SOME IMPROVEMENTS FOR A CLASS OF THE CAFFARELLI-KOHN-NIRENBERG INEQUALITIES

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Abstract. In this paper, we concern a weighted version of the Hardy inequality, which is a special case of the more general Caffarelli-Kohn-Nirenberg inequalities. We improve the inequality on the whole space or on a bounded domain by adding various remainder terms. On the whole space, we show the existence of a remainder term which has the form of ratio of two weighted integrals. Also we give a simple derivation of the remainder term involving a distance from the manifold of the "virtual extremals". Finally on a bounded domain, we prove the existence of remainder terms involving the gradient of functions.

1. Introduction

In this paper, we are concerned with the weighted version of the Hardy inequality:

$$\int_\Omega |\nabla u|^p |x|^{-pa} dx \ge \left(\frac{N-p-pa}{p} \right)^p \int_\Omega \frac{|u|^p}{|x|^{p(a+1)}} dx \tag{1.1}$$

for all $u \in C_0^\infty(\Omega)$, where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \ge 3)$ with $0 \in \Omega$, or $\Omega = \mathbb{R}^N$, $1 < p < N$ and $-\infty < a < \frac{N-p}{p}$. Actually, much more general weighted type inequalities are shown by Caffarelli, Kohn and Nirenberg [4] and (1.1) is one of the special cases. Let $D_{0,a}^{1,p}(\Omega)$ and $W_{0,a}^{1,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to each norm

$$\|u\|_{D_{0,a}^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p |x|^{-pa} dx \right)^{1/p},$$

$$\|u\|_{W_{0,a}^{1,p}(\Omega)} = \left(\int_\Omega \left(|\nabla u|^p + |u|^p \right) |x|^{-pa} dx \right)^{1/p},$$

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respectively. Then (1.1) holds true for all $u \in D^{1,p}_{0,a}(\Omega)$. If Ω is bounded, the Poincaré type inequality implies that $D^{1,p}_{0,a}(\Omega) = W^{1,p}_{0,a}(\Omega)$. Also the constant $\left(\frac{N-p-pa}{p}\right)^p$ in (1.1) is known optimal and never attained in $D^{1,p}_{0,a}(\Omega)$.

When $a = 0$, (1.1) becomes the classical Hardy inequality

$$\int_\Omega |\nabla u|^p dx \ge \left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx, \tag{1.2}$$

again the equality in (1.2) is never achieved by any function in $D^{1,p}_{0,0}(\Omega)$. There are many papers up to now that treat the improvement of (1.2) when Ω is a smooth bounded domain (see [2], [3], [6], [8], [10], [11], [12], [17], [18], and references therein). On the other hand, when $\Omega = \mathbb{R}^N$, Ghoussoub and Moradifam [11] show that there is no strictly positive $V \in C^1((0,\infty))$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \ge \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} V(|x|)|u|^2 dx$$

holds for all $u \in W^{1,2}(\mathbb{R}^N)$. Therefore we cannot expect the same type of remainder terms as in the bounded domain case, one of the reasons of what is a lack of the Poincaré inequality in the whole space. Instead, Cianchi and Ferone [7] provided the following "non-standard" remainder term: Let $p^* = \frac{Np}{N-p}$ be the critical Sobolev exponent, $u_0(x) = |x|^{-\frac{N-p}{p}}$ for $x \in \mathbb{R}^N$, and define

$$d_p(u) = \inf_{c\in\mathbb{R}} \frac{\|u - cu_0\|_{L^{p^*,\infty}(\mathbb{R}^N)}}{\|u\|_{L^{p^*,p}(\mathbb{R}^N)}} \quad (1 < p < N).$$

Here $L^{\tau,\sigma}(\mathbb{R}^N)$ $(0 < \tau \le \infty, 1 \le \sigma \le \infty)$ is the Lorentz space with the norm

$$\|u\|_{L^{\tau,\sigma}(\mathbb{R}^N)} = \|s^{\frac{1}{\tau}-\frac{1}{\sigma}} u^*(\cdot)\|_{L^\sigma(0,\infty)},$$

where u^* denotes the (one-dimensional) decreasing rearrangement of u. Then in [7] it is shown that for any $1 < p < N$ there exists a constant $C = C(p, N)$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \ge \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \left(1 + C d_p(u)^{2p^*}\right)$$

holds for every real-valued weakly differentiable function u in $\mathbb{R}^N$ decaying to zero at infinity with $|\nabla u| \in L^p(\mathbb{R}^N)$. Recently, the authors of this paper have succeeded to obtain a remainder term for the classical Hardy inequality (1.2) on the whole space [16]. Our method consists of the well-known Brezis-Vázquez transformation [3] and the use of the Gagliardo-Nirenberg inequality.

Concerning the inequality (1.1), on the other hand, Wang and Willem [19] obtained the following improved version of (1.1) on a bounded domain $\Omega \subset \mathbb{R}^N$: Assume $\Omega \subset\subset B_R(0)$ for some $R > 0$. Then

$$\begin{aligned}&\int_\Omega |\nabla u|^2 |x|^{-2a} dx - \left(\frac{N-2-2a}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^{2(a+1)}} dx\\ &\geq C \int_\Omega \left(\log\left(\frac{R}{|x|}\right)\right)^{-2} |x|^{-2a} |\nabla u|^2 dx\end{aligned}$$

holds for all $u \in D^{1,2}_{0,a}(\Omega)$, where $C = C(\Omega, a)$ is a positive constant. Their method consists of the use of a conformal transformation introduced by Catrina and Wang [5], which transforms the problem on a bounded domain of $\mathbb{R}^N$ to that of a cylinder $\mathcal{C} = \mathbb{R} \times S^{N-1}$; see §4. Later, Abdellaoui, Colorado and Peral [1] obtained the following improvement of (1.1): Let $|\Omega|$ denote the volume of Ω. Then for all $1 < q < p$, there exists a positive constant $C = C(N, p, q, a, |\Omega|)$ such that the inequality

$$\begin{aligned}&\int_\Omega |\nabla u|^p |x|^{-pa} dx - \left(\frac{N-p-pa}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^{p(a+1)}} dx\\ &\geq C \left(\int_\Omega |x|^{-ra} |\nabla u|^q dx\right)^{p/q}\end{aligned}$$

holds true for all $u \in D^{1,p}_{0,a}(\Omega)$, where r is any number such that $q \leq r < \infty$ if $a \leq 0$ and $1 \leq r < p + \rho$ for some positive constant ρ when $a > 0$. Their method is based on the Picone type inequality for the operator $\operatorname{div}(|x|^{-pa}|\nabla u|^{p-2}\nabla u)$. Note that from the proof of [1], the constant $C(N, p, q, a, |\Omega|) \to 0$ as $|\Omega| \to \infty$.

