GENERIC TORUS ORBIT CLOSURES IN SCHUBERT VARIETIES

EUNJEONG LEE AND MIKIYA MASUDA

Abstract. The closure of a generic torus orbit in the flag variety G/B of type A_{n-1} is known to be a permutohedral variety and well studied. In this paper we introduce the notion of a generic torus orbit in the Schubert variety X_w ($w \in \mathfrak{S}_n$) and study its closure Y_w. We identify the maximal cone in the fan of Y_w corresponding to a fixed point uB ($u \leq w$), associate a graph $\Gamma_w(u)$ to each $u \leq w$, and show that Y_w is smooth at uB if and only if $\Gamma_w(u)$ is a forest. We also introduce a polynomial $A_w(t)$ for each w, which agrees with the Eulerian polynomial when w is the longest element of $\mathfrak{S}_n$, and show that the Poincaré polynomial of Y_w agrees with $A_w(t^2)$ when Y_w is smooth.

1. Introduction

Let $G = \mathrm{GL}_n(\mathbb{C})$, $B \subset G$ the Borel subgroup of upper triangular matrices, and $T \subset B$ the torus subgroup of diagonal matrices. The left multiplication by T on G induces the T-action on the flag variety G/B. The set of T-fixed points in G/B bijectively corresponds to the symmetric group $\mathfrak{S}_n$ on the set $\{1, 2, \dots, n\}$.

Let $\mathcal{O}$ be a T-orbit and $\overline{\mathcal{O}}$ its closure. It is known that $\overline{\mathcal{O}}$ is normal ([10, Proposition 4.8]), so $\overline{\mathcal{O}}$ is a toric variety. When $\mathcal{O}$ is *generic*, which means that the closure $\overline{\mathcal{O}}$ contains all the T-fixed points in G/B, $\overline{\mathcal{O}}$ is known to be the permutohedral variety of complex dimension $n-1$. The maximal cones in the fan of the permutohedral variety are Weyl chambers in type A_{n-1}. The permutohedral variety appears in many mathematics and is well studied (see, for instance, [1], [13], [16], [18], and [20]). It is smooth and its Poincaré polynomial is given by $A_n(t^2)$ where $A_n(t)$ denotes the Eulerian polynomial associated to $\mathfrak{S}_n$.

For an element $w \in \mathfrak{S}_n$ we denote by X_w the *Schubert variety* $\overline{BwB/B}$ in G/B. The T-action on G/B leaves X_w invariant. We say that a T-orbit $\mathcal{O}$ in X_w is *generic in* X_w if the closure $\overline{\mathcal{O}}$ contains all the T-fixed points in X_w. Here the T-fixed points in X_w are uB's for $u \leq w$ in Bruhat order. When w is the longest element w_0 of $\mathfrak{S}_n$, our notion of *generic in* X_w agrees with the *generic* mentioned in the above paragraph but otherwise different.

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In this paper we study the closure, denoted by Y_w, of a generic T-orbit in X_w. First we identify maximal cones in the fan of Y_w. To each $v \in \mathfrak{S}_n$ we associate a cone in $\mathbb{R}^n$:

$$C(v) := \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid a_{v(1)} \le a_{v(2)} \le \cdots \le a_{v(n)}\}.$$

These cones projected on the quotient vector space $\mathbb{R}^n/\mathbb{R}(1,\dots,1)$ are exactly the maximal cones in the fan of the permutohedral variety Y_{w_0}. To describe the maximal cones in the fan of Y_w, we introduce an operation $v \to v'$ on $\mathfrak{S}_n$ (depending on w), which satisfies the following properties:

$$v' \le w \text{ for any } v, \quad v' = v \text{ if } v \le w, \quad \text{and} \quad w_0' = w.$$

(See Section 3 for the definition of the operation.) With this understood we have

Theorem 1.1 (Corollary 3.7)**.** *The maximal cone in the fan of Y_w corresponding to a fixed point uB in Y_w is the union of $C(v)$'s with $v' = u$ projected on the quotient vector space $\mathbb{R}^n/F_w$, where the linear subspace F_w of $\mathbb{R}^n$ is determined by the subtorus of T which fixes Y_w pointwise* (*see Remark 3.1 for F_w*)*.*

Using this description of the maximal cones (in fact, considering their dual cones), we obtain a criterion of smoothness of Y_w in terms of graphs. Indeed, we associate a graph $\Gamma_w(u)$ to each $u \le w$ and prove the following.

