} \|\Delta u\|_r} \right)^{\frac{2}{q-2}} \tag{1.10}$$

holds for all $u \in W^{k,p}(\mathbb{R}^N)$ with $\Delta u \in L^r(\mathbb{R}^N)$, $u \neq 0$. The constants F_k ($k = 2, 3$) are explicitly given as $F_k = E_k C_{k,p}^{\frac{2(p-1)}{q-2}} H^{\frac{-2}{q-2}}$ where H is the positive constant in the Hardy-Littlewood-Sobolev inequality (2.10) below.

Remark 3. The remainder term of the inequality (1.9) is scale invariant under the scaling (1.8) on $\mathbb{R}^N$: $u_\lambda(x) = \lambda^{-\frac{N-kp}{p}} u(y)$, $y = \frac{x}{\lambda}$, $x \in \mathbb{R}^N$. Indeed, for $a, b \in \mathbb{R}$, we have

$$\int_{\mathbb{R}^N} |u_\lambda(x)|^a |x|^b dx = \lambda^{-\left(\frac{N-kp}{p}\right)a+b+N} \int_{\mathbb{R}^N} |u(y)|^a |y|^b dy. \tag{1.11}$$

Thus by taking $a = \frac{pq}{2}$ and $b = \alpha_k$, or $a = p$ and $b = 2 - kp$ in (1.11), we have

$$\int_{\mathbb{R}^N} |u_\lambda(x)|^{\frac{pq}{2}} |x|^{\alpha_k} dx = \lambda^2 \int_{\mathbb{R}^N} |u(y)|^{\frac{pq}{2}} |y|^{\alpha_k} dy,$$
$$\int_{\mathbb{R}^N} |u_\lambda(x)|^p |x|^{2-kp} dx = \lambda^2 \int_{\mathbb{R}^N} |u(y)|^p |y|^{2-kp} dy.$$

Therefore the remainder term in the inequality (1.9) has the scale invariance.

Remark 4. If $\alpha_k \leq 0$ in Theorem 2, then $u^{\#}$ in the RHS of (1.10) can be replaced by u thanks to the Hardy-Littlewood inequality: $\int_{\mathbb{R}^N} g^{\#} h^{\#} \geq \int_{\mathbb{R}^N} gh$ (see e.g., [17]), and the fact $(|x|^{\alpha_k})^{\#} = |x|^{\alpha_k}$.

2. Proofs of Main results

Next simple lemma is used to prove Theorem 1.

Lemma 5. *Let $p \geq 1$ and $a, b \in \mathbb{R}$. Then it holds*

$$|a-b|^p - |a|^p \geq -p|a|^{p-2}ab.$$

Proof of Lemma 5. First, we assume $a \geq 0$. We use the mean value theorem for the function $f(t) = (a-t)^p$, which is defined for $t \leq a$. When $b \leq a$, we have

$$f(b) - f(0) = (a-b)^p - a^p = pc^{p-1}(-b) \geq -pa^{p-1}b,$$

where $c \in \mathbb{R}$ satisfies $0 \leq a-b \leq c \leq a$ if $b \geq 0$, or $0 \leq a \leq c \leq a-b$ if $b \leq 0$. When $b \geq a$, then $2a - b \leq a$ and we have

$$f(2a-b) - f(0) = (b-a)^p - a^p = pc^{p-1}(b-2a) \geq -pa^{p-1}b,$$

where $c \in \mathbb{R}$ satisfies $0 \leq a \leq c \leq b-a$ if $b - 2a \geq 0$, or $0 \leq b-a \leq c \leq a$ if $b - 2a \leq 0$. This implies the result when $a \geq 0$.

The case when $a \leq 0$ follows by considering $a = -\tilde{a}$, $\tilde{a} \geq 0$ and $b = -\tilde{b}$, $\tilde{b} \in \mathbb{R}$. □

Proof of Theorem 1. We show the inequality (1.9) for all radial function $u \in W^{k,p}(\mathbb{R}^N)$. By density argument, we may assume $0 \leq u \in C_0^\infty(\mathbb{R}^N)$ without loss of generality.