In this paper, firstly we improve the inequality (1.1) when $\Omega = \mathbb{R}^N$ by adding a remainder term of the form of ratio of two weighted integrals to (1.1).

Theorem 1. *(A remainder term of the form of ratio on the whole space) Let $N \geq 3$, $2 \leq p < N$ and $-\infty < a < \frac{N-p}{p}$. For given $n \in \mathbb{N}$, $t \in (0, 1)$ and $\gamma < \min\{1-t, \frac{p-n}{p}\}$, set $\delta = n - N + \frac{n}{1-t-\gamma}\left(\gamma + \frac{N-p-pa}{p}\right)$.*

Then there exists a constant $C > 0$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left(\frac{N-p-pa}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx$$
$$\geq C \frac{\left(\int_{\mathbb{R}^N} |x|^\delta |u|^{\frac{n}{1-t-\gamma}} dx \right)^{\frac{p(1-t-\gamma)}{nt}}}{\left(\int_{\mathbb{R}^N} |x|^{-pa} |u|^p dx \right)^{\frac{1-t}{t}}} \tag{1.3}$$

holds for all radially symmetric function $u \in W_{0,a}^{1,p}(\mathbb{R}^N)$, $u \not\equiv 0$.

Remark 1. If we put $F_{A,B}(u) = \int_{\mathbb{R}^N} |u|^A |x|^B dx$ for constants A, B, and put $u_\lambda(x) = \lambda^C u(\lambda x)$ for $x \in \mathbb{R}^N$ and $C \in \mathbb{R}$, then a simple computation shows that $F_{A,B}(u_\lambda) = \lambda^{AC-B-N} F_{A,B}(u)$. The remainder term in Theorem 1 can be written as

$$R(u) = \frac{\left(\int_{\mathbb{R}^N} |u|^{\frac{n}{1-t-\gamma}} |x|^\delta dx \right)^{\frac{p(1-t-\gamma)}{nt}}}{\left(\int_{\mathbb{R}^N} |u|^p |x|^{-pa} dx \right)^{\frac{1-t}{t}}} = \frac{\left\{ F_{\frac{n}{1-t-\gamma},\delta}(u) \right\}^{\frac{p(1-t-\gamma)}{nt}}}{\left\{ F_{p,-pa}(u) \right\}^{\frac{1-t}{t}}},$$

therefore it satisfies

$$R(u_\lambda) = \frac{\left\{ F_{\frac{n}{1-t-\gamma},\delta}(u_\lambda) \right\}^{\frac{p(1-t-\gamma)}{nt}}}{\left\{ F_{p,-pa}(u_\lambda) \right\}^{\frac{1-t}{t}}} = \frac{\left\{ \lambda^{(\frac{n}{1-t-\gamma})C-\delta-N} F_{\frac{n}{1-t-\gamma},\delta}(u) \right\}^{\frac{p(1-t-\gamma)}{nt}}}{\left\{ \lambda^{pC+pa-N} F_{p,-pa}(u) \right\}^{\frac{1-t}{t}}}.$$

If we put $C = \frac{N-p-pa}{p}$, we see

$$R(u_\lambda) = \frac{\left\{ \lambda^{(\frac{n}{1-t-\gamma})(\frac{N-p-pa}{p})-\delta-N} \right\}^{\frac{p(1-t-\gamma)}{nt}}}{\left\{ \lambda^{p(\frac{N-p-pa}{p})+pa-N} \right\}^{\frac{1-t}{t}}} R(u) = \left(\frac{\lambda^{-p(\frac{1-t}{t})}}{\lambda^{-p(\frac{1-t}{t})}} \right) R(u)$$
$$= R(u).$$

Thus the remainder term is invariant under the scaling $u(x) \mapsto u_\lambda(x) = \lambda^{\frac{N-p-pa}{p}} u(\lambda x)$.

Remark 2. If we choose n, t, γ satisfying $\frac{n}{1-t-\gamma} = p$, then (1.3) in Theorem 1 also holds even for non-radial functions. Indeed, for a non-radial function u, let us consider the radial function

$$U(r) = \left(\omega_N^{-1} \int_{S^{N-1}} |u(r\omega)|^p dS_\omega \right)^{\frac{1}{p}}.$$

Then Hölder's inequality implies that

$$U'(r) \leq \left(\omega_N^{-1} \int_{S^{N-1}} |\partial_r u(r\omega)|^p dS_\omega \right)^{\frac{1}{p}},$$

so we obtain

$$\omega_N \int_0^\infty |U'(r)|^p r^{N-1-pa}\, dr \le \int_{\mathbb{R}^N} \left| \nabla u \cdot \frac{x}{|x|} \right|^p |x|^{-pa} dx. \tag{1.4}$$

Also we have

$$\int_{\mathbb{R}^N} |U(|x|)|^p |x|^A dx = \int_{\mathbb{R}^N} |u|^p |x|^A dx \tag{1.5}$$

for any $A \in \mathbb{R}$. Thus when $\frac{n}{1-t-\gamma} = p$, (1.3) for U, (1.4) and (1.5) imply that the same inequality holds for all non-radial functions.

Note that the standard rearrangement argument, see for example [13], is not applicable because of the presence of weights, since a may be negative and $|x|^{-p(a+1)}$ may be an increasing function. Thus, differently from [16], here we use another type of pointwise estimate for the expansion of $|a-b|^p$, see Lemma 1. The use of full version of the Caffarelli-Kohn-Nirenberg inequality, see Proposition 1, is another key point. Since the technique used here is different from that of [16], the remainder term obtained in Theorem 1 is also different from that in [16] even when $a = 0$.

In §3, we improve (1.1) on the whole space by adding a remainder term which involves a distance from "the manifold of the virtual extremals" $\{cu_a \,|\, c \in \mathbb{R}\}$, where $u_a(x) = |x|^{-\frac{N-p-pa}{p}}$. For the proof, differently from that of Cianchi-Ferone [7], we use a new inequality recently obtained by Machihara, Ozawa and Wadade [14]. It is surprising for the authors that a direct use of the Machihara-Ozawa-Wadade inequality leads to the existence of "non-standard" type remainder terms for (1.1) very simply, at least in the radially symmetric case.