Theorem 1.2 (Corollary 7.12)**.** *The generic torus orbit closure Y_w in the Schubert variety X_w is smooth at a fixed point uB in Y_w if and only if the graph $\Gamma_w(u)$ is a forest. Therefore, Y_w is smooth if and only if $\Gamma_w(u)$ is a forest for every $u \le w$.*

To our surprise, the graph $\Gamma_w(u)$ with $u = w$, abbreviated as Γ_w, has been studied in [7] (see also [24]) and the following three conditions are shown to be equivalent:

(1) Γ_w is a forest.
(2) w avoids the patterns 4231 and $45\bar{3}12$ (see Section 6).
(3) X_w is factorial.

Looking at examples of $\Gamma_w(u)$, we conjecture that if Γ_w is a forest, then so is $\Gamma_w(u)$ for any $u \le w$, in other words, Y_w is smooth if Y_w is smooth at wB (Conjecture 7.16). One can also see that the fixed point idB in Y_w, where id denotes the identity permutation in $\mathfrak{S}_n$, is smooth in Y_w for any $w \in \mathfrak{S}_n$ (Corollary 7.12). These are in sharp contrast with the Schubert variety X_w because the fixed point wB is smooth in X_w for any $w \in \mathfrak{S}_n$ and X_w is smooth if and only if idB is smooth in X_w (see [5, p. 208]). This sharp contrast shows that if X_w is a toric variety, i.e. $X_w = Y_w$, then X_w is smooth because it is smooth at idB. Moreover, the smoothness at idB implies the result in [17] that X_w is a toric variety if and only if w is a product of distinct simple reflections (Corollary 7.14).

There is a moment map $\mu\colon G/B \to \mathbb{R}^n$ whose image is the convex hull of points $(u^{-1}(1), \dots, u^{-1}(n))$ in $\mathbb{R}^n$ for all $u \in \mathfrak{S}_n$, that is, the permutohedron of dimension $n-1$. It follows from [2, Theorem 2] that $\mu(Y_w)$ is the convex hull of points $(u^{-1}(1), \dots, u^{-1}(n))$ in $\mathbb{R}^n$ for all $u \le w$, which is the Bruhat interval polytope $Q_{id,w^{-1}}$ in [22]. One can see that the graph $\Gamma_w(u)$ is a forest for every $u \le w$ if and only if the polytope $Q_{id,w^{-1}}$ is simple. Our conjecture is equivalent to saying that the polytope $Q_{id,w^{-1}}$ is simple if $\mu(wB) = (w^{-1}(1), \dots, w^{-1}(n))$ is a simple vertex.

We also study the Poincaré polynomial of Y_w. We introduce a polynomial $A_w(t)$ for each w in a purely combinatorial way. The polynomial $A_w(t)$ agrees with the

Eulerian polynomial $A_n(t)$ when $w = w_0$, so the following theorem generalizes the known result on the Poincaré polynomial of the permutohedral variety Y_{w_0}.

Theorem 1.3 (Theorem 8.3)**.** *If Y_w is smooth, then the Poincaré polynomial of Y_w agrees with $A_w(t^2)$ and hence the polynomial $A_w(t)$ is palindromic and unimodal.*

When $w = 4231$ or 3412, Y_w is singular and $A_w(t)$ is not palindromic but $A_w(t^2)$ still agrees with the Poincaré polynomial of Y_w, so it would be interesting to ask whether $A_w(t^2)$ agrees with the Poincaré polynomial of Y_w for any $w \in \mathfrak{S}_n$ and palindromicity of $A_w(t)$ implies smoothness of Y_w.

This paper is organized as follows. In Section 2, we discuss about generic orbits and generic points in the Schubert variety X_w. In Section 3 we introduce the operation on $\mathfrak{S}_n$ and prove Theorem 1.1. In Section 4 we show that the set $\{v \in \mathfrak{S}_n \mid v' = u\}$ for $u \le w$ is an interval of the right weak Bruhat order. In Section 5 we identify the cone dual to the maximal cone in the fan of Y_w corresponding to wB. In Section 6 we associate the graph Γ_w to the dual cone and discuss simpliciality of the dual cone. In addition, we prove that simpliciality and smoothness are same in our case. In Section 7 we discuss the smoothness of Y_w at the other fixed points uB $(u < w)$. Indeed, we identify the cone dual to the maximal cone in the fan of Y_w corresponding to uB, introduce the graph $\Gamma_w(u)$, and prove Theorem 1.2. We also discuss when X_w is a toric variety. In Section 8 we introduce the polynomial $A_w(t)$ and prove Theorem 1.3. In Appendix A we give an alternative proof to the pattern avoidance criterion of when Γ_w is a forest. In Appendix B we compute Poincaré polynomials of Y_w when $w = 4231$ and 3412 using retraction sequences of polytopes.

