Radial compactness of the embedding from $W^{1,p}_{\rm rad}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ and its application to nonlinear elliptic problem with variable critical exponent

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Abstract

We consider on compactness for the embedding from radial Sobolev spaces $W^{1,p}_{\rm rad}(\mathbb{R}^N)$ to *variable exponent* Lebesgue spaces $L^{q(x)}(\mathbb{R}^N)$. In particular, we point out that the behavior of $q(x)$ at infinity plays an essential role on compactness. As an application we prove the existence of solutions of the quasi-linear elliptic equation with a variable critical exponent.

Keywords: Compact embedding, Radial Sobolev space, Variable exponent, Nonlinear elliptic problem, Whole space
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1. Introduction and main results

Sobolev-type embedding has been studied by many researchers so far. As well known result there is a continuous embedding from $W^{1,p}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ for $N \geq 2$, $1 < p < N$, and $q \in [p, p^*]$, where p^* is $pN/(N-p)$. In addition, this embedding is not compact since these two function spaces have a property of invariance on translation. On the other hand, the embedding from $W^{1,p}_{\rm rad}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ is compact for $q \in (p, p^*)$ (see [13], [17]), where $W^{1,p}_{\rm rad}(\mathbb{R}^N)$ is the set of radially symmetric functions in $W^{1,p}(\mathbb{R}^N)$. Note that even radial Sobolev spaces $W^{1,p}_{\rm rad}(\mathbb{R}^N)$, it is not compact for $q = p$ and $q = p^*$. Related results are obtained in [6], [8], and so on.

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Generalized Sobolev spaces $W^{k,p(x)}(\Omega)$ with variable exponents $p(x)$ have also studied so far. For a domain $\Omega \subset \mathbb{R}^N$ and a function $p \in L^\infty(\Omega)$ with $p(x) \geq 1$ we set

$$L^{p(x)}(\Omega) = \left\{ u \text{ is a real measurable function on } \Omega \,\middle|\, \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$
$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \,\middle|\, D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k \right\}.$$

These $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ are Banach spaces with the following norms:

$$\|u\|_{p(x)} = \inf \left\{ \lambda > 0 \,\middle|\, \int_\Omega \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\}, \ \|u\|_{W^{k,p(x)}} = \|u\|_{p(x)} + \sum_{|\alpha| \leq k} \|D^\alpha u\|_{p(x)}.$$

When Ω is a bounded domain with the cone property, some results concerning $W^{k,p(x)}(\Omega)$ are obtained by [12], [9], and [14]. One of the results in [9] is the existence of the compact embedding. They consider the situation when $p(x)$ is uniformly continuous on $\overline{\Omega}$ and $1 < \operatorname{ess\,inf}_{\overline{\Omega}} p(x) \leq \operatorname{ess\,sup}_{\overline{\Omega}} p(x) < N/k$. Under this situation there exists a compact embedding from $W^{k,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ for $q(x)$ satisfying $p(x) \leq q(x)$ a.e. in Ω and $\operatorname{ess\,inf}_{\overline{\Omega}} p^*(x) - q(x) > 0$, where $p^*(x) = Np(x)/(N-kp(x))$. On the other hand, for $W^{1,p}(\Omega)$ Kurata and Shioji [12] consider the critical case, that is $\operatorname{ess\,sup}_{\overline{\Omega}} q(x) = p^*$. They showed that if there exist $x_0 \in \Omega, C_0 > 0, \eta > 0$, and $0 < \ell < 1$ such that $\operatorname{ess\,sup}_{\Omega \setminus B_\eta(x_0)} q(x) < p^*$ and

$$q(x) \leq p^* - \frac{C_0}{|\log|x - x_0||^\ell} \quad \text{for a.e. } x \in \Omega \cap B_\eta(x_0),$$

then the embedding from $W^{1,p}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact. Conversely, if

$$q(x) \geq p^* - \frac{C_0}{|\log|x - x_0||} \quad \text{for a.e. } x \in \Omega \cap B_\eta(x_0),$$

then the embedding from $W^{1,p}(\Omega)$ to $L^{q(x)}(\Omega)$ is not compact.

When $\Omega = \mathbb{R}^N$ and conditions of $p(x)$ are same as those of bounded domain case, the compact embedding from $W^{1,p(x)}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is obtained for $q(x)$ satisfying $\operatorname{ess\,inf}_{\mathbb{R}^N} q(x) - p(x) > 0$ and $\operatorname{ess\,inf}_{\mathbb{R}^N} p^*(x) - q(x) > 0$ by [10]. However, the critical case, that is $\operatorname{ess\,inf}_{\mathbb{R}^N} q(x) - p(x) = 0$ or $\operatorname{ess\,inf}_{\mathbb{R}^N} p^*(x) - q(x) = 0$, is not treated even if $p(x) \equiv p$.

In this paper, we fix $p(x) \equiv p$ and we investigate the case when $\operatorname{ess\,inf}_{\mathbb{R}^N} q(x) = p$ and $\operatorname{ess\,sup}_{\mathbb{R}^N} q(x) = p^*$. Our first purpose is to obtain a sufficiently condition of compactness and non-compactness.

Before introducing main results, we fix several notations. B_R denote a open ball centered 0 with radius R. ω_{N-1} is an area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Throughout this paper we assume that $q(x) \in L^\infty(\mathbb{R}^N)$ and $q(x) \geq 1$ for a.e. $x \in \mathbb{R}^N$. A letter C denotes various positive constant. If u is a radial function in $\mathbb{R}^N$, then we can write as $u(x) = \tilde{u}(|x|)$ by some function $\tilde{u} = \tilde{u}(r)$ in $\mathbb{R}_+$. For simplicity we write $u(x) = u(|x|)$ with admitting some ambiguity.

Theorem 1. *(Non-compactness) If there exist positive constants R, C_0 and a open set Γ in $\mathbb{S}^{N-1}$ such that*

$$q(x) \leq p + \frac{C_0}{|\log|x||} \quad \text{for } x \in (R, +\infty) \times \Gamma, \tag{1}$$

then the embedding from $W^{1,p}_{rad}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is not compact.

Theorem 2. *(Compactness) If there exist positive constants r, R, C_0, C_1, and $k, l \in (0,1)$ such that*

$$q(x) \leq p^* - \frac{C_0}{|\log|x||^k} \quad \text{for } x \in B_r, \tag{2}$$

$$q(x) \geq p + \frac{C_1}{|\log|x||^\ell} \quad \text{for } x \in \mathbb{R}^N \setminus B_R, \tag{3}$$

then the embedding from $W^{1,p}_{rad}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is compact.

Remark 1. *In Theorem 2, we don't need the constraint $p \leq q(x) \leq p^*$. $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \subset L^{q(x)}(\mathbb{R}^N)$ holds whenever $q(x)$ satisfies $q(x) \leq p^*$ in B_r and $q(x) \geq p$ in $\mathbb{R}^N \setminus B_R$. Concerning the continuous embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ for a constant q, the constraint $q \in [p, p^*]$ comes from (2) and (3).*

As an application of Theorem 2, we discuss the existence of a weak solution of the following nonlinear elliptic equation under the hypotheses (2), (3) in Theorem 2.

$$\begin{cases} -\Delta_p u + u^{p-1} = u^{q(x)-1}, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N, \\ u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N), \end{cases} \tag{4}$$

where $\Delta_p u = \mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$ is $p-$Laplacian. Note that in the non-critical case, that is $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} q(x) > p$, existence of solutions to a quasi-linear equation similar type to (4) has already studied by [2]. However, different from [2], there is a possibility of $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} q(x) = p$ under the hypothesis (3). This condition causes some difficulties to show the existence of solution to (4). Before introducing our result, we state several difficulties of our problem.