In §4, we will show some improvements of (1.1) when $p = 2$:

$$\int_\Omega |\nabla u|^2 |x|^{-2a} dx \ge \left(\frac{N-2-2a}{2} \right)^2 \int_\Omega \frac{|u|^2}{|x|^{2(a+1)}} dx,$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \ge 3$). In this part, the method of the proof is to combine two ideas: One is to transform the problem to the one on a cylinder, which was initiated by Catrina and Wang [5] in this context, and the another is to improve the one-dimensional Hardy (weighted Poincaré) inequality by the method of Picone's identity see [11], [12] for $p = 2$ case and [1] for $p \neq 2$ case.

2. Proof of Theorem 1.

In this section, we prove Theorem 1. For the proof, we need the following lemma.

Lemma 1 ([9])**.** *Let $p \geq 2$ and a, b be real numbers. Then there exists $c_p > 0$ such that*

$$|a-b|^p \geq |a|^p - p|a|^{p-2}ab + c_p|b|^p$$

holds true. c_p is explicitly given as $c_p = \min_{0<t\leq 1/2}\left((1-t)^p - t^p + pt^{p-1}\right)$ and sharp in this inequality.

We recall here the following inequality obtained by Caffarelli, Kohn and Nirenberg [4] in its full version:

Proposition 1. *(Caffarelli-Kohn-Nirenberg [4]) Let $n \in \mathbb{N}$ and let $p, q, r, \alpha, \beta, \sigma$ and t be real constants such that $p, q \geq 1$, $r > 0$, $0 \leq t \leq 1$, and*

$$\frac{1}{p} + \frac{\alpha}{n}, \quad \frac{1}{q} + \frac{\beta}{n}, \quad \frac{1}{r} + \frac{\gamma}{n} > 0$$

where $\gamma = t\sigma + (1-t)\beta$. Then there exists a positive constant C such that

$$\left\| |x|^\gamma u \right\|_{L^r(\mathbb{R}^n)} \leq C \left\| |x|^\alpha |\nabla u| \right\|^t_{L^p(\mathbb{R}^n)} \left\| |x|^\beta u \right\|^{1-t}_{L^q(\mathbb{R}^n)} \tag{2.1}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$, if and only if the following conditions hold:

(1) *(balance of dimension)*

$$\frac{1}{r} + \frac{\gamma}{n} = t\left(\frac{1}{p} + \frac{\alpha - 1}{n}\right) + (1-t)\left(\frac{1}{q} + \frac{\beta}{n}\right),$$

(2) $0 \leq \alpha - \sigma$ *if* $t > 0$,
(3) $1 \geq \alpha - \sigma$ *if* $t > 0$ *and* $\frac{1}{r} + \frac{\gamma}{n} = \frac{1}{p} + \frac{\alpha-1}{n}$.

Proof of Theorem 1. We show Theorem 1 for a radial function $u \in C_0^\infty(\mathbb{R}^N)$. Then a density argument implies the desired result. Since u is radial, u can be written as $u(x) = \tilde{u}(|x|)$ for some function $\tilde{u} \in C_0^\infty([0, +\infty))$. Appealing to Brezis-Vázquez's idea, we put

$$\tilde{v}(r) = r^{\frac{N-p-pa}{p}} \tilde{u}(r). \tag{2.2}$$

We see $\tilde{v}(0) = 0$ since $a < \frac{N-p}{p}$ and $\tilde{v}(+\infty) = 0$, since $\tilde{u} \equiv 0$ near $r = +\infty$. Put $v(y) = \tilde{v}(|y|)$ for $y \in \mathbb{R}^n$, $n \in \mathbb{N}$. Calculation shows that

$$\begin{aligned}
J &:= \int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left(\frac{N-p-pa}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx \\
&= \omega_N \int_0^\infty |\tilde{u}'(r)|^p r^{-pa} r^{N-1} dr - \omega_N \left(\frac{N-p-pa}{p}\right)^p \int_0^\infty |\tilde{u}(r)|^p r^{-p(a+1)} r^{N-1} dr \\
&= \omega_N \int_0^\infty \left| \left(\frac{N-p-pa}{p}\right) r^{-\frac{N-pa}{p}} \tilde{v}(r) - r^{-\frac{N-p-pa}{p}} \tilde{v}'(r) \right|^p r^{N-1-pa} dr \\
&- \omega_N \left(\frac{N-p-pa}{p}\right)^p \int_0^\infty |\tilde{v}(r)|^p r^{-1} dr.
\end{aligned}$$

We apply Lemma 1 to the integrand of the first term:

$$\begin{aligned}
&\left| \left(\frac{N-p-pa}{p}\right) r^{-\frac{N-pa}{p}} \tilde{v}(r) - r^{-\frac{N-p-pa}{p}} \tilde{v}'(r) \right|^p r^{N-1-pa} \\
&\geq \Bigg[\left(\frac{N-p-pa}{p}\right)^p r^{-N+pa} |\tilde{v}(r)|^p \\
&- p \left(\frac{N-p-pa}{p}\right)^{p-1} |\tilde{v}(r)|^{p-2} \tilde{v}(r) \tilde{v}'(r) r^{-\left(\frac{N-pa}{p}\right)(p-1)} r^{-\left(\frac{N-p-pa}{p}\right)} \\
&+ c_p |\tilde{v}'(r)|^p r^{-N+p+pa} \Bigg] r^{N-1-pa} \\
&= \left(\frac{N-p-pa}{p}\right)^p r^{-1} |\tilde{v}(r)|^p - p \left(\frac{N-p-pa}{p}\right)^{p-1} |\tilde{v}(r)|^{p-2} \tilde{v}(r) \tilde{v}'(r) \\
&+ c_p |\tilde{v}'(r)|^p r^{p-1}.
\end{aligned}$$

By using the fact $\tilde{v}(0) = \tilde{v}(+\infty) = 0$ and $p \geq 2$, we see

$$p \int_0^\infty |\tilde{v}(r)|^{p-2} \tilde{v}(r) \tilde{v}'(r)\, dr = \int_0^\infty \frac{d}{dr} \left(|\tilde{v}(r)|^p \right) dr = 0.$$

Finally, we note that the terms involving $\int_0^\infty |\tilde{v}(r)|^p r^{-1} dr$ cancel out by subtracting each other. Thus we obtain

$$J \geq c_p \omega_N \int_0^\infty |\tilde{v}'(r)|^p \, r^{p-1}\, dr = c_p \frac{\omega_N}{\omega_n} \int_{\mathbb{R}^n} |\nabla v(y)|^p |y|^{p-n} dy. \tag{2.3}$$