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2. Generic points in Schubert varieties

Let $G = \mathrm{GL}_n(\mathbb{C})$, $B \subset G$ the Borel subgroup of upper triangular matrices and $T \subset B$ the torus subgroup of diagonal matrices. Let $\mathfrak{S}_n$ be the symmetric group on the set $\{1, 2, \dots, n\}$. An element w of $\mathfrak{S}_n$ defines a permutation matrix $[e_{w(1)} \ \cdots \ e_{w(n)}]$ where $e_1, \dots, e_n$ denote the standard column vectors in $\mathbb{R}^n$. Through this correspondence, we think of an element of $\mathfrak{S}_n$ as an element of G. For an element $w \in \mathfrak{S}_n$ we denote the *Schubert variety* $\overline{BwB/B}$ in the flag variety G/B by X_w. The left multiplication by T on G induces the T-action on G/B which leaves X_w invariant. The set of T-fixed points in G/B bijectively corresponds to the symmetric group $\mathfrak{S}_n$ through the map $v \in \mathfrak{S}_n \to vB \in G/B$ and vB lies in X_w if and only if $v \le w$ in Bruhat order (see [14, §10.5]).

Definition 2.1. We call a T-orbit in X_w *generic* if its closure contains all the T-fixed points in X_w and call a point in X_w *generic* if it is in a generic T-orbit. We will denote the closure of a generic T-orbit in X_w by Y_w.

We describe some generic points in the Schubert variety X_w using the Plücker coordinates. To introduce the Plücker coordinate, we define the set

$$I_{d,n} = \{\underline{i} = (i_1, \dots, i_d) \in \mathbb{Z}^d \mid 1 \le i_1 < \cdots < i_d \le n\}.$$

For an element $x = (x_{ij}) \in G = \mathrm{GL}_n(\mathbb{C})$, the $\underline{i}$th Plücker coordinate $p_{\underline{i}}(x)$ of x is given by the $d \times d$ minor of x, with row indices $i_1, \dots, i_d$ and the column indices $1, \dots, d$ for $\underline{i} = (i_1, \dots, i_d) \in I_{d,n}$. Then the Plücker embedding ψ is defined to be

$$\psi \colon G/B \to \prod_{d=1}^{n-1} \mathbb{C}P^{\binom{n}{d}-1}, \quad xB \mapsto \prod_{d=1}^{n-1} (p_{\underline{i}}(x))_{\underline{i} \in I_{d,n}}. \tag{2.1}$$

The map ψ is T-equivariant with respect to the action of T on $\prod_{d=1}^{n-1} \mathbb{C}P^{\binom{n}{d}-1}$ given by

$$(t_1, \dots, t_n) \cdot (p_{\underline{i}})_{\underline{i} \in I_{d,n}} := (t_{i_1} \cdots t_{i_d} \cdot p_{\underline{i}})_{\underline{i} \in I_{d,n}}$$

for $(t_1, \dots, t_n) \in T$ and $\underline{i} = (i_1, \dots, i_d)$.

Example 2.2. Suppose that $G = \mathrm{GL}_3(\mathbb{C})$. Then the Plücker embedding $\psi \colon G/B \to \mathbb{C}P^{\binom{3}{1}-1} \times \mathbb{C}P^{\binom{3}{2}-1}$ maps an element $x = (x_{ij}) \in \mathrm{GL}_3(\mathbb{C})$ to

$$\begin{aligned}&([p_1(x), p_2(x), p_3(x)], [p_{1,2}(x), p_{1,3}(x), p_{2,3}(x)]) \\ &\quad = ([x_{11}, x_{21}, x_{31}], [x_{11}x_{22} - x_{21}x_{12}, x_{11}x_{32} - x_{31}x_{12}, x_{21}x_{32} - x_{31}x_{22}]).\end{aligned}$$

Since the action of T on $\mathrm{GL}_3(\mathbb{C})$ is given by

$$(t_1, t_2, t_3) \cdot \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} t_1x_{11} & t_1x_{12} & t_1x_{13} \\ t_2x_{21} & t_2x_{22} & t_2x_{23} \\ t_3x_{31} & t_3x_{32} & t_3x_{33} \end{pmatrix},$$

one can easily check that the map ψ is T-equivariant.