Mountain pass method which has been introduced by Ambrosetti and Rabinowitz [1] is useful to show the existence of nonlinear elliptic equations. However, in (4) with the case $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} q(x) = p$, we cannot confirm whether the energy functional J (see Section 4) corresponding to (4) satisfies the "Palais-Smale condition" or not. Besides that, satisfying the mountain pass structure for J is not trivial since we can not apply the fibering map method directly. To overcome these difficulties, in Section 3, we construct a solution of (4) as a limit of mountain pass solutions of some elliptic equations approaching (4) in the sense of energy functional. In Section 4, we show an another proof by using the variant of the mountain pass theorem. More precisely, by introducing the condition (C) (see Section 4) defined in [5] or [3] instead of the "Palais-Smale condition", we obtain a solution of (4) in a different way from Section 3.

Theorem 3. *Assume that $q(x)$ satisfies the hypotheses (2), (3) in Theorem 2 and $\operatorname{ess\,inf}_{x\in B_R} q(x) > p$. Then there exists a nontrivial weak solution $u \in W^{1,p}_{rad}(\mathbb{R}^N)$ of (4) in the sense of*

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u\nabla\phi + u^{p-1}\phi - u^{q(x)-1}\phi\right) dx = 0 \tag{5}$$

for any $\phi \in W^{1,p}_{rad}(\mathbb{R}^N)$.

Remark 2. *If $q(x)$ is radially symmetric satisfying the hypotheses of Theorem 3, then we can show that the weak solution u obtained in Theorem 3 satisfies $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N \setminus \{0\})$ and $u(x) > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$. Indeed, since u and $q(x)$ are radially symmetric, it follows that for all $\phi \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$*

$$\int_0^\infty \left(|u'(r)|^{p-2}u'(r)\phi(r) + u^{p-1}\phi - u^{q(r)-1}\phi\right) r^{N-1}dr = 0,$$

where $r = |x|$. If for any $\psi \in C^\infty_c(\mathbb{R}^N)$ we consider the radial function $\Psi(r) = \int_{\omega\in\mathbb{S}^{N-1}} \psi(r\omega)\, dS_\omega$, then we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u\nabla\psi + u^{p-1}\psi - u^{q(x)-1}\psi\right) dx \\
&= \int_0^\infty \left(|u'(r)|^{p-2}u'(r)\Psi(r) + u^{p-1}\Psi - u^{q(r)-1}\Psi\right) r^{N-1}dr = 0.
\end{aligned}$$

Therefore we see that u satisfies (5) even for non-radial functions ϕ. Finally, by Corollary of Theorem 2 in [7] we have $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N \setminus \{0\})$. And also, by Theorem 2.5.1 in [15] we have $u(x) > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

2. Compactness and non-compactness of the embedding

We prove Theorem 1 and Theorem 2. Before beginning the proof we recall the pointwise estimate and the compactness theorem introduced in [13], and [17] ($p = 2$). For the reader's convenience, the proofs are in Appendix.

Proposition 1. *For any $u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ we have*

$$|u(x)| \le \left(\frac{p}{\omega_{N-1}}\right)^{\frac{1}{p}} |x|^{-\frac{N-1}{p}} \|u\|^{\frac{p-1}{p}}_{L^p(\mathbb{R}^N)} \|\nabla u\|^{\frac{1}{p}}_{L^p(\mathbb{R}^N)}. \tag{6}$$

Proposition 2. *The embedding from $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ is compact for $q \in (p, p^*)$.*

Proof of Theorem 1. We shall show Theorem 1 in the same way as [12]. Set $r(x) = q(x) - p$ for $x \in \mathbb{R}^N$. Let $\phi \in C^\infty_c(\mathbb{R}^N)$ be a radial function satisfying $\phi \equiv 1$ on $B_{\frac{1}{2}}$ and $\text{supp}\phi \subset B_1$. For $m \in \mathbb{N}$, we define $\phi_m(x) = m^{-\frac{N}{p}} \phi(\frac{x}{m})$. Then for any $m \in \mathbb{N}$ we obtain

$$\|\phi_m\|_{L^p(\mathbb{R}^N)} = \|\phi\|_{L^p(B_1)}, \quad \|\nabla \phi_m\|_{L^p(\mathbb{R}^N)} = m^{-1} \|\nabla \phi\|_{L^p(B_1)}.$$

Since $\{\phi_m\}_{m=1}^\infty$ is a bounded sequence in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ and $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ is reflexive (see e.g. Proposition 3.20. in [4]), there exist a weakly convergent subsequence $\{\phi_{m_j}\}_{j=1}^\infty$ and $\phi_\infty \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ such that $\phi_{m_j} \rightharpoonup \phi_\infty$ in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ as $j \to \infty$. By compactness of the embedding from $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ to $L^r(\mathbb{R}^N)$ for $p < r < p^*$, we have $\phi_{m_j} \to \phi_\infty$ in $L^r(\mathbb{R}^N)$ and $\phi_{m_j} \to \phi_\infty$ a.e. in $\mathbb{R}^N$ which yields that $\phi_\infty \equiv 0$. On the other hand, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\phi_m(x)|^{q(x)}\, dx &= \int_{B_m} m^{-\frac{N}{p}(p+r(x))} \left|\phi\left(\frac{x}{m}\right)\right|^{q(x)} dx \\
&= \int_{B_1} m^{-\frac{N}{p} r(my)} |\phi(y)|^{q(my)}\, dy \\
&\ge \int_{B_{\frac{1}{2}} \setminus B_{\frac{1}{4}}} m^{-\frac{N}{p} r(my)}\, dy.
\end{aligned}$$

Since Γ is open in $\mathbb{S}^{N-1}$, there exists a open disk $D \subset \mathbb{S}^{N-1}$ such that $D \subset \Gamma$. By using the polar coordinates as $y = s\omega\, (s > 0, \omega \in \mathbb{S}^{N-1})$ we obtain

$$\int_{\mathbb{R}^N} |\phi_m(x)|^{q(x)}\, dx \ge \int_{s=\frac{1}{4}}^{\frac{1}{2}} \int_{\omega \in D} m^{-\frac{N}{p} r(ms\omega)}\, s^{N-1} ds dS_\omega.$$

By the assumption (1), we obtain $r(ms\omega) \leq C_0|\log ms|^{-1}$ for large m, $s \in (1/4, 1/2)$, and $\omega \in D \subset \Gamma$. Moreover for $s \in (1/4, 1/2)$ and large m, it holds $\log ms = \log m + \log s \geq \frac{1}{2}\log m$ which yields that

$$r(ms\omega) \leq \frac{2C_0}{\log m}.$$

Therefore we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\phi_m(x)|^{q(x)}\, dx &\geq \int_{s=\frac{1}{4}}^{\frac{1}{2}} \int_{\omega\in D} e^{-\frac{N}{p}\log m\,\frac{2C_0}{\log m}}\, s^{N-1} ds dS_{\omega} \\ &= \mathcal{L}^{N-1}(D)\, e^{-\frac{2C_0 N}{p}}\, \frac{2^{-N} - 4^{-N}}{N} > 0\end{aligned}$$

for large m, where $\mathcal{L}^d$ is the $d-$dimensional Lebesgue measure. Thus, if we assume the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is compact, then we have $\int_{\mathbb{R}^N} |\phi_\infty|^{q(x)}\, dx > 0$ which contradicts $\phi_\infty \equiv 0$. Hence the embedding from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $L^{q(x)}(\mathbb{R}^N)$ is not compact. □