First, note that the inequality

$$|u|_{k,p}^p = |\Delta u|_{k-2,p}^p \geq C_{k-2,p}^p \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{(k-2)p}}\, dx \tag{2.1}$$

holds from Rellich's inequality (1.7). Actually when $k = 2$, this is the equality. Thus, in order to prove Theorem, it is enough to show the RHS of (2.1) is bounded from below by the RHS of (1.9).

Since u is radial, u can be written as $u(x) = \tilde{u}(|x|)$ where $0 \leq \tilde{u} \in C_0^\infty([0, +\infty))$. We define the new function v as follows:

$$\tilde{v}(r) = r^{\frac{N-kp}{p}} \tilde{u}(r), \quad r \in [0, \infty), \quad \text{and} \qquad v(y) = \tilde{v}(|y|), \quad y \in \mathbb{R}^2.$$

Note that $\tilde{v}(0) = 0$ and also $\tilde{v}(+\infty) = 0$ since the support of u is compact. We claim that if $u \in W^{k,p}(\mathbb{R}^N)$, then $v \in L^p(\mathbb{R}^2)$. Indeed, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} |v(y)|^p dy &= \omega_2 \int_0^\infty |\tilde{v}(r)|^p r dr \\
&= \omega_2 \int_0^\infty |\tilde{u}(r)|^p r^{N-kp+1}\, dr = \frac{\omega_2}{\omega_N} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp-2}}\, dx \\
&\le \frac{\omega_2}{\omega_N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\, dx \right)^{\frac{kp-2}{kp}} \left(\int_{\mathbb{R}^N} |u|^p\, dx \right)^{\frac{2}{kp}} \\
&\le \frac{\omega_2}{\omega_N} C_{k,p}^{\frac{2-kp}{k}} |u|_{k,p}^{\frac{kp-2}{k}} \left(\int_{\mathbb{R}^N} |u|^p\, dx \right)^{\frac{2}{kp}} < \infty, \qquad (2.2)
\end{aligned}$$

here we have used Hölder's inequality, Rellich's inequality (1.7), and the assumption $u \in W^{k,p}(\mathbb{R}^N)$. Therefore we have checked $v \in L^p(\mathbb{R}^2)$.

For $k \ge 2, k \in \mathbb{N}$ and $k < kp < N$, put

$$\begin{aligned}
\theta_k &= \theta(k, N, p) = 2k + \frac{N(p-2)}{p}, \quad \text{and} \\
\Delta_{\theta_k} f &= f''(r) + \frac{\theta_k - 1}{r} f'(r)
\end{aligned}$$

for smooth radial functions $f = f(r)$. Define

$$A_{k,p} = \frac{(N-kp)[(k-2)p + (p-1)N]}{p^2}.$$

Then a direct calculation shows that

$$-\Delta \tilde{u} = r^{k-2-\frac{N}{p}} \left(A_{k,p} \tilde{v}(r) - r^2 \Delta_{\theta_k} \tilde{v}(r) \right).$$

Now applying Lemma 5 with the choice

$$a = A_{k,p} \tilde{v}(r) \quad \text{and} \quad b = r^2 \Delta_{\theta_k} \tilde{v}(r),$$

and using the fact $\int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\tilde{v}'dr = 0$ since $\tilde{v}(0) = \tilde{v}(+\infty) = 0$, we have

$$
\begin{aligned}
J &= \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{(k-2)p}}\,dx - A_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\,dx \\
&= \omega_N \int_0^\infty |-\Delta\tilde{u}(r)|^p\, r^{N-1-(k-2)p}\,dr - A_{k,p}^p \omega_N \int_0^\infty |\tilde{u}(r)|^p r^{N-kp-1}\,dr \\
&= \omega_N \int_0^\infty \left(\left|A_{k,p}\tilde{v}(r) - r^2\Delta_{\theta_k}\tilde{v}(r)\right|^p - (A_{k,p}\tilde{v}(r))^p\right) r^{-1}\,dr \\
&\geq -p\omega_N A_{k,p}^{p-1} \int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\Delta_{\theta_k}\tilde{v}\, r\,dr \\
&= -p\omega_N A_{k,p}^{p-1} \int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\left(\tilde{v}'' + \frac{\theta_k - 1}{r}\tilde{v}'\right) r\,dr \\
&= -p\omega_N A_{k,p}^{p-1} \int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\tilde{v}'' r\,dr. \qquad (2.3)
\end{aligned}
$$