Given $\underline{i} = (i_1, \dots, i_d), \underline{j} = (j_1, \dots, j_d) \in I_{d,n}$, define a partial order $\geq$ on $I_{d,n}$ by

$$\underline{i} \geq \underline{j} \iff i_t \geq j_t \text{ for all } 1 \leq t \leq d. \tag{2.2}$$

With this partial order, it is known from [5, Theorem 3.2.10] that the ideal sheaf of the Schubert variety X_w is generated by $\{p_{\underline{i}} \mid \underline{i} \in I_w\}$ where

$$I_w := \bigcup_{1 \leq d \leq n-1} \{\underline{i} \in I_{d,n} \mid w^{(d)} \not\geq \underline{i}\}.$$

Here $w^{(d)}$ denotes the ordered d-tuple obtained from $\{w(1), \dots, w(d)\}$ by arranging its elements in ascending order. It follows that

$$\underline{i} \in I_w \implies x_{\underline{i}} := p_{\underline{i}}(x) = 0 \quad \text{for any } x \in X_w. \tag{2.3}$$

Now we introduce

$$J_w := \bigcup_{1 \leq d \leq n-1} \{\underline{j} \in I_{d,n} \mid \underline{j} = v^{(d)} \text{ for some } v \leq w\}. \tag{2.4}$$

Then $p_{\underline{j}}$ for $\underline{j} \in J_w$ is not identically zero on X_w because if $\underline{j} = v^{(d)}$ for some $v \leq w$, then $p_{\underline{j}}(vB) = \pm 1$. Since the set of points $x \in X_w$ with $p_{\underline{j}}(x) = 0$ is of codimension one in X_w and $|J_w|$ is finite, there exists a point $x \in X_w$ such that $x_{\underline{j}} \neq 0$ for any $\underline{j} \in J_w$. We will see in Proposition 3.8 that such a point x is generic in X_w.

Example 2.3. Let $w = 312 \in \mathfrak{S}_3$ in the one-line notation. Then we have $w^{(1)} = (3)$, $w^{(2)} = (1,3)$, $w^{(3)} = (1,2,3)$. Hence $p_{\underline{i}} = 0$ on X_w if $\underline{i} = (2,3)$, and the following points

$$x = \begin{pmatrix} \alpha & 1 & 0 \\ \beta & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B \qquad (\alpha, \beta \in \mathbb{C} \backslash \{0\})$$

satisfy that $x_{\underline{j}} \neq 0$ for any $\underline{j} \in J_w = \{(1),(2),(3),(1,2),(1,3)\}$.

3. Fan of the generic torus orbit closure Y_w

The set $\mathrm{Hom}(\mathbb{C}^*, T)$ of algebraic homomorphisms from $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ to T forms an abelian group under the multiplication of T and $\mathrm{Hom}(\mathbb{C}^*, T)$ is isomorphic to $\mathbb{Z}^n$ through the correspondence

$$(a_1,\dots,a_n) \in \mathbb{Z}^n \to \big(t \mapsto (t^{a_1},\dots,t^{a_n})\big) \in \mathrm{Hom}(\mathbb{C}^*, T). \tag{3.1}$$

In the following we identify $\mathrm{Hom}(\mathbb{C}^*, T)$ with $\mathbb{Z}^n$ through (3.1) and $\mathrm{Hom}(\mathbb{C}^*, T)\otimes\mathbb{R}$ with $\mathbb{R}^n$ but we often denote the element of $\mathrm{Hom}(\mathbb{C}^*, T)$ corresponding to $\mathbf{a} \in \mathbb{Z}^n$ by $\lambda^{\mathbf{a}}$.