From now on, we estimate the right hand side of (2.3) by using the Caffarelli-Kohn-Nirenberg inequality (2.1) (Proposition 1) for v on $\mathbb{R}^n$. We take

$$q = p \quad \text{and} \quad \alpha = \beta = \frac{p-n}{p}.$$

By these choices, we see

$$\frac{1}{p} + \frac{\alpha}{n} = \frac{1}{q} + \frac{\beta}{n} = \frac{1}{n} > 0.$$

The first condition of the Proposition 1 (balance of dimension) reduces to

$$\frac{1}{r} + \frac{\gamma}{n} = \frac{1-t}{n}, \quad \text{i.e.,} \quad r = \frac{n}{1-t-\gamma}.$$

Thus $\frac{1}{r} + \frac{\gamma}{n} > 0$ if $t < 1$, and by the assumption $\gamma < \min\{1-t, \frac{p-n}{p}\}$, r is positive and the condition $\alpha \geq \sigma = \frac{\gamma-(1-t)\beta}{t}$ when $t > 0$ is fulfilled. Also under the choice $0 < t < 1$, we see $0 = \frac{1}{p} + \frac{\alpha-1}{n} \neq \frac{1}{r} + \frac{\gamma}{n} = \frac{1-t}{n}$, thus we do not need to consider the third condition. In conclusion, we assure that the following inequality holds true for v:

$$\left\| |y|^{\gamma} v \right\|_{L^{\frac{n}{1-t-\gamma}}(\mathbb{R}^n)} \leq C \left\| |y|^{\frac{p-n}{p}} |\nabla v| \right\|_{L^p(\mathbb{R}^n)}^{t} \left\| |y|^{\frac{p-n}{p}} v \right\|_{L^p(\mathbb{R}^n)}^{1-t},$$

that is,

$$\left(\int_{\mathbb{R}^n} |y|^{\frac{n\gamma}{1-t-\gamma}} |v|^{\frac{n}{1-t-\gamma}} dy \right)^{\frac{p(1-t-\gamma)}{nt}} \leq C \left(\int_{\mathbb{R}^n} |y|^{p-n} |\nabla v|^p dy \right) \left(\int_{\mathbb{R}^n} |y|^{p-n} |v|^p dy \right)^{\frac{1-t}{t}}.$$

Combining this to (2.3), we have

$$J \geq C' \frac{\left(\int_{\mathbb{R}^n} |y|^{\frac{n\gamma}{1-t-\gamma}} |v|^{\frac{n}{1-t-\gamma}} dy \right)^{\frac{p(1-t-\gamma)}{nt}}}{\left(\int_{\mathbb{R}^n} |y|^{p-n} |v|^p dy \right)^{\frac{1-t}{t}}} \tag{2.4}$$

where $C' = C^{-1} c_p \frac{\omega_N}{\omega_n}$. By the definition $v(y) = \tilde{v}(|y|)$, $y \in \mathbb{R}^n$ and the assumption that $u \in W_{0,a}^{1,p}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^n} |y|^{p-n} |v|^p dy = \omega_n \int_0^\infty r^{N-p-pa} |\tilde{u}(r)|^p r^{p-n} r^{n-1} dr$$
$$= \frac{\omega_n}{\omega_N} \int_{\mathbb{R}^N} |u|^p |x|^{-pa} dx < \infty.$$

On the other hand,

$$\int_{\mathbb{R}^n} |y|^{\frac{n\gamma}{1-t-\gamma}} |v|^{\frac{n}{1-t-\gamma}} dy = \omega_n \int_0^\infty \left(r^{\frac{N-p-pa}{p}} |\tilde{u}(r)| \right)^{\frac{n}{1-t-\gamma}} r^{\frac{n\gamma}{1-t-\gamma}} r^{n-1} dr$$
$$= \omega_n \int_0^\infty |\tilde{u}(r)|^{\frac{n}{1-t-\gamma}} r^{\frac{n}{1-t-\gamma}\left(\gamma + \frac{N-p-pa}{p}\right) + n - N} r^{N-1} dr$$
$$= \frac{\omega_n}{\omega_N} \int_{\mathbb{R}^N} |u|^{\frac{n}{1-t-\gamma}} |x|^{\delta} dx.$$

Inserting these into (2.4), we obtain the desired conclusion. $\square$

Remark 3. If we assume that the function $u \in W^{1,p}_{0,a}(\mathbb{R}^N)$ is nonnegative, radially symmetric and radially decreasing, then we can apply the same argument in [16] also in our situation. In this case we obtain the following theorem, the proof of it is exactly the same as in [16].

Theorem 2. *For given $N \geq 3$, $2 \leq p < N$ and $q > 2$, set $\alpha = \alpha(p,q,N) = 2 - N + \frac{q(N-p-pa)}{2}$. Then there exists $D = D(p,q,N) > 0$ such that the inequality*

$$\int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left(\frac{N-p-pa}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx$$
$$\geq D \left(\frac{\int_{\mathbb{R}^N} |u|^{\frac{pq}{2}} |x|^{\alpha} dx}{\int_{\mathbb{R}^N} |u|^p |x|^{2-p-pa} dx} \right)^{\frac{2}{q-2}}$$

holds for all nonnegative, radially symmetric and radially decreasing function $u \in W^{1,p}_{0,a}(\mathbb{R}^N)$, $u \not\equiv 0$.

3. A remainder term involving a distance from the virtual extremals

For $-\infty < a < \frac{N-p}{p}$, let $u_a(x) = |x|^{-\frac{N-p-pa}{p}}$. Note that u_a is a solution to the Euler-Lagrange equation associated with the best constant of the inequality (1.1)

$$-\mathrm{div}\left(|x|^{-pa}|\nabla u|^{p-2}\nabla u\right) = \left(\frac{N-p-pa}{p}\right)^p |x|^{-p(a+1)} u^{p-1},$$
$$u \geq 0 \quad \text{in } \mathbb{R}^N,$$

however, $u_a \notin D^{1,p}_{0,a}(B_R(0))$ for any $R > 0$. Thus u_a is not a genuine minimizer for the best constant of (1.1) in the admissible class $D^{1,p}_{0,a}(\mathbb{R}^N)$, but just approximates the non-existing extremals on the whole space.

In this section, we prove an improved version of (1.1), which involves a sort of the "distance" of the associated function u from the one-dimensional space of "virtual extremals" $\{cu_a \mid c \in \mathbb{R}\}$. For the (sub-critical, also the critical) Hardy, or higher order Hardy-Rellich inequalities, see [15].