Moreover we observe that

$$
\begin{aligned}
-\int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\tilde{v}'' r\,dr &= (p-1)\int_0^\infty |\tilde{v}|^{p-2}(\tilde{v}')^2 r\,dr + \int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\tilde{v}'\,dr \\
&= \frac{4(p-1)}{p^2}\int_0^\infty |(|\tilde{v}|^{\frac{p-2}{2}}\tilde{v})'|^2 r\,dr \\
&= \frac{4(p-1)}{p^2\omega_2}\int_{\mathbb{R}^2} |\nabla(|v|^{\frac{p-2}{2}}v)|^2\,dy. \qquad (2.4)
\end{aligned}
$$

Now, we apply the Gagliardo-Nirenberg inequality to $|v|^{\frac{p-2}{2}}v \in L^2(\mathbb{R}^2)$: there exists a constant $C(q) > 0$ such that it holds

$$
\||v|^{\frac{p}{2}}\|_{L^q(\mathbb{R}^2)} \leq C(q)\,\||v|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^{\frac{2}{q}}\|\nabla(|v|^{\frac{p-2}{2}}v)\|_{L^2(\mathbb{R}^2)}^{\frac{q-2}{q}}. \qquad (2.5)
$$

Combining (2.3), (2.4) and (2.5), we obtain

$$
\begin{aligned}
J &\geq \frac{4(p-1)\omega_N A_{k,p}^{p-1}}{p\omega_2} C(q)^{-\frac{2q}{q-2}} \left(\frac{\int_{\mathbb{R}^2} |v(y)|^{\frac{pq}{2}}dy}{\int_{\mathbb{R}^2} |v(y)|^p dy}\right)^{\frac{2}{q-2}} \\
&= \frac{4(p-1)\omega_N A_{k,p}^{p-1}}{p\omega_2} C(q)^{-\frac{2q}{q-2}} \left(\frac{\int_{\mathbb{R}^N} |u|^{\frac{pq}{2}}|x|^{\alpha_k}dx}{\int_{\mathbb{R}^N} |u|^p|x|^{2-kp}dx}\right)^{\frac{2}{q-2}}. \qquad (2.6)
\end{aligned}
$$

Consequently, from (2.1), the definition of J, (2.6) and $C_{k-2,p}A_{k,p} = C_{k,p}$, we obtain

$$\begin{aligned}
|u|_{k,p}^p &\geq C_{k-2,p}^p \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{(k-2)p}}\, dx \\
&= C_{k-2,p}^p \left(A_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\, dx + J \right) \\
&\geq C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\, dx + E_k \left(\frac{\int_{\mathbb{R}^N} |u|^{\frac{pq}{2}} |x|^{\alpha_k}\, dx}{\int_{\mathbb{R}^N} |u|^p |x|^{2-kp}\, dx} \right)^{\frac{2}{q-2}}
\end{aligned}$$

where

$$\begin{aligned}
E_k &= C_{k-2,p}^p \frac{4\omega_N (p-1) A_{k,p}^{p-1}}{\omega_2 p} C(q)^{-\frac{2q}{q-2}} \\
&= \frac{4\omega_N (p-1)}{\omega_2 p} C_{k-2,p} C_{k,p}^{p-1} C(q)^{-\frac{2q}{q-2}}.
\end{aligned}$$