Proof of Theorem 2. We assume that $r < R$ without loss of generality. Let $\{u_m\}_{m=1}^\infty$ be a bounded sequence in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. We shall show the existence of a strongly convergence subsequence of $\{u_m\}_{m=1}^\infty$ in $L^{q(x)}(\mathbb{R}^N)$. By the reflexivity of $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, there exist a subsequence $\{u_{m_j}\}_{j=1}^\infty$ and $u_0 \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_{m_j} \rightharpoonup u_0$ in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ as $j \to \infty$. Especially it also holds that $u_{m_j} \rightharpoonup u_0$ in $W^{1,p}(\mathbb{R}^N)$ as $j \to \infty$. And also, by Proposition 2 we have $u_{m_j} \to u_0$ in $L^q(\mathbb{R}^N)$ for any $q \in (p, p^*)$ and

$$u_{m_j} \to u_0 \quad \text{a.e. in } \mathbb{R}^N \text{ as } j \to \infty. \tag{7}$$

Furthermore, $\{u_{m_j}|_{B_r}\}_{j=1}^\infty \subset W^{1,p}(B_r)$ is a bounded sequence and the embedding from $W^{1,p}(B_r)$ to $L^{q(x)}(B_r)$ is compact by the assumption (2) (see Remark 2 in [12]). Thus there exist a subsequence of $\{u_{m_j}|_{B_r}\}_{j=1}^\infty$ (we use $\{u_{m_j}|_{B_r}\}_{j=1}^\infty$ again for simplicity) and $v_0 \in L^{q(x)}(B_r)$ such that the followings hold true:

$$\begin{aligned} u_{m_j}|_{B_r} &\rightharpoonup v_0 \quad \text{in } W^{1,p}(B_r),\\ u_{m_j}|_{B_r} &\to v_0 \quad \text{in } L^{q(x)}(B_r),\\ u_{m_j}|_{B_r} &\to v_0 \quad \text{in } L^{p}(B_r),\\ u_{m_j}|_{B_r} &\to v_0 \quad \text{a.e. in } B_r \text{ as } j \to \infty. \end{aligned} \tag{8}$$

By (7) and (8), we can check that $u_0|_{B_r} = v_0$ a.e. in B_r which yields that

$$u_{m_j}|_{B_r} \to u_0|_{B_r} \quad \text{in } L^{q(x)}(B_r) \text{ as } j \to \infty. \tag{9}$$

In the similar way as above, we also obtain the followings

$$\begin{aligned} u_{m_j}|_{B_K\setminus B_r} &\rightharpoonup u_0|_{B_K\setminus B_r} && \text{in } W^{1,p}_{\text{rad}}(B_K \setminus B_r), \\ u_{m_j}|_{B_K\setminus B_r} &\to u_0|_{B_K\setminus B_r} && \text{in } L^q(B_K \setminus B_r), \\ u_{m_j}|_{B_K\setminus B_r} &\to u_0|_{B_K\setminus B_r} && \text{a.e. in } B_K \setminus B_r \end{aligned} \tag{10}$$

for any $K > 0$ and any $q \geq 1$ as $j \to \infty$ since the embedding from $W^{1,p}_{\text{rad}}(B_K \setminus B_r)$ to $L^q(B_K \setminus B_r)$ is compact for any K, q.

Set $v_{m_j} := u_{m_j} - u_0$. In order to make good use of (9) and (10) we divide $\int_{\mathbb{R}^N} |v_{m_j}(x)|^{q(x)}dx$ into three terms as follows:

$$\begin{aligned} &\int_{\mathbb{R}^N} |v_{m_j}(x)|^{q(x)}dx \qquad (11)\\ =\ &\int_{B_r} |v_{m_j}(x)|^{q(x)}dx + \int_{B_K\setminus B_r} |v_{m_j}(x)|^{q(x)}dx + \int_{\mathbb{R}^N\setminus B_K} |v_{m_j}(x)|^{q(x)}dx \\ =:\ &I_1(j) + I_2(j,K) + I_3(j,K), \end{aligned}$$

where K is sufficiently large.

Firstly, by (9) we have

$$I_1(j) = o(1) \text{ as } j \to \infty. \tag{12}$$

Next, for $I_2(j,K)$ we have

$$I_2(j,K) = \int_{B_K\setminus B_r} |v_{m_j}(x)|^{q(x)}dx \leq \int_{B_K\setminus B_r} |v_{m_j}(x)|\, dx + \int_{B_K\setminus B_r} |v_{m_j}(x)|^{\|q\|_{L^\infty(\mathbb{R}^N)}}\, dx.$$

Thus, by (10) we obtain

$$I_2(j,K) = o(1) \text{ as } j \to \infty \text{ for fixed } K > 0. \tag{13}$$

Finally we shall estimate $I_3(j,K)$. Since

$$|v_{m_j}(x)| \leq \left(\frac{p}{\omega_{N-1}}\right)^{\frac{1}{p}} \|v_{m_j}\|_{W^{1,p}(\mathbb{R}^N)}|x|^{-\frac{N-1}{p}} \leq C|x|^{-\frac{N-1}{p}}$$

by Proposition 1 and the boundedness of $\{v_{m_j}\}_{j=1}^{\infty}$, we can assume $|v_{m_j}(x)| \le 1$ for $x \in \mathbb{R}^N \setminus B_K$ with large K. Therefore by the assumption (3) we obtain

$$\begin{aligned}
I_3(j,K) &= \int_{\mathbb{R}^N \setminus B_K} |v_{m_j}|^{q(x)} dx \le \int_{\mathbb{R}^N \setminus B_K} |v_{m_j}|^{p + C_1(\log|x|)^{-\ell}} dx \\
&\le \sum_{n=2}^{\infty} \int_{B_{K^n} \setminus B_{K^{n-1}}} |v_{m_j}|^{p + C_1(n \log K)^{-\ell}} dx \\
&\le \sum_{n=2}^{\infty} \int_{B_{K^n} \setminus B_{K^{n-1}}} |v_{m_j}|^p \left(C|x|^{-\frac{N-1}{p}}\right)^{C_1(n\log K)^{-\ell}} dx \\
&\le C^{C_1(2\log K)^{-\ell}} \|v_{m_j}\|^p_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^{\infty} K^{-\frac{N-1}{p}(n-1)C_1(n\log K)^{-\ell}} \\
&\le C \sum_{n=2}^{\infty} \delta_1^{(n-1)^{1-\ell}} = C \sum_{n=1}^{\infty} \delta_1^{n^{1-\ell}},
\end{aligned}$$

where $\delta_1 = \delta_1(K) := K^{-\frac{N-1}{p}C_1(\log K)^{-\ell}} \to 0$ as $K \to \infty$. Since $\sum_{n=1}^{\infty} \delta_1^{n^{1-\ell}} = \delta_1 + \int_1^{\infty} \delta_1^{x^{1-\ell}} dx < \infty$ for each $\delta_1 \in (0,1)$, we have

$$\sum_{n=1}^{\infty} \delta_1^{n^{1-\ell}} \to 0 \quad \text{as } K \to \infty.$$

Hence we have

$$I_3(j,K) = o(1) \text{ uniformly in } j \text{ as } K \to \infty. \tag{14}$$

We go back (11) and by (12), (13), and (14) we have

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} |v_{m_j}(x)|^{q(x)} dx = 0.$$

As a consequence we obtain $u_{m_j} \to u_0$ in $L^{q(x)}(\mathbb{R}^N)$. □

3. Approximation method : Proof of Theorem 3

In this section, we show Theorem 3 by using Theorem 2. First, we prepare the mountain pass theorem (Theorem 4) introduced in [16], [18], and so on which are based on [1]. Let V be a Banach space and $E \in C^1(V, \mathbb{R})$. We define a Palais-Smale sequence for E as $\{u_m\} \subset V$ satisfying $|E(u_m)| \le c$ uniformly in m, and

$E'(u_m) \to 0$ in V^*, where $E'(\cdot)$ is Fréchet derivative and V^* is the dual space of V. We say that E satisfies (P.-S.) condition if any Palais-Smale sequence has a strongly convergent subsequence.