For $R > 0$, let us define

$$d_R(f,g) = \left(\int_{\mathbb{R}^N} \frac{|f(x) - g(x)|^p}{\left|\log \frac{R}{|x|}\right|^p |x|^{p(a+1)}} dx \right)^{1/p} \tag{3.1}$$

for functions f, g, for which the right hand side is finite.

Theorem 3. *Let $N \geq 3$, $2 \leq p < N$ and assume $-\infty < a < \frac{N-p}{p}$. Then*

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left(\frac{N-p-pa}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx \\ &\geq c_p \left(\frac{p-1}{p} \right)^p \sup_{R>0} \inf_{c \in \mathbb{R}} d_R(u, cu_a)^p \end{aligned} \tag{3.2}$$

holds for any radially symmetric function $u \in D^{1,p}_{0,a}(\mathbb{R}^N)$. Here c_p is a constant in Lemma 1 and $d_R(\cdot, \ \cdot)$ is defined in (3.1).

For the proof, we need the following result.

Proposition 2. *(Machihara-Ozawa-Wadade* [14]*:Theorem 1.1) Let $N \in \mathbb{N}$, $1 < \alpha < \infty$ and $\max\{1, \alpha - 1\} < \beta < \infty$. Then for any $R > 0$, the inequality*

$$\int_{\mathbb{R}^N} \frac{|f(x) - f(R\frac{x}{|x|})|^\beta}{\left|\log \frac{R}{|x|}\right|^\alpha |x|^N} dx \leq \left(\frac{\beta}{\alpha - 1} \right)^\beta \int_{\mathbb{R}^N} \frac{\left| \frac{x}{|x|} \cdot \nabla f(x) \right|^\beta}{|x|^{N-\beta} \left|\log \frac{R}{|x|}\right|^{\alpha - \beta}} dx \tag{3.3}$$

holds for all $f \in W^1 L_{N,\beta,\frac{\beta-\alpha}{\beta}}(\mathbb{R}^N)$. Also the constant $\left(\frac{\beta}{\alpha-1}\right)^\beta$ is best possible in (3.3).

Remark 4. Here, $W^1 L_{p,q,\lambda}(\mathbb{R}^N)$ denotes the Sobolev-Lorentz-Zygmund spaces. For the precise definition of these spaces, we refer to [14]. However, we note that if $\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) |x|^{p-N} dx < \infty$, then the function $u \in W^1 L_{N,p,0}(\mathbb{R}^N)$.

Proof of Theorem 3. First, we prove Theorem for a radial function $u \in C_0^\infty(\mathbb{R}^N)$. Let $u \in C_0^\infty(\mathbb{R}^N)$, $u(x) = \tilde{u}(r), r = |x|$ be a radial function. Define $v(x) = \tilde{v}(|x|)$ for $x \in \mathbb{R}^N$ where $\tilde{v}(r)$ is defined in (2.2). As in the proof of Theorem 1, we obtain

$$\begin{aligned} J(u) &= \int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left(\frac{N-p-pa}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx \\ &\geq c_p \omega_N \int_0^\infty |\tilde{v}_r'|^p r^{p-1} dr = c_p \int_{\mathbb{R}^N} |\nabla v|^p |x|^{p-N} dx, \end{aligned} \tag{3.4}$$

since we assume $p \geq 2$, see (2.3). Here we claim that if $u \in C_0^\infty(\mathbb{R}^N) \subset W^{1,p}_{0,a}(\mathbb{R}^N)$, v satisfies

$$\int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p) |x|^{p-N} dx < \infty.$$

In particular, $v \in W^1 L_{N,p,0}(\mathbb{R}^N)$ by the above remark. Indeed, from (3.4), we have $\int_{\mathbb{R}^N} |\nabla v|^p |x|^{p-N} dx \le J(u)/c_p < \infty$. Also by the definition of v, we see $\int_{\mathbb{R}^N} |x|^{p-N}|v|^p dx = \int_{\mathbb{R}^N} |x|^{-pa}|u|^p dx < \infty$. Thus we have obtained the claim.

By the claim, we can apply Proposition 2 to $v \in W^1 L_{N,p,0}(\mathbb{R}^N)$. Then we derive

$$
\begin{aligned}
J(u) &\ge c_p \int_{\mathbb{R}^N} |\nabla v|^p |x|^{p-N} dx \ge c_p \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^N} \frac{\left| v(x) - v(\frac{Rx}{|x|})\right|^p}{\left|\log \frac{R}{|x|}\right|^p |x|^N} dx \\
&= c_p \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^N} \frac{\left| |x|^{\frac{N-p-pa}{p}} u(x) - R^{\frac{N-p-pa}{p}} u(\frac{Rx}{|x|})\right|^p}{\left|\log \frac{R}{|x|}\right|^p |x|^N} dx \\
&= c_p \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^N} \frac{\left| u(x) - R^{\frac{N-p-pa}{p}} \tilde{u}(R) |x|^{-\frac{N-p-pa}{p}}\right|^p}{\left|\log \frac{R}{|x|}\right|^p |x|^{p(a+1)}} dx \qquad (3.5) \\
&\ge c_p \left(\frac{p-1}{p}\right)^p \inf_{c\in\mathbb{R}} \int_{\mathbb{R}^N} \frac{\left| u(x) - c|x|^{-\frac{N-p-pa}{p}}\right|^p}{\left|\log \frac{R}{|x|}\right|^p |x|^{p(a+1)}} dx
\end{aligned}
$$

for any $R > 0$, here we have used (3.4) in the first inequality, (3.3) with $f = v$ and $\alpha = \beta = p$ in the second inequality, the definition of v in the first equality, and the fact that u is radially symmetric in the second equality. This proves Theorem for a radial function $u \in C_0^\infty(\mathbb{R}^N)$.