This proves Theorem 1. □

Proof of Theorem 2. First, we treat the case $k = 2$. We show the inequality

$$\int_{\mathbb{R}^N} |\Delta u|^p\, dx \geq C_{2,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}}\, dx + F_2 \left(\frac{\int_{\mathbb{R}^N} |u^{\#}|^{\frac{pq}{2}} |x|^{\alpha_2}\, dx}{\|\Delta u\|_p^{p-1} \|\Delta u\|_r} \right)^{\frac{2}{q-2}} \tag{2.7}$$

for all $u \in W^{2,p}(\mathbb{R}^N) \cap D^{2,r}(\mathbb{R}^N)$. Set $f = -\Delta u \in L^p(\mathbb{R}^N)$ and $w(x) = \frac{1}{(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{f^{\#}(y)}{|x-y|^{N-2}}\, dy$. Since $w(Ox) = w(x)$ for any $O \in O(N)$, the group of orthogonal matrices in $\mathbb{R}^N$, we see w is a radial function. Also since $f^{\#} \in L^p(\mathbb{R}^N)$, the Calderon-Zygmund inequality (see [16] Theorem 9.9.) implies that $w \in D^{2,p}(\mathbb{R}^N)$ and satisfies $-\Delta w = f^{\#}$ a.e. in $\mathbb{R}^N$. Therefore we have

$$\|\Delta w\|_p = \|\Delta u\|_p. \tag{2.8}$$

By Talenti's comparison principle [23], we know $w \geq u^{\#} \geq 0$. Hence we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |w|^{\beta} |x|^{\gamma}\, dx &\geq \int_{\mathbb{R}^N} |u^{\#}|^{\beta} |x|^{\gamma}\, dx \quad \text{if } \beta \geq 0, \\
&\geq \int_{\mathbb{R}^N} |u|^{\beta} |x|^{\gamma}\, dx \quad \text{if } \beta \geq 0 \text{ and } \gamma \leq 0.
\end{aligned} \tag{2.9}$$

where the second inequality comes from the Hardy-Littlewood inequality. Furthermore there exists a constant $H > 0$ such that the inequality

$$\|w\|_p \leq H \|f^{\#}\|_r = H \|(-\Delta u)^{\#}\|_r = H \|(-\Delta u)\|_r \tag{2.10}$$

holds from the Hardy-Littlewood-Sobolev inequality, where $\frac{1}{p} = \frac{1}{r} - \frac{2}{N}$. From (2.8), Theorem 1, (2.9) and (2.10), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta u|^p\,dx &= \int_{\mathbb{R}^N} |\Delta w|^p\,dx \\ &\geq C_{2,p}^p \int_{\mathbb{R}^N} \frac{|w|^p}{|x|^{2p}}\,dx + E_2\left(\frac{\int_{\mathbb{R}^N} |w|^{\frac{pq}{2}}|x|^{\alpha_2}\,dx}{\int_{\mathbb{R}^N} |w|^p|x|^{2-2p}\,dx}\right)^{\frac{2}{q-2}} \\ &\geq C_{2,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}}\,dx + E_2\left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}}|x|^{\alpha_2}\,dx}{C_{2,p}^{1-p}\|\Delta w\|_p^{p-1}\|w\|_p}\right)^{\frac{2}{q-2}} \\ &\geq C_{2,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}}\,dx + F_2\left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}}|x|^{\alpha_2}\,dx}{\|\Delta u\|_p^{p-1}\|\Delta u\|_r}\right)^{\frac{2}{q-2}},\end{aligned}$$

where $F_2 = E_2 C_{2,p}^{\frac{2(p-1)}{q-2}} H^{\frac{-2}{q-2}}$, which concludes (2.7).

Next, we treat the case $k = 3$. As before, set $f = -\Delta u \in W^{1,p}(\mathbb{R}^N)$ and $w(x) = \frac{1}{(N-2)\omega_N}\int_{\mathbb{R}^N} \frac{f^\#(y)}{|x-y|^{N-2}}\,dy$. Again we obtain $w \in D^{2,p}(\mathbb{R}^N)$ and $-\Delta w = f^\#$ a.e. in $\mathbb{R}^N$. By Pólya-Szegö inequality (see e.g., [17]), we have

$$|u|_{3,p}^p = \int_{\mathbb{R}^N} |\nabla\Delta u|^p dx = \int_{\mathbb{R}^N} |\nabla f|^p dx \geq \int_{\mathbb{R}^N} |\nabla f^\#|^p dx = |w|_{3,p}^p.$$

This inequality corresponds to (2.8). The rest of the proof will be done by the same argument as above. □