Theorem 4 ([16], [18])**.** *Suppose $E \in C^1(V, \mathbb{R})$ satisfies (P.-S.) condition. Assume that*

(i) E(0)=0

(ii) There exist $\rho > 0,\ \alpha > 0$ such that $E(u) \geq \alpha$ for any $u \in V$ with $\|u\| = \rho$.

(iii) There exists $u_1 \in V$ such that $\|u_1\| \geq \rho$ and $E(u_1) < \alpha$.

Define

$$P = \{\, p \in C([0,1], V) \mid p(0) = 0,\ p(1) = u_1 \,\}.$$

Then

$$\beta = \inf_{p \in P} \sup_{0 \leq t \leq 1} E(p(t))$$

is a critical value.

Proof of Theorem 3. *Step 1.* We may assume that R in the hypotheses of Theorem 2 is sufficiently large such that $\operatorname{ess\,inf}_{x \in B_R} q(x) = p + C_1(\log R)^{-\ell}$ without loss of generality. For $m \in \mathbb{N}$ let $\{R_m\}$ be a sequence such that $R_1 = R$, $R_m \to \infty$ as $m \to \infty$. Then we set functions as

$$q_m(x) = \begin{cases} q(x) & \text{if} \quad q(x) \geq p + C_1(\log R_m)^{-\ell}, \\ p + C_1(\log R_m)^{-\ell} & \text{if} \quad q(x) < p + C_1(\log R_m)^{-\ell}. \end{cases}$$

Define a functional J_m from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $\mathbb{R}$ by

$$J_m(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p)\, dx - \int_{\mathbb{R}^N} \frac{1}{q_m(x)} u_+^{q_m(x)} dx.$$

We can check that $J_m \in C^1(W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N), \mathbb{R})$. Moreover, for each m, J_m satisfies as follows:

(i) J_m satisfies (P.-S.) condition.

(ii) $J_m(0) = 0$,

(iii) There exist positive constants α, ρ such that $J_m(u) \geq \alpha$ for any $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p}(\mathbb{R}^N)} = \rho$,

(iv) There exists $v \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $\|v\|_{W^{1,p}(\mathbb{R}^N)} \geq \rho$, $J_m(v) < \alpha$.

By Theorem 4 there exists a critical point $u_m \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ of J_m such that

$$J_m(u_m) = \beta_m,$$

where β_m is defined in the same way as β in Theorem 4. Thus u_m is a nontrivial weak solution of

$$-\Delta_p w + |w|^{p-2}w = w_+^{q_m(x)-1} \quad \text{in} \quad \mathbb{R}^N. \tag{15}$$

We can also see that $u_m \geq 0$ by multiplying both sides of (15) by $(u_m)_-$.

Proposition 3. *$\{u_m\}$ is bounded in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$.*

We will prove this proposition at last of this section.

Step 2. Since $\{u_m\}$ is a bounded sequence, there exists $u_0 \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_m \rightharpoonup u_0$ weakly in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. Put

$$G_m = \left\langle |\nabla u_m|^{p-2}\nabla u_m - |\nabla u_0|^{p-2}\nabla u_0, \nabla u_m - \nabla u_0 \right\rangle_{\mathbb{R}^N} + (u_m^{p-1} - u_0^{p-1})(u_m - u_0).$$

Then we have

$$\int_{\mathbb{R}^N} G_m dx = \int_{\mathbb{R}^N} (|\nabla u_m|^p + u_m^p)\, dx - \int_{\mathbb{R}^N} (|\nabla u_m|^{p-2}\nabla u_m \nabla u_0 + u_m^{p-1}u_0)\, dx + h_m,$$

where $h_m = \int_{\mathbb{R}^N} \left[|\nabla u_0|^{p-2}\nabla u_0(\nabla u_0 - \nabla u_m) + u_0^{p-1}(u_0 - u_m) \right] dx = o(1)$ as $m \to \infty$. Moreover, from (22) and (23) in the proof of Proposition 3 it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|\nabla u_m|^p + u_m^p)\, dx - \int_{\mathbb{R}^N} (|\nabla u_m|^{p-2}\nabla u_m \nabla u_0 + u_m^{p-1}u_0)\, dx \\
&= \int_{\mathbb{R}^N} (u_m)_+^{q_m(x)-1}\, ((u_m)_+ - u_0)\, dx \\
&\leq C_H\, ||u_m^{q_m(x)-1}||_{\frac{q(x)}{q(x)-1}} ||u_m - u_0||_{q(x)} \\
&= C_H\, ||u_m||_{q(x)}\, ||u_m - u_0||_{q(x)},
\end{aligned}$$

where C_H is a positive constant due to the generalized Hölder inequality (see e.g. [11] Theorem 2.1). By the boundedness of $\{u_m\}$ in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ and Theorem 2 we have $||u_m||_{q(x)}||u_m - u_0||_{q(x)} = o(1)$ as $m \to \infty$. Hence

$$\int_{\mathbb{R}^N} G_m dx = o(1) \tag{16}$$

as $m \to \infty$. Recall that for $p \geq 1$, $a, b \in \mathbb{R}^d$ we have

$$\left\langle |b|^{p-2}b - |a|^{p-2}a, b - a \right\rangle \geq \begin{cases} 2^{2-p}|b-a|^p & \text{if} \quad p \geq 2, \\ (p-1)|b-a|^2(1+|a|^2+|b|^2)^{\frac{p-2}{2}} & \text{if} \quad 1 \leq p \leq 2. \end{cases}$$

From this inequality and (16) it follows that

$$\int_{\mathbb{R}^N} (|\nabla u_m - \nabla u_0|^p + |u_m - u_0|^p) dx = o(1)$$

which is equivalent to $u_m \to u_0$ strongly in $W^{1,p}(\mathbb{R}^N)$. Thus u_0 satisfies

$$-\Delta_p u_0 + u_0^{p-1} = u_0^{q(x)-1}, \ u_0 \geq 0 \quad \text{in} \quad \mathbb{R}^N.$$

Step 3. Finally, we have to show $u_0 \not\equiv 0$. From the boundedness of $\{u_m\}$ and Proposition 1, we see that $u_m \leq 1$ in $\mathbb{R}^N \setminus B_L$ for large L. Therefore we have

$$\int_{\mathbb{R}^N} (|\nabla u_m|^p + u_m^p) dx = \int_{\mathbb{R}^N} u_m^{q_m(x)} dx \leq \int_{\mathbb{R}^N} u_m^p dx + \int_{B_r} u_m^{p^*} dx + \int_{B_L \setminus B_r} u_m^{||q||_{L^\infty(\mathbb{R}^N)}} dx. \tag{17}$$

By the Sobolev inequality it follows that

$$\int_{B_r} u_m^{p^*} dx \leq \int_{\mathbb{R}^N} u_m^{p^*} dx \leq S^{-\frac{p^*}{p}} \left(\int_{\mathbb{R}^N} |\nabla u_m|^p dx \right)^{\frac{p^*}{p}}. \tag{18}$$