Next, we prove Theorem for a radial function $u \in D^{1,p}_{0,a}(\mathbb{R}^N)$. Here we follow an argument by Machihara, Ozawa and Wadade [14]. Let $\{u_m\}_{m=1}^\infty \subset C_0^\infty(\mathbb{R}^N)$ be a sequence of radially symmetric functions such that $u_m \to u$ in $\mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N)$ as $m \to \infty$. Then there exists a subsequence $\{u_{m_j}\}_{j=1}^\infty$ such that

$$
\frac{u_{m_j}}{|x|^{a+1}} \to \frac{u}{|x|^{a+1}} \quad \text{in } L^p(\mathbb{R}^N),
$$
$$
u_{m_j} \to u \quad \text{a.e. in } \mathbb{R}^N
$$

by (1.1). Now, define

$$
w_R(x) = \frac{w(x) - R^{\frac{N-p-pa}{p}} \tilde{w}(R) |x|^{-\frac{N-p-pa}{p}}}{|x|^{a+1} |\log \frac{R}{|x|}|}
$$

for a radial function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $w(x) = \tilde{w}(|x|)$, and $R > 0$. Since the inequality (3.5) holds for $u_{m_j} - u_{m_k} \in C_0^\infty(\mathbb{R}^N)$, we observe that $\{(u_{m_j})_R\}_{j=1}^\infty$ is a Cauchy sequence in $L^p(\mathbb{R}^N)$. Since $(u_{m_j})_R \to u_R$ a.e.

in $\mathbb{R}^N$, we easily see that $(u_{m_j})_R \to u_R$ in $L^p(\mathbb{R}^N)$ as $j \to \infty$. Therefore, for any $R > 0$, we have

$$\begin{aligned} J(u) &= \lim_{j\to\infty} J(u_{m_j}) \\ &\geq c_p \left(\frac{p-1}{p}\right)^p \lim_{j\to\infty} \int_{\mathbb{R}^N} \frac{|u_{m_j}(x) - R^{\frac{N-p-pa}{p}} \widetilde{u_{m_j}}(R)|x|^{-\frac{N-p-pa}{p}}|^p}{\left|\log \frac{R}{|x|}\right|^p |x|^{p(a+1)}} dx \\ &= c_p \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u(x) - R^{\frac{N-p-pa}{p}} \tilde{u}(R)|x|^{-\frac{N-p-pa}{p}}|^p}{\left|\log \frac{R}{|x|}\right|^p |x|^{p(a+1)}} dx \end{aligned}$$

for all radial functions $u \in \mathcal{D}^{1,p}_{0,a}(\mathbb{R}^N)$, here we have used (3.5) for $u_{m_j} \in C_0^\infty(\mathbb{R}^N)$. As before, this ends the proof. □

4. Improved Caffarelli-Kohn-Nirenberg type inequalities on a bounded domain

In this section, we revisit the idea by Catrina and Wang [5], Wang-Willem [19] to improve the inequality

$$\int_\Omega |\nabla u|^2 |x|^{-2a} dx \geq \left(\frac{N-2-2a}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^{2(a+1)}} dx \tag{4.1}$$

on a smooth bounded domain Ω in $\mathbb{R}^N$ $(N \geq 3)$ with $0 \in \Omega$. Here u is a function in $C_0^\infty(\Omega \backslash \{0\})$ and $-\infty < a < \frac{N-2}{2}$ is always assumed. We can check that under the assumption $a < \frac{N-2}{2}$, $D^{1,2}_{0,a}(\Omega)$ is identical to the completion of $C_0^\infty(\Omega\backslash\{0\})$ with respect to the norm $\|\cdot\|_{D^{1,2}_{0,a}(\Omega)}$. The idea of Catrina and Wang consists of the use of a conformal transformation which converts a problem to an equivalent one in a domain on a cylinder $\mathcal{C} = \mathbb{R} \times S^{N-1}$. More precisely, for a function $u \in C_0^\infty(\Omega \setminus \{0\})$, let us associate $v \in C_0^\infty(\tilde{\Omega})$ by the transformation

$$u(x) = |x|^{-\frac{N-2-2a}{2}} v\left(-\log|x|, \frac{x}{|x|}\right), \tag{4.2}$$

where $\tilde{\Omega}$ is a domain on the cylinder $\mathcal{C}$ defined as

$$\tilde{\Omega} = \left\{ (t,\theta) \in \mathbb{R} \times S^{N-1} \ : \ t = -\log|x|, \theta = \frac{x}{|x|}, x \in \Omega \right\}.$$

Catrina and Wang proved in [5] that when $\Omega = \mathbb{R}^N$ and $a < \frac{N-2}{2}$, the transformation (4.2) provides an isomorphism between two Hilbert spaces $D^{1,2}_{0,a}(\mathbb{R}^N)$ and $H^1(\mathcal{C})$, the inner product of the latter is given by

$$(v,w)_{H^1(\mathcal{C})} = \int_{\mathcal{C}} \left\{ \nabla v \cdot \nabla w + \left(\frac{N-2-2a}{2} \right)^2 vw \right\} d\mu.$$

Here $|\nabla v|^2 = v_t^2 + |\nabla_\theta v|^2$ and $d\mu = dtdS_\theta = |x|^{-N}dx$ are the length of the gradient vector and the volume element on $\mathcal{C}$. By a direct computation, we see

$$\nabla u(x) = -|x|^{-\frac{N-2a}{2}} \left[\left\{ v_t(t,\theta) + \left(\frac{N-2-2a}{2} \right) v \right\} \theta + \nabla_\theta v(t,\theta) \right],$$

and since $\langle \theta, \nabla_\theta v \rangle \equiv 0$, it holds

$$|\nabla u(x)|^2 = |x|^{-N+2a} \left[\left\{ v_t(t,\theta) + \left(\frac{N-2-2a}{2} \right) v \right\}^2 + |\nabla_\theta v(t,\theta)|^2 \right]. \tag{4.3}$$

Furthermore, we have

$$\int_{\tilde{\Omega}} |\nabla v|^2 d\mu = \int_\Omega |\nabla u|^2 |x|^{-2a} dx - \left(\frac{N-2-2a}{2} \right)^2 \int_\Omega \frac{u^2}{|x|^{2(a+1)}} dx, \tag{4.4}$$

$$\int_{\tilde{\Omega}} v^2 d\mu = \int_\Omega \frac{u^2}{|x|^{2(a+1)}} dx. \tag{4.5}$$

On the other hand, Wang and Willem [19] proved the weighted Poincaré inequality of the form

$$\int_{\tilde{\Omega}} |\nabla v|^2 d\mu \geq \frac{1}{4} \int_{\tilde{\Omega}} \frac{v^2}{t^2} d\mu$$

for $v \in C_0^\infty(\tilde{\Omega})$. Using this inequality and the transformation (4.2), they obtained the improved inequality mentioned in the Introduction, See [5], [19].

Following their arguments, we prove the next theorem.