Remark 6. Up to now, we do not obtain the result for $k \geq 4$ in Theorem 2. For example, put $f = -\Delta u \in W^{2,p}(\mathbb{R}^N)$ for $u \in W^{4,p}(\mathbb{R}^N)$. Since we do not know the validity of the inequality

$$\int_{\mathbb{R}^N} |\Delta f|^p\,dx \geq \int_{\mathbb{R}^N} |\Delta f^\#|^p\,dx,$$

the argument of the proof of Theorem 2 does not work for $k = 4$ case. Instead, if we define $f = (-\Delta)^2 u \in L^p(\mathbb{R}^N)$ and $w(x) = C_N \int_{\mathbb{R}^N} \frac{f^\#(y)}{|x-y|^{N-4}}\,dy$, then we obtain $(-\Delta)^2 w = f^\#$ in $\mathbb{R}^N$ and $|u|_{4,p}^p = |w|_{4,p}^p$. However in this case, we do not know whether the comparison $u^\# \leq w$ hold or not, which violates the proof of Theorem 2.

3. Appendix

Davies-Hinz [10] showed that the constant $C_{k,p}^p$ in the inequality (1.7) is optimal when $\Omega = \mathbb{R}^N$. In this Appendix, we will show the fact when Ω is a general bounded domain.

Proposition 7. *Let $k \in \mathbb{N}$, $k < kp < N$ and let Ω be a bounded domain with $0 \in \Omega$ in $\mathbb{R}^N$. Then the constant $C_{k,p}^p$ in the inequality (1.7) is optimal. That is*

$$\inf_{0 \neq u \in W_0^{k,p}(\Omega)} \frac{|u|_{k,p}^p}{\int_\Omega \frac{|u(x)|^p}{|x|^{kp}}\, dx} = C_{k,p}^p.$$

Proof of Proposition 7. By the scaling (1.8) and zero extension, we may assume $B_1(0) \subset\subset \Omega$ without loss of generality. First, we show the optimality of $C_{k,p}^p$ in the even case $k = 2m$, $m \in \mathbb{N}$. For $0 < \varepsilon \ll 1$, we define the function $u_\varepsilon \in W_0^{2m,p}(\Omega)$ as follows:

$$u_\varepsilon(x) = \begin{cases} \varepsilon^{-\frac{N-2mp}{p}} \log \frac{1}{\varepsilon}, & \text{if } 0 \le |x| \le \varepsilon \\ |x|^{-\frac{N-2mp}{p}} \log \frac{1}{|x|}, & \text{if } \varepsilon \le |x| \le 1, \\ 0, & \text{if } x \in \Omega \setminus B_1(0). \end{cases}$$

Let $\alpha = \frac{N-2mp}{p}$. By using the formula

$$\Delta r^{-x} = x(x-N+2) r^{-x-2},$$
$$\Delta \left(r^{-x} \log \frac{1}{r} \right) = x(x-N+2) r^{-x-2} \log \frac{1}{r} + (2x-N+2) r^{-x-2},$$

we compute that

$$\Delta^m u_\varepsilon = \begin{cases} 0, & \text{if } 0 \le |x| \le \varepsilon, \\ A_m |x|^{-(\alpha+2m)} \log \frac{1}{|x|} + B_m |x|^{-(\alpha+2m)}, & \text{if } \varepsilon \le |x| \le 1, \\ 0, & \text{if } x \in \Omega \setminus B_1(0), \end{cases}$$

where A_m and B_m are determined by the iterative formula:

$$A_1 = \alpha(\alpha - N + 2),$$
$$A_{j+1} = (\alpha + 2j)(\alpha + 2(j+1) - N) A_j, \quad j = 1, 2, \dots,$$
$$B_1 = 2\alpha - N + 2,$$
$$B_{j+1} = (\alpha + 2j)(\alpha + 2(j+1) - N) B_j + 2\alpha + 2(2j+1) - N \quad j = 1, 2, \dots.$$

Thus we have

$$A_m = \prod_{j=0}^{m-1} (\alpha + 2j)(\alpha + 2(j+1) - N), \quad |A_m| = C_{2m,p}.$$