Remark 3.1. Since the action of T on Y_w is not effective, the ambient space of the fan of Y_w is the quotient of $\mathrm{Hom}(\mathbb{C}^*, T)\otimes\mathbb{R}$ by the subspace $\mathrm{Hom}(\mathbb{C}^*, T_w)\otimes\mathbb{R}$, where T_w is the toral subgroup of T which fixes Y_w pointwise (F_w in Theorem 1.1 is $\mathrm{Hom}(\mathbb{C}^*, T_w)\otimes\mathbb{R}$). However, for simplicity, we will think of $\mathbb{R}^n$ as the ambient space of the fan of Y_w throughout the paper.

We will find the maximal cones in the fan of Y_w using the Orbit-Cone correspondence, see [12, Proposition 3.2.2.]. Namely, we observe the limit point $\lim_{t\to 0}\lambda^{\mathbf{a}}(t)\cdot x$ for a generic point $x \in X_w$ and $\mathbf{a} \in \mathbb{Z}^n$. It is known and not difficult to see that Y_{w_0}, where w_0 is the longest element of $\mathfrak{S}_n$, is the permutohedral variety of complex dimension $n-1$ and the cones

$$C(v) := \{(a_1,\dots,a_n) \in \mathbb{R}^n \mid a_{v(1)} \le a_{v(2)} \le \cdots \le a_{v(n)}\} \tag{3.2}$$

for $v \in \mathfrak{S}_n$ are the maximal cones in the fan of Y_{w_0}. Unless $w = w_0$, we will see that a maximal cone in the fan of Y_w is the union of some of $C(v)$'s (Corollary 3.7). Here is an example which shows how to find the maximal cones in the fan of Y_w.

Example 3.2. Suppose that $G = \mathrm{GL}_3(\mathbb{C})$ and $w = 312 \in \mathfrak{S}_3$. Take a (generic) point x in Example 2.3. Then Example 2.2 shows that for $\mathbf{a} = [a_1, a_2, a_3] \in \mathbb{Z}^3$, the corresponding curve in the product $\mathbb{C}P^2 \times \mathbb{C}P^2$ of projective spaces is given by

$$\begin{aligned}\lambda^{\mathbf{a}}(t)\cdot\psi(x) &= (t^{a_1}, t^{a_2}, t^{a_3})\cdot([\alpha,\beta,1],[-\beta,-1,0])\\ &= ([t^{a_1}\alpha, t^{a_2}\beta, t^{a_3}],[-t^{a_1+a_2}\beta, -t^{a_1+a_3}, 0]).\end{aligned}$$

Take $v = 321$. Then for any $\mathbf{a} = (a_1,a_2,a_3) \in \mathrm{Int}(C(321))\cap\mathbb{Z}^3$ (so that $a_3 < a_2 < a_1$), we have that

$$\begin{aligned}\lim_{t\to 0}\lambda^{\mathbf{a}}(t)\cdot\psi(x) &= \lim_{t\to 0}([t^{a_1}\alpha, t^{a_2}\beta, t^{a_3}],[-t^{a_1+a_2}\beta, -t^{a_1+a_3}, 0])\\ &= \lim_{t\to 0}([t^{a_1-a_3}\alpha, t^{a_2-a_3}\beta, 1],[t^{a_2-a_3}\beta, 1, 0])\\ &= ([0,0,1],[0,1,0])\\ &= \psi(312B).\end{aligned}$$

A similar argument shows that the limit point corresponding to each $C(v)$ is as follows:

$C(123)\colon ([1,0,0],[1,0,0])$, $C(132)\colon ([1,0,0],[0,1,0])$, $C(213)\colon ([0,1,0],[1,0,0])$,
$C(231)\colon ([0,1,0],[1,0,0])$, $C(312)\colon ([0,0,1],[0,1,0])$, $C(321)\colon ([0,0,1],[0,1,0])$.

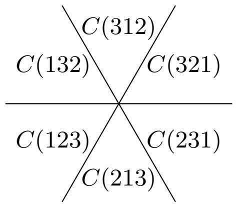


FIGURE 1. Cones $C(v)$ for $v \in \mathfrak{S}_3$.

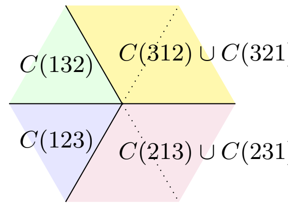


FIGURE 2. The fan of Y_{312}.