Theorem 4. *Let $V = V(t)$ is nonnegative, monotone decreasing function on $t \in (0,+\infty)$ and assume that there exists a strictly positive function $\phi = \phi(t) \in C^2((0,+\infty))$ such that $-\phi''(t) \geq V(t)\phi(t)$ holds on $(0,+\infty)$. Let Ω be a domain in $\mathbb{R}^N$ such that $\Omega \subset\subset B_R(0)$ for some*

$R > 0$. *Then there exists a constant* $C = C(a, \Omega, V) > 0$ *such that*

$$\int_\Omega |\nabla u|^2 |x|^{-2a} dx - \left(\frac{N-2-2a}{2} \right)^2 \int_\Omega \frac{|u|^2}{|x|^{2(a+1)}} dx$$

$$\geq C \int_\Omega V \left(\log \left(\frac{R}{|x|} \right) \right) |x|^{-2a} |\nabla u|^2 dx \tag{4.6}$$

holds for all $u \in D^{1,2}_{0,a}(\Omega)$.

For example, $V(t) = \frac{1}{4t^2}$ satisfies the assumption of Theorem 4 with $\phi(t) = t^{1/2}$, $t \in (0, +\infty)$.

As a corollary, we obtain the following improved inequalities.

Corollary 1. *Let* Ω *be a domain in* $\mathbb{R}^N$ *such that* $\Omega \subset\subset B_R(0)$ *for some* $R > 0$. *Assume* $-\infty < a < \frac{N-2}{2}$. *Then for any integer* $k \in \mathbb{N}$, *there exists a constant* $C = C(a, \Omega, k) > 0$ *such that the followings hold.*

(i) Define the functions $X_i = X_i(s)$ *for* $s \in (0, 1]$ *iteratively as* $X_1(s) = (1 - \log s)^{-1}$, $X_i(s) = X_1(X_{i-1}(s))$ *for* $i \geq 2$. *Then*

$$\int_\Omega |\nabla u|^2 |x|^{-2a} dx - \left(\frac{N-2-2a}{2} \right)^2 \int_\Omega \frac{|u|^2}{|x|^{2(a+1)}} dx$$

$$\geq C \sum_{i=1}^{k} \int_\Omega \frac{1}{|x|^{2a}} X_1^2 \left(\frac{|x|}{R} \right) X_2^2 \left(\frac{|x|}{R} \right) \cdots X_i^2 \left(\frac{|x|}{R} \right) |\nabla u|^2 dx$$

holds for all $u \in D^{1,2}_{0,a}(\Omega)$.

(ii) Let $e_{-1} = 0$, $e_0 = 1$, $e_i = e^{e_{i-1}}$ *for* $i \geq 1$, *and define functions iteratively as* $\log^{(1)} s = \log s$, $\log^{(i)}(s) = \log^{(1)} \left(\log^{(i-1)} s \right)$ *for* $i \geq 2$, *those are well defined when* $s > e_{i-2}$ *and positive when* $s > e_{i-1}$. *For any integer* $k \in \mathbb{N}$, *take* $\rho \geq Re_{k-1}$. *Then there exists a constant* $C = C(a, \Omega, k) > 0$ *such that*

$$\int_\Omega |\nabla u|^2 |x|^{-2a} dx - \left(\frac{N-2-2a}{2} \right)^2 \int_\Omega \frac{|u|^2}{|x|^{2(a+1)}} dx$$

$$\geq C \sum_{i=1}^{k} \int_\Omega \frac{1}{|x|^{2a}} \left(\log^{(1)} \frac{\rho}{|x|} \right)^{-2} \left(\log^{(2)} \frac{\rho}{|x|} \right)^{-2} \cdots \left(\log^{(i)} \frac{\rho}{|x|} \right)^{-2} |\nabla u|^2 dx$$

holds for all $u \in D^{1,2}_{0,a}(\Omega)$.

For the proof of Theorem 4, first we show a lemma.

Lemma 2. *(The weighted Poincaré inequality on a cylinder) Let* $\tilde{\Omega} \subset \mathcal{C}_+$ *be a bounded domain where* $\mathcal{C}_+ = \{(t, \theta) \in \mathbb{R} \times S^{N-1} : t > 0\}$. *Let*

$V = V(t)$ satisfy the assumptions in Theorem 4. Then it holds

$$\int_{\tilde{\Omega}} |\nabla v|^2 d\mu \geq \int_{\tilde{\Omega}} V(t) v^2 d\mu \tag{4.7}$$

for all $v \in C_0^\infty(\tilde{\Omega})$.

Proof. Put $\psi(t,\theta) = v(t,\theta)/\phi(t)$ where $\phi(t) > 0$, $\phi = \phi(t) \in C^2((0,+\infty))$ such that $-\phi''(t) \geq V(t)\phi(t)$ on $(0,+\infty)$. Then $\psi \equiv 0$ near $t = 0$ and $t = \infty$, since $v \in C_0^\infty(\tilde{\Omega})$. Thus

$$\begin{aligned}
&\int_0^\infty v_t^2(t,\theta)dt = \int_0^\infty (\phi_t\psi + \phi\psi_t)^2\, dt = \int_0^\infty \left(\phi_t^2\psi^2 + \phi^2\psi_t^2 + \phi\phi_t(\psi^2)_t\right) dt \\
&= \int_0^\infty \left(\phi_t^2\psi^2 + \phi^2\psi_t^2\right) dt + \left[\psi^2\phi\phi_t\right]_0^\infty - \int_0^\infty \psi^2(\phi\phi_t)_t dt \\
&= \int_0^\infty \phi^2\psi_t^2 dt - \int_0^\infty \psi^2\phi\phi_{tt} dt \geq -\int_0^\infty \psi^2\phi\phi_{tt} dt \\
&= \int_0^\infty v^2\left(-\frac{\phi''(t)}{\phi(t)}\right) dt \geq \int_0^\infty V(t)v^2 dt.
\end{aligned}$$

Integrating both sides on S^{N-1} with respect to dS_θ and adding $\int_{\tilde{\Omega}} |\nabla_\theta v|^2 dt dS_\theta$ to the right hand side, we obtain (4.7). □

The inequality (4.7) also holds for $v \in H_0^1(\tilde{\Omega})$ by a density argument.