Moreover, we have

$$\begin{aligned} \int_{B_L \setminus B_r} u_m^{||q||_{L^\infty(\mathbb{R}^N)}} dx &\leq C \left[\int_{B_L \setminus B_r} (|\nabla u_m|^p + |u_m|^p) dx \right]^{\frac{||q||_{L^\infty(\mathbb{R}^N)}}{p}} \\ &\leq C \left[\int_{B_L \setminus B_r} |\nabla u_m|^p + \left(\int_{B_L \setminus B_r} |u_m|^{p^*} dx \right)^{\frac{p}{p^*}} |B_L \setminus B_r|^{1-\frac{p}{p^*}} \right]^{\frac{||q||_{L^\infty(\mathbb{R}^N)}}{p}} \\ &\leq C \left(\int_{\mathbb{R}^N} |\nabla u_m|^p \right)^{\frac{||q||_{L^\infty(\mathbb{R}^N)}}{p}}. \end{aligned} \tag{19}$$

Put $q_* := \min\{p^*, ||q||_{L^\infty(\mathbb{R}^N)}\}$. From (17), (18), and (19), we obtain

$$C \leq \left(\int_{\mathbb{R}^N} |\nabla u_m|^p \right)^{\frac{q_* - p}{p}},$$

where we used that $u_m \not\equiv 0$. By Theorem 2 we have

$$\begin{aligned} C &\leq \lim_{m\to\infty} \int_{\mathbb{R}^N} |\nabla u_m|^p dx \\ &= \lim_{m\to\infty} \int_{\mathbb{R}^N} (-u_m^p + u_m^{q_m(x)}) dx \\ &\leq \int_{\mathbb{R}^N} u_0^{q(x)} dx. \end{aligned}$$

Consequently we have $u_0 \not\equiv 0$. □

Proof of Proposition 3. We take a smooth radial function $\hat{u} > 0$ on $\mathbb{R}^N$. Since

$$\begin{aligned} J_m(K\hat{u}) &\leq \frac{K^p}{p} \int_{\mathbb{R}^N} (|\nabla \hat{u}|^p + |\hat{u}|^p)\, dx - \int_{B_R} \frac{K^{q(x)}}{q(x)} \hat{u}_+^{q(x)} dx \\ &\leq \frac{K^p}{p} \int_{\mathbb{R}^N} (|\nabla \hat{u}|^p + |\hat{u}|^p)\, dx - \frac{K^{p+C_1(\log R)^\ell}}{\operatorname{ess\,sup}_{B_R} q(x)} \int_{B_R} \hat{u}_+^{q(x)} dx \to -\infty \end{aligned}$$

as $K \to +\infty$, there exists $\hat{K} > 0$ independent of m such that $J_m(\hat{K}\hat{u}) < 0$. If we set $\hat{p}(t) = t\hat{K}\hat{u}$ for $t \in [0,1]$, then we see that

$$\hat{p} \in \hat{P} = \left\{ p \in C([0,1], W^{1,p}_{\text{rad}}(\mathbb{R}^N)) \;\middle|\; p(0) = 0,\ p(1) = \hat{K}\hat{u} \right\}.$$

Moreover, we have

$$\begin{aligned} \beta_m &= \inf_{p\in\hat{P}} \max_{0\leq t\leq 1} J_m(p(t)) \leq \max_{0\leq t\leq 1} J_m(\hat{p}(t)) \\ &= \max_{0\leq t\leq \hat{K}} \left[\frac{t^p}{p} \int_{\mathbb{R}^N} (|\nabla \hat{u}|^p + |\hat{u}|^p)\, dx - \int_{B_R} \frac{t^{q(x)}}{q(x)} \hat{u}_+^{q(x)} dx \right] \leq C. \end{aligned} \tag{20}$$

On the other hand, since u_m is a critical point of J_m at β_m we have

$$\beta_m = \frac{1}{p} \int_{\mathbb{R}^N} (\,|\nabla u_m|^p + |u_m|^p\,)\, dx - \int_{\mathbb{R}^N} \frac{1}{q_m(x)} (u_m)_+^{q_m(x)} dx \tag{21}$$

and for any $\phi \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (\,|\nabla u_m|^{p-2} \nabla u_m \nabla \phi \;+\; |u_m|^{p-2} u_m \phi\,)\, dx - \int_{\mathbb{R}^N} (u_m)_+^{q_m(x)-1} \phi\, dx = 0. \tag{22}$$

In particular,

$$\int_{\mathbb{R}^N} (|\nabla u_m|^p + |u_m|^p\,)\, dx - \int_{\mathbb{R}^N} (u_m)_+^{q_m(x)} dx = 0. \tag{23}$$

From (20), (21), and (23), it follows that

$$\int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q_m(x)} \right) (u_m)_+^{q_m(x)} dx \leq C.$$

Furthermore, by $q(x) \leq q_m(x)$ we have

$$\int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q(x)} \right) (u_m)_+^{q_m(x)} dx \leq C. \tag{24}$$

Thus for any $L > 0$ there exists a positive constant $C(L)$ such that

$$\int_{B_L} (u_m)_+^{q_m(x)} dx \leq C(L). \tag{25}$$

Here, we take a constant $R_0 > R$ sufficiently large (This R_0 will be chosen again later) and we have

$$\|u_m\|_{W^{1,p}(\mathbb{R}^N)}^p \leq C(R_0) + \int_{\mathbb{R}^N \setminus B_{R_0}} (u_m)_+^{q_m(x)} dx \tag{26}$$

by (23) and (25). Set $\delta = C_1(\log R_0)^{-\ell}$ and $A_n := B_{R_0^n} \setminus B_{R_0^{n-1}}$. Then we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_{R_0}} (u_m)_+^{q_m(x)} dx \\
&= \int_{\{q_m(x) > p+\delta\}} (u_m)_+^{q_m(x)} dx + \int_{\{q_m(x) \leq p+\delta\}} (u_m)_+^{q_m(x)} dx \\
&\leq \int_{\{q_m(x) > p+\delta\}} (u_m)_+^{q_m(x)} dx + \int_{\{q(x) \leq p+\delta\}} (u_m)_+^{q_m(x)} dx \\
&\leq \int_{\{q_m(x) > p+\delta\}} (u_m)_+^{q_m(x)} dx + \sum_{n=2}^{\infty} \int_{A_n} (u_m)_+^{p+C_1(n\log R_0)^{-\ell}} dx + \sum_{n=2}^{\infty} \int_{A_n} (u_m)_+^{p+\delta} dx \\
&=: L_1 + L_2 + L_3,
\end{aligned}$$

where third inequality comes from the assumption (3). We shall estimate L_1, L_2, and L_3. For L_1, by (24) we have

$$L_1 \leq \left(\frac{1}{p} - \frac{1}{p+\delta} \right)^{-1} \int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q_m(x)} \right) (u_m)_+^{q_m(x)} dx = C. \tag{27}$$

In order to estimate L_2 and L_3, we prepare an estimate of $\|u_m\|_{W^{1,p}(A_n)}$. For each $n \in \mathbb{N}$ and small $\varepsilon > 0$, we take $\xi_\varepsilon = \xi_{n,\varepsilon} \in C_c^\infty(\mathbb{R}^N)$ such that

$$0 \le \xi_\varepsilon \le 1 \text{ in } \mathbb{R}^N, \quad \xi_\varepsilon = 1 \text{ in } A_{n,\varepsilon}, \quad \xi_\varepsilon = 0 \text{ in } \mathbb{R}^N \setminus A_n, \quad |\nabla \xi_\varepsilon| \le \frac{C}{\varepsilon},$$

where $A_{n,\varepsilon} = \{\, x \in A_n \mid \mathrm{dist}(x, A_n) \ge \varepsilon \,\}$. In (22), by replacing ϕ with $u_m \xi_\varepsilon$ and letting $\varepsilon \to 0$, we have

$$\|u_m\|^p_{W^{1,p}(A_n)} = \int_{A_n} (u_m)_+^{q_m} dx \quad \text{for each } n \in \mathbb{N}.$$