We compute

$$\int_\Omega |\Delta^m u_\varepsilon(x)|^p\,dx = \omega_N \int_\varepsilon^1 \left|A_m \log\frac{1}{r} + B_m\right|^p r^{-(\alpha+2m)p+N-1} dr$$

$$= \omega_N \left(\frac{1}{A_m}\right) \int_{B_m}^{B_m+A_m\log\frac{1}{\varepsilon}} |t|^p\,dt$$

$$= \omega_N \left(\frac{1}{A_m(p+1)}\right)\left(\left|B_m + A_m\log\frac{1}{\varepsilon}\right|^p (B_m + A_m\log\frac{1}{\varepsilon}) - |B_m|^p B_m\right). \tag{3.1}$$

On the other hand, we have

$$\int_\Omega \frac{|u_\varepsilon(x)|^p}{|x|^{2mp}}\,dx$$

$$= \omega_N\, \varepsilon^{-\alpha p}\left(\log\frac{1}{\varepsilon}\right)^p \int_0^\varepsilon r^{N-2mp-1}\,dr + \omega_N \int_\varepsilon^1 r^{-1}\left(\log\frac{1}{r}\right)^p\,dr$$

$$= \omega_N\,\frac{\varepsilon^{N-2mp}}{N-2mp}\left(\log\frac{1}{\varepsilon}\right)^p + \omega_N \int_0^{\log\frac{1}{\varepsilon}} t^p\,dt$$

$$= \omega_N\,\frac{\varepsilon^{N-2mp}}{N-2mp}\left(\log\frac{1}{\varepsilon}\right)^p + \omega_N \frac{1}{p+1}\left(\log\frac{1}{\varepsilon}\right)^{p+1}. \tag{3.2}$$

By (3.1), (3.2) and the fact $|A_m| = C_{2m,p}$, we obtain

$$\frac{\int_{B_1(0)} |\Delta^m u_\varepsilon(x)|^p\,dx}{\int_{B_1(0)} \frac{|u_\varepsilon(x)|^p}{|x|^{2mp}}\,dx} \to |A_m|^p = C_{2m,p}^p \text{ as } \varepsilon \to 0,$$

which implies the optimality of $C_{2m,p}^p$.

Next, in the odd case $k = 2m+1$, $m \in \mathbb{N}$, we consider the function $u_\varepsilon \in W_0^{2m+1,p}(B_1(0))$ as follows:

$$u_\varepsilon(x) = \begin{cases} \varepsilon^{-\frac{N-(2m+1)p}{p}} \log\frac{1}{\varepsilon}, & \text{if } 0 \le |x| \le \varepsilon,\\ |x|^{-\frac{N-(2m+1)p}{p}} \log\frac{1}{|x|}, & \text{if } \varepsilon \le |x| \le 1,\\ 0, & \text{if } x \in \Omega \setminus B_1(0). \end{cases}$$

Let $\beta = \frac{N-(2m+1)p}{p}$. Note that

$$\nabla(\Delta^m u_\varepsilon) = \begin{cases} 0, \quad \text{if } 0 \le |x| \le \varepsilon,\\ |x|^{-(\beta+2m-2)} x\left\{-A_m(\beta+2m)\log\frac{1}{|x|} - (A_m + (\beta+2m)B_m)\right\},\\ \qquad \text{if } \varepsilon \le |x| \le 1,\\ 0, \quad \text{if } x \in \Omega \setminus B_1(0). \end{cases}$$

If we make a calculation similar to the even case, we obtain

$$\frac{\int_\Omega |\nabla(\Delta^m u_\varepsilon)(x)|^p \, dx}{\int_\Omega \frac{|u_\varepsilon(x)|^p}{|x|^{(2m+1)p}} \, dx} \to |A_m|^p (\beta + 2m)^p \text{ as } \varepsilon \to 0,$$

which implies the optimality of $C^p_{2m+1,p}$ by $\beta + 2m = \frac{N-p}{p}$ and $C^p_{2m+1,p} = \left(\frac{N-p}{p}\right)^p C^p_{2m,p} = |A_m|^p(\beta + 2m)^p$. □

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