Proof of Theorem 4. By using a scaling $x \mapsto x/R$, it is enough to prove Theorem when $R = 1$. In this case, $\tilde{\Omega} \subset \mathcal{C}_+$ where $\mathcal{C}_+ = \{(t,\theta) \in \mathbb{R} \times S^{N-1} : t > 0\}$. Now, we will prove

$$\int_{\tilde{\Omega}} |\nabla v|^2 d\mu \geq C \int_{\tilde{\Omega}} V(t) \left[|\nabla_\theta v|^2 + \left\{ v_t(t,\theta) + \left(\frac{N-2-2a}{2}\right) v \right\}^2 \right] d\mu \tag{4.8}$$

for some $C > 0$ independent of v. Indeed, since $r_0 = \sup_{x\in\Omega} |x| < 1$, we have $t = -\log|x| \geq -\log r_0 > 0$ on $\tilde{\Omega}$. Then the positivity and the decreasing property of V imply that

$$
\begin{aligned}
&\int_{\tilde{\Omega}} V(t) \left[|\nabla_\theta v|^2 + \left\{ v_t(t,\theta) + \left(\frac{N-2-2a}{2} \right) v \right\}^2 \right] d\mu \\
&\leq 2 \int_{\tilde{\Omega}} V(t) |\nabla v|^2 d\mu + 2 \left(\frac{N-2-2a}{2} \right)^2 \int_{\tilde{\Omega}} V(t) v^2 d\mu \\
&\leq 2V(-\log r_0) \int_{\tilde{\Omega}} |\nabla v|^2 d\mu + 2 \left(\frac{N-2-2a}{2} \right)^2 \int_{\tilde{\Omega}} V(t) v^2 d\mu \\
&\leq \left(2V(-\log r_0) + 2 \left(\frac{N-2-2a}{2} \right)^2 \right) \int_{\tilde{\Omega}} |\nabla v|^2 d\mu,
\end{aligned}
$$

where we have used the weighted Poincaré inequality (4.7) in the last inequality. Thus we get (4.8). Simple computation using (4.3), (4.4), (4.5) in (4.8) yields the desired inequality (4.6). □

Remark 5. The constant C in Theorem 4 can be chosen that

$$
C = \left(2V(-\log r_0) + 2 \left(\frac{N-2-2a}{2} \right)^2 \right)^{-1}.
$$

Proof of Corollary 1. We follow the computation in [17].

(i) We may assume $R = 1$. Note that X_i is well-defined and $X_i(0) = 0$, $X_i(1) = 1$, $0 < X_i(s) < 1$ for $s \in (0,1)$. We compute

$$
\begin{aligned}
&X_1'(s) = \frac{1}{s} X_1(s)^2, \quad X_i'(s) = \frac{X_1(s) \cdots X_{i-1}(s)}{s} X_i(s)^2, \\
&(X_1 \cdots X_k)'(s) = \frac{(X_1 \cdots X_k)(s)}{s} \left\{ X_1(s) + (X_1 X_2)(s) + \cdots + (X_1 \cdots X_k)(s) \right\}.
\end{aligned}
$$

Define $Y_i(t) = X_i(e^{-t})$ for $t \in [0, +\infty)$. Then we obtain

$$
\begin{aligned}
&Y_1'(t) = -Y_1(t)^2, \quad Y_i'(t) = -\left(Y_1 \cdots Y_i \right)(t) Y_i(t), \\
&(Y_1 \cdots Y_k)'(t) = (Y_1 \cdots Y_k)(t) \left\{ Y_1(t) + (Y_1 Y_2)(t) + \cdots + (Y_1 \cdots Y_k)(t) \right\}.
\end{aligned}
$$

Now, define

$$
\phi_k(t) = (Y_1 \cdots Y_k)^{-1/2}(t) > 0.
$$

Then by differentiating $\log \phi_k(t) = -\frac{1}{2} \sum_{i=1}^k \log Y_i(t)$, we check that

$$
\phi_k'(t) = \frac{\phi_k(t)}{2} \sum_{i=1}^k \left(Y_1 \cdots Y_i \right)(t), \quad \phi_k''(t) = -\frac{\phi_k(t)}{4} \sum_{i=1}^k \left(Y_1 \cdots Y_i \right)^2(t),
$$

thus ϕ_k is a solution of $-\frac{\phi_k''(t)}{\phi_k(t)} = \frac{1}{4}\sum_{i=1}^k (Y_1 \cdots Y_i)^2(t)$. It is easy to check that $V_k(t) = \frac{1}{4}\sum_{i=1}^k (Y_1 \cdots Y_i)^2(t)$ is nonnegative and decreasing on $(0, +\infty)$. Applying Theorem 4, we obtain the result.

(ii) We compute

$$\left(\log^{(i)} s\right)' = s^{-1}\left(\log^{(1)} s\right)^{-1}\left(\log^{(2)} s\right)^{-1} \cdots \left(\log^{(i-1)} s\right)^{-1}$$

for $i = 1, 2, \cdots, k$. Put $L_i(t) = \log^{(i)}(\rho e^t)$ and $W_i(t) = (L_1 \cdots L_i)^{-1}(t)$ for $i = 1, 2, \cdots, k$. We see $L_i(t) > 0$ if $\rho > e^{-t}e_{i-1}$ and obtain

$$L_1'(t) = 1, \quad L_i'(t) = (L_1 \cdots L_{i-1})^{-1}(t) \quad \text{for } i = 2, 3, \ldots, k,$$

$$W_i'(t) = -W_i(t)\sum_{j=1}^i (L_1 \cdots L_j)^{-1}(t).$$

Define

$$\psi_k(t) = (L_1 \cdots L_k)^{1/2}(t) > 0.$$

Then we check that

$$\psi_k'(t) = \frac{\psi_k(t)}{2}\sum_{i=1}^k (L_1 \cdots L_i)^{-1}(t) = \frac{\psi_k(t)}{2}\sum_{i=1}^k W_i(t),$$

$$\psi_k''(t) = \frac{\psi_k(t)}{4}\left(\sum_{i=1}^k W_i(t)\right)^2 - \frac{\psi_k(t)}{2}\sum_{i=1}^k W_i(t)\sum_{j=1}^i W_j(t),$$

$$-\frac{\psi_k''}{\psi_k}(t) = \frac{1}{4}\sum_{i=1}^k W_i(t)^2.$$

On the way of computation, we have used the identity

$$2\sum_{i=1}^k a_i \sum_{j=1}^i a_j - \left(\sum_{i=1}^k a_i\right)\left(\sum_{j=1}^k a_j\right) = \sum_{i=1}^k a_i^2$$

for any $a_1, \cdots, a_k \in \mathbb{R}$. Again $V_k(t) = \frac{1}{4}\sum_{i=1}^k (L_1 \cdots L_i)^{-2}(t)$ is nonnegative and decreasing on $(0, +\infty)$, so we may apply Theorem 4 to obtain the result. $\square$

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