From this equality and (24), we have

$$\begin{aligned} \|u_m\|^p_{W^{1,p}(A_n)} &\le \left(\frac{1}{p} - \frac{1}{p + C_1(\log R_0^n)^{-\ell}}\right)^{-1} \int_{A_n} \left(\frac{1}{p} - \frac{1}{q(x)}\right)(u_m)_+^{q_m(x)} dx \\ &\le C\,(\log R_0^n\,)^{\ell}. \end{aligned} \tag{28}$$

For L_2, by using (28) and Proposition 1, we have

$$\begin{aligned} L_2 &\le C \sum_{n=2}^{\infty} \|u_m\|^{C_1(n\log R_0)^{-\ell}}_{W^{1,p}(\mathbb{R}^N)} R_0^{\left(-\frac{N-1}{p}\right)(n-1)C_1(n\log R_0)^{-\ell}} \int_{A_n} u_n^p dx \\ &= C \sum_{n=2}^{\infty} \|u_m\|^{C_1(n\log R_0)^{-\ell}}_{W^{1,p}(\mathbb{R}^N)} R^{-\frac{N-1}{p}C_1(2\log R_0)^{-\ell}(n-1)^{1-\ell}} \|u_n\|^p_{W^{1,p}(A_n)} \\ &\le C\|u_n\|^p_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^{\infty} \delta_2^{(n-1)^{1-\ell}} \|u_n\|^{C_1(n\log R_0)^{-\ell}}_{W^{1,p}(A_n)} \\ &\le C\|u_n\|^p_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^{\infty} \delta_2^{(n-1)^{1-\ell}} (n\log R_0)^{\frac{C_1\ell}{p}(n\log R_0)^{-\ell}}, \end{aligned}$$

where $\delta_2 = \delta_2(R_0) = R_0^{-(N-1)C_1(2\log R_0)^{-\ell}/p}$. In the same way as the proof of Theorem 2, we observe that $\sum_{n=2}^\infty \delta_2^{(n-1)^{1-\ell}} \to 0$ as $R_0 \to \infty$. Moreover, since

$$(n\log R_0)^{\frac{C_1\ell}{p}(n\log R_0)^{-\ell}} \to 1 \text{ as } n \to \infty \text{ or } R_0 \to \infty,$$

there exists a positive constant $\tilde{C}$ which is independent of n and R_0 such that $(n\log R_0)^{C_1\ell(n\log R_0)^{-\ell}/p} \le \tilde{C}$. Hence, for sufficiently large R_0 we have

$$L_2 \le \frac{1}{3}\|u_m\|^p_{W^{1,p}(\mathbb{R}^N)}. \tag{29}$$

In the same way as L_2, we obtain the estimate of L_3 as follows.

$$\begin{aligned}
L_3 &\le C\|u_m\|^{\delta}_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^{\infty} R_0^{\left(-\frac{N-1}{p}\right)(n-1)\delta} \int_{A_n} u_m^p dx \\
&\le C\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \sum_{n=2}^{\infty} R_0^{\left(-\frac{N-1}{p}\right)(n-1)\delta} (n\log R_0)^{\frac{\ell}{p}\delta} \\
&= C\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} (\log R_0)^{\frac{\ell\delta}{p}} \sum_{n=2}^{\infty} (n^{\ell} R_0^{-(n-1)(N-1)})^{\frac{\delta}{p}} \\
&= C\|u_n\|^{p}_{W^{1,p}(\mathbb{R}^N)} (\log R_0)^{\frac{\ell\delta}{p}} \sum_{n=2}^{\infty} \left[n^{\ell} R_0^{-(n-1)} R_0^{-(n-1)(N-2)}\right]^{\frac{\delta}{p}} \\
&\le C\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \sum_{n=1}^{\infty} \delta_3^{n(N-2)},
\end{aligned}$$

where $\delta_3 = \delta_3(R_0) = R_0^{-\delta/p}$. We can easily check that $\sum_{n=1}^{\infty} \delta_3^{n(N-2)} < \infty$ which yields that $\sum_{n=1}^{\infty} \delta_3^{n(N-2)} \to 0$ as $R_0 \to \infty$. Therefore for sufficiently large R_0 we have

$$L_3 \le \frac{1}{3}\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)}. \tag{30}$$

From (26), (27), (29), and (30) we have

$$\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)} \le C + \frac{2}{3}\|u_m\|^{p}_{W^{1,p}(\mathbb{R}^N)}.$$

As a consequence u_m is bounded. □

4. Mountain pass theorem under the condition (C) : Proof of Theorem 3

In this section, we show Theorem 3 by a different method from Section 3.

Cerami [5] and Bartolo-Benci-Fortunato [3] have proposed a variant of (P.-S.) condition. In this paper, we use the condition (C) introduced by [5] and [3] and the mountain pass theorem under the condition (C) (Theorem 6). Let V be a real Banach space and $E \in C^1(V, \mathbb{R})$. First, we define the condition (C) based on [5] and [3].

Definition 5 ([5], [3] Definition 1.1.)**.** We say that E satisfies the condition (C) in $(c_1, c_2), (-\infty \le c_1 < c_2 \le +\infty)$, if

(i) every bounded sequence $\{u_k\} \subset E^{-1}((c_1, c_2))$, for which $\{E(u_k)\}$ is bounded and $E'(u_k) \to 0$, possesses a convergent subsequence, and

(ii) for any $c \in (c_1, c_2)$ there exist $\sigma, \rho, \alpha > 0$ such that $[c - \sigma, c + \sigma] \subset (c_1, c_2)$ and for any $u \in E^{-1}([c - \sigma, c + \sigma])$ with $\|u\| \geq \rho$, $\|E'(u)\|_* \|u\| \geq \alpha$.

Theorem 6 (Mountain pass theorem under the condition (C))**.** *Let E satisfy the condition (C) in* $(0, +\infty)$*. Assume that*

(i) E(0)=0

(ii) There exist $\rho > 0,\ \alpha > 0$ *such that* $E(u) \geq \alpha$ *for any* $u \in V$ *with* $\|u\| = \rho$*.*

(iii) There exists $u_1 \in V$ *such that* $\|u_1\| \geq \rho$ *and* $E(u_1) < \alpha$*.*

Define

$$P = \{\, p \in C([0, 1], V) \mid p(0) = 0,\ p(1) = u_1 \,\}.$$

Then

$$\beta = \inf_{p \in P} \sup_{0 \leq t \leq 1} E(p(t)) \geq \alpha$$

is a critical value.

For $c \in \mathbb{R}$, we set

$$E_c = \{\, u \in V \mid E(u) < c \,\},\ K_c = \{\, u \in V \mid E'(u) = 0, E(u) = c \,\}.$$

Note that Theorem 6 can be shown in the same way as the proof of Theorem 6.1 in p.109 in [18] by substituting the following deformation theorem under the condition (C) for Theorem 3.4 in p.83 in [18].

Theorem 7 ([3] Theorem 1.3.)**.** *Let E satisfy the condition (C) in* (c_1, c_2)*. If* $\beta \in (c_1, c_2)$ *and N is any neighborhood of* K_β*, there exist a bounded homeomorphism* η *of V onto V and constants* $\overline{\varepsilon} > \varepsilon > 0$ *such that* $[\beta - \overline{\varepsilon}, \beta + \overline{\varepsilon}] \subset (c_1, c_2)$*, satisfying the following properties*

(I) $\eta(E_{\beta+\varepsilon} \setminus N) \subset E_{\beta-\varepsilon}$

(II) $\eta(E_{\beta+\varepsilon}) \subset E_{\beta-\varepsilon}$ *if* $K_\beta = \emptyset$

(III) $\eta(u) = u$ *if* $|E(u) - \beta| \geq \overline{\varepsilon}$*.*

We set a energy functional from $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to $\mathbb{R}$ as

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) - \int_{\mathbb{R}^N} \frac{1}{q(x)} u_+^{q(x)} dx.$$

We can check that $J \in C^1(W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N), \mathbb{R})$.

Proposition 4. *Assume that $q(x)$ satisfies the hypotheses (2), (3) in Theorem 2 and* $\operatorname{ess\,inf}_{x\in B_R} q(x) > p$*. Then J satisfies the condition (C) on $\mathbb{R}$.*

Proof. We take $c_1, c_2 \in \mathbb{R}$ with $c_1 < c_2$ arbitrary. First, we shall show that J satisfies (i) in Definition 5. Let $\{u_m\} \subset W^{1,p}_{\rm rad}(\mathbb{R}^N)$ be a bounded sequence satisfying that $J(u_m) \in (c_1, c_2)$ and $\|J'(u_m)\|_* \to 0$ as $m \to +\infty$. Then the following holds true for any $\phi \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} (\,|\nabla u_m|^{p-2}\nabla u_m \nabla\phi \;+\; |u_m|^{p-2}u_m\phi\,)\,dx - \int_{\mathbb{R}^N} (u_m)_+^{q(x)-1}\phi\,dx = o(1). \tag{31}$$

In particular, since $\{u_m\}$ is bounded it follows that

$$\int_{\mathbb{R}^N} (|\nabla u_m|^p + |u_m|^p\,)\,dx - \int_{\mathbb{R}^N} (u_m)_+^{q(x)} dx = o(1). \tag{32}$$

Likewise since $\{u_m\}$ is bounded, there exists a subsequence written as $\{u_m\}$ for simplicity and $u_0 \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$ such that $u_m \rightharpoonup u_0$ weakly in $W^{1,p}(\mathbb{R}^N)$. Put

$$G_m = \Big\langle |\nabla u_m|^{p-2}\nabla u_m - |\nabla u_0|^{p-2}\nabla u_0, \nabla u_m - \nabla u_0 \Big\rangle_{\mathbb{R}^N} + (u_m^{p-1} - u_0^{p-1})(u_m - u_0)$$

as in Section 3. In the same way as Step 2 in the proof of Theorem 3 in Section 3 by substituting (31), (32) for (22), (23) respectively we have

$$\int_{\mathbb{R}^N} G_m\,dx = o(1)$$

as $m \to \infty$ by Theorem 2. Recalling that

$$\Big\langle |b|^{p-2}b - |a|^{p-2}a, b-a \Big\rangle \geq \begin{cases} 2^{2-p}|b-a|^p & \text{if} \quad p \geq 2, \\ (p-1)|b-a|^2(1+|a|^2+|b|^2)^{\frac{p-2}{2}} & \text{if} \quad 1 \leq p \leq 2, \end{cases}$$

and consequently we have

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} (|\nabla(u_m - u_0)|^p + |u_m - u_0|^p)\,dx \leq C \lim_{m\to\infty} \int_{\mathbb{R}^N} G_m\,dx = 0.$$

This implies that $u_m \to u_0$ strongly in $W^{1,p}(\mathbb{R}^N)$.

Next, we shall show (ii). For any $c \in (c_1, c_2)$, we take some σ with $[c - \sigma, c + \sigma] \subset (c_1, c_2)$. We will choose suitable $\rho > 0$ again later. By deriving a contradiction, we show that there exists $\alpha > 0$ such that for any $u \in J^{-1}([c-\sigma, c+$

$\sigma]$) with $\|u\| \geq \rho$, $\|J'(u)\|_* \|u\| \geq \alpha$. We assume that there exists $\{u_m\} \subset W^{1,p}_{\rm rad}(\mathbb{R}^N)$ such that $u_m \in J^{-1}([c-\sigma, c+\sigma])$ with $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} \geq \rho$, and $\|J'(u_m)\|_* \|u_m\|_{W^{1,p}(\mathbb{R}^N)} =: \alpha_m \to 0$ as $m \to +\infty$. Since $J'(u_m)u_m \to 0$ as $m \to +\infty$, we have

$$\left| \|u_m\|^p_{W^{1,p}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} (u_m)_+^{q(x)}\, dx \right| \leq \alpha_m$$

which yields that

$$\begin{aligned} c + \sigma &\geq J(u_m) \\ &\geq \int_{\mathbb{R}^N} \left(\frac{1}{p} - \frac{1}{q(x)}\right) (u_m)_+^{q(x)} dx - \alpha_m. \end{aligned} \tag{33}$$

Moreover, in the same way as the proof of Proposition 3, we have

$$\|u_m\|^p_{W^{1,p}(A_n)} \leq (c + \sigma + \alpha_m) \frac{p}{C_1} \left(p + C_1 (2 \log R_0)^\ell\right) (\log R_0^n)^\ell, \tag{34}$$

where $A_n := B_{R_0^n} \setminus B_{R_0^{n-1}}$ for $n \geq 2$ and R_0 is the same as the proof of Proposition 3. By substituting (33), (34) for (24), (28), we obtain the following estimates:

$$\begin{aligned} \|u_m\|^p_{W^{1,p}(\mathbb{R}^N)} - \alpha_m &\leq \int_{B_{R_0}} (u_m)_+^{q(x)} dx + \int_{\mathbb{R}^N \setminus B_{R_0}} (u_m)_+^{q(x)} dx \\ &\leq C(R_0)(c + \sigma + \alpha_m) + \frac{2}{3} \|u_m\|^p_{W^{1,p}(\mathbb{R}^N)}, \end{aligned}$$

where $C(R_0)$ is a positive constant independent of ρ. Therefore we have

$$\begin{aligned} \|u_m\|^p_{W^{1,p}(\mathbb{R}^N)} &\leq 3\{\, \alpha_m + C(R_0)\,(c + \sigma + \alpha_m)\,\} \\ &\leq 3\{\, 1 + C(R_0)\,(c_2 + 1)\,\} \end{aligned} \tag{35}$$

for large m. If we choose sufficiently large ρ satisfying $\rho > 3^{1/p}\{\, 1 + C(R_0)(c_2 + 1)\,\}^{1/p}$, then we see that (35) contradicts $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} \geq \rho$.

The proof of Proposition 4 is now complete. □

Proposition 5. *Assume that $q(x)$ satisfies the hypotheses (2), (3) in Theorem 2 and $\operatorname{ess\,inf}_{x \in B_R} q(x) > p$. Then J has the mountain pass geometry, that is J satisfies (i), (ii) and (iii) in Theorem 6.*

Proof. (i) is obvious. We prove (ii). Let S be the best constant of the Sobolev inequality : $S\|v\|^p_{L^{p^*}(\mathbb{R}^N)} \leq \|\nabla v\|^p_{L^p(\mathbb{R}^N)}$ for $v \in C_c^\infty(\mathbb{R}^N)$. Set $q^* = \max\{p^*, p^2, \|q\|_{L^\infty(\mathbb{R}^N)}\}$.

Note that $q^* \geq p^* > pN/(N-1)$. For $u \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p}(\mathbb{R}^N)} = \gamma$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{1}{q(x)} u_+^{q(x)} dx &\leq \frac{1}{p}\int_{\mathbb{R}^N} |u|^p + \frac{1}{p}\left[\int_{B_r} |u|^{p^*} dx + \int_{\mathbb{R}^N\setminus B_r} |u|^{q^*} dx\right] \\
&\leq \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx + \frac{1}{p}\left[\left(S^{-1}\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{p^*}{p}} + \|u\|_{L^p(\mathbb{R}^N)}^{q^*\frac{p-1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^N)}^{\frac{q^*}{p}} K(r)\right] \\
&\leq \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx \left[S^{-\frac{p^*}{p}} \gamma^{p^*-p} + K(r)\gamma^{q^*-p}\right],
\end{aligned}$$

where $K(r) = (p/\omega_{N-1})^{q^*/p} \int_{\mathbb{R}^N\setminus B_r} |x|^{-q^*(N-1)/p} dx < \infty$ and the second inequality comes from Proposition 1. From this if γ is sufficiently small, we have

$$J(u) \geq \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx \left[1 - S^{-\frac{p^*}{p}}\gamma^{p^*-p} - K(r)\gamma^{q^*-p}\right] > 0. \qquad (36)$$

For $\{u_m\} \subset W^{1,p}_{\rm rad}(\mathbb{R}^N)$ and γ satisfying $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} = \gamma$ and (36), we assume that $J(u_m) \to 0$ and derive a contradiction. From (36) it follows that $\int_{\mathbb{R}^N} |\nabla u_m|^p dx \to 0$. In addition, for sufficiently large R we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{1}{q(x)} (u_m)_+^{q(x)} dx &\leq \frac{1}{p}\left(\int_{B_r} |u_m|^{q(x)} dx + \int_{B_R\setminus B_r} |u_m|^{q(x)} dx + \int_{\mathbb{R}^N\setminus B_R} |u_m|^{q(x)} dx\right) \\
&\leq \frac{1}{p}\left[\int_{B_r} |u_m|^{p^*} + \int_{B_R} |u_m|^p dx + \int_{\mathbb{R}^N\setminus B_r} |u_m|^{q^*} + \int_{\mathbb{R}^N\setminus B_R} |u_m|^{p+C_1(\log|x|)^{-\ell}} dx\right] \\
&= \frac{1}{p}(H_1 + H_2 + H_3 + H_4).
\end{aligned}$$

By using the estimates in the calculation of $\int_{\mathbb{R}^N} (u)_+^{q(x)}/q(x) dx$ to show (36) we have $H_1 = o(1)$ and $H_3 = o(1)$ as $m \to \infty$. For H_2 we have

$$H_2 \leq |B_R|^{1-\frac{p}{p^*}} S^{-1} \int_{\mathbb{R}^N} |\nabla u_m|^p = o(1).$$

We can show that H_4 is bounded uniformly for m and $H_4 \to 0$ as $R \to \infty$ in the same way as the estimate of $I_3(j, K)$ in the proof of Theorem 2. Therefore

$$\int_{\mathbb{R}^N} \frac{1}{q(x)} |u_m|^{q(x)} dx \to 0$$

as $m \to \infty$, and which implies $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} \to 0$ since $J(u_m) \to 0$ as $m \to \infty$. This contradicts $\|u_m\|_{W^{1,p}(\mathbb{R}^N)} = \gamma$.

Finally, we prove (iii). We take a smooth radial function v such that $\|v\|_{W^{1,p}(\mathbb{R}^N)} = \gamma$, $v > 0$ in B_R, where R is in the hypothesis (3). Recalling that $\underline{q} := \text{ess inf}_{x \in B_R} q(x) > p$. By taking sufficiently large t we have

$$\begin{aligned} J(tv) &= \frac{t^p}{p} \int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p) dx - \int_{\mathbb{R}^N} \frac{t^{q(x)}}{q(x)} v_+^{q(x)} dx \\ &\leq \frac{t^p}{p} \int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p) dx - t^{\underline{q}} \int_{B_R} \frac{1}{q(x)} v_+^{q(x)} dx \\ &< 0. \end{aligned}$$

Since $\|tv\|_{W^{1,p}(\mathbb{R}^N)} > \gamma$ we prove (iii). □

Proof of Theorem 3. From Proposition 4, Proposition 5, and Theorem 6, we can show the existence of a nontrivial critical point $u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ which is a weak solution to $-\Delta_p u + |u|^{p-2}u = u_+^{q(x)-1}$ in $\mathbb{R}^N$. Then we also see that $u \geq 0$ in $\mathbb{R}^N$. □

5. Appendix

In this section we show Proposition 1 and Proposition 2.

Proof of Proposition 1. It is sufficiently to show (6) holds for $f \in C_c^\infty(\mathbb{R}^N)$ with radially symmetric. We have that

$$r^{N-1}|f(r)|^p = -\int_r^\infty \frac{d}{ds}\left(s^{N-1}|f(s)|^p\right) ds.$$

By direct calculation we have

$$(s^{N-1}|f(s)|^p)' = (N-1)s^{N-2}|f(s)|^p + ps^{N-1}|f(s)|^{p-2} f(s) f(s)'.$$

Thus it follows

$$\begin{aligned} r^{N-1}|f(r)|^p &= -(N-1)\int_r^\infty s^{N-2}|f(s)|^p ds - p\int_r^\infty s^{N-1}|f(s)|^{p-2} f(s) f(s)' ds \\ &\leq p\int_r^\infty s^{N-1}|f(s)|^{p-1}|f(s)'| ds \\ &\leq \frac{p}{\omega_{N-1}} \|f\|^{p-1}_{L^p(\mathbb{R}^N)} \|\nabla f\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

Consequently (6) follows immediately. □

Proof of Proposition 2. By (6) we have

$$\int_{\mathbb{R}^N\setminus B_R} |u|^q dx \leq C_u \int_{\mathbb{R}^N\setminus B_R} |x|^{-\frac{N-1}{p}q} = C_u \int_R^\infty r^{-(N-1)\left(\frac{q}{p}-1\right)} dr,$$

where $C_u = \left(\frac{p}{\omega_{N-1}}\right)^{q/p} \|u\|^{q(p-1)/p}_{L^p(\mathbb{R}^N)} \|\nabla u\|^{q/p}_{L^p(\mathbb{R}^N)}$. When $(N-1)(q/p-1) > 1$, that is, $q > pN/(N-1)$ we have

$$\int_{\mathbb{R}^N\setminus B_R} |u|^q dx \leq C_u R^{-(N-1)\left(\frac{q}{p}-1\right)+1}.$$

Let $\{u_m\}$ be a sequence such that $u_m \rightharpoonup 0$ weakly in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$. Firstly we show that the case of $q \in (pN/(N-1), p^*)$. In this case we have

$$\int_{\mathbb{R}^N} |u_m|^q dx \leq \int_{B_R} |u_m|^q dx + C_{u_m} R^{-(N-1)\left(\frac{q}{p}-1\right)+1}.$$

Since C_{u_m} is bounded from above uniformly, letting $m \to \infty$ and $R \to \infty$ we have $u_m \to 0$ strongly in $L^q(\mathbb{R}^N)$.

Next, for $q \in (p, pN/(N-1)]$ using interpolation of L^q space, we have

$$\|u_m\|_{L^q(\mathbb{R}^N)} \leq \|u_m\|^{\lambda}_{L^p(\mathbb{R}^N)} \|u_m\|^{1-\lambda}_{L^r(\mathbb{R}^N)},$$

where $r \in (pN/(N-1), p^*)$. Since $\|u_m\|_{L^r(\mathbb{R}^N)} \to 0$ and $\|u_m\|_{L^p(\mathbb{R}^N)}$ is bounded we have $\|u_m\|_{L^q(\mathbb{R}^N)} \to 0$. □

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