Hence there are four limit points and accordingly the fan of Y_w consists of four maximal cones:

$$C(123),\ C(132),\ C(213) \cup C(231),\ C(312) \cup C(321).$$

See Figure 2. We note that Y_w is a Hirzebruch surface $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. In this case, the Schubert variety X_{312} itself is a toric variety, i.e. $X_{312} = Y_{312}$. We will discuss when X_w is a toric variety in general in Section 7.

Motivated by the above observation, we introduce an operation on $\mathfrak{S}_n$ with respect to w.

Definition 3.3. Fix $w \in \mathfrak{S}_n$. For $v \in \mathfrak{S}_n$, we inductively choose

$$\begin{aligned}
i_1 &:= \min\{i \in [n] \mid v(i) \leq w(1) = w^{(1)}\},\\
i_2 &:= \min\{i \in [n] \setminus \{i_1\} \mid \{v(i_1), v(i)\}\uparrow \leq w^{(2)}\},\\
&\ \vdots\\
i_n &:= \min\{i \in [n] \setminus \{i_1, \dots, i_{n-1}\} \mid \{v(i_1), \dots, v(i_{n-1}), v(i)\}\uparrow \leq w^{(n)}\},
\end{aligned}$$

and define

$$v' := v(i_1)v(i_2)\cdots v(i_n) \in \mathfrak{S}_n.$$

Here, $\{a_1, \dots, a_d\}\uparrow$ denotes the ordered d-tuple obtained from $\{a_1, \dots, a_d\}$ by arranging its elements in ascending order and $w^{(d)} = \{w(1), \dots, w(d)\}\uparrow$.

The operation v' for $v \in \mathfrak{S}_n$ will play an important role in our argument. It depends on w but since w is fixed throughout the paper, we will not specify w in the notation of the operation for simplicity. As is well-known,

$$v \leq w \iff v^{(d)} \leq w^{(d)} \quad \text{for all } 1 \leq d \leq n-1 \tag{3.3}$$

(see [5, (3.2.5)]), so the following lemma immediately follows from the definition of the operation.

Lemma 3.4. *The operation v' for $v \in \mathfrak{S}_n$ has the following properties:*

(1) *$v' \leq w$ for any $v \in \mathfrak{S}_n$,*
(2) *$v' = v$ if $v \leq w$,*
(3) *$w_0' = w$ where w_0 is the longest element of $\mathfrak{S}_n$ as before.*

Example 3.5. Let $w = 3412$ and $v = 4123$. Then $i_1, \dots, i_4$ are given as follows:

$$\begin{aligned} i_1 &= \min\{i \in [4] \mid v(i) \leq w(1) = 3\} = 2, \\ i_2 &= \min\{i \in [4] \setminus \{2\} \mid \{1, v(i)\} \uparrow \leq (3,4)\} = 1, \\ i_3 &= \min\{i \in [4] \setminus \{1,2\} \mid \{1,4,v(i)\} \uparrow \leq (1,3,4)\} = 3, \\ i_4 &= 4. \end{aligned}$$

Hence $v' = v(2)v(1)v(3)v(4) = 1423$.

The following proposition is a key observation.

Proposition 3.6. *Let x be a point in X_w with $x_{\underline{j}} \neq 0$ for any $\underline{j} \in J_w$ (see* (2.4) *for J_w). Then for any $v \in \mathfrak{S}_n$ and any $\mathbf{a} \in \mathrm{Int}(C(v)) \cap \mathbb{Z}^n$, we have*

$$\lim_{t \to 0} \lambda^{\mathbf{a}}(t) \cdot x = v'B.$$

Proof. Suppose that $v \leq w$. Then, since $v^{(d)} \leq w^{(d)}$ for any $1 \leq d \leq n-1$ by (3.3), we have $v^{(d)} \in J_w$ and hence

$$x_{v^{(d)}} \neq 0 \quad \text{for all } 1 \leq d \leq n-1.$$

On the other hand, since $\mathbf{a} = (a_1, \dots, a_n) \in \mathrm{Int}(C(v)) \cap \mathbb{Z}^n$, we have $a_{v(1)} < a_{v(2)} < \cdots < a_{v(n)}$. Therefore the sum $\sum_{k=1}^d a_{v(k)}$ is smallest among the sum of arbitrary d elements in $a_1, \dots, a_n$. Then, the same argument as in Example 3.2 shows that

$$\lim_{t \to 0} \lambda^{\mathbf{a}}(t) \cdot x = vB.$$

Here $v' = v$ by Lemma 3.4 because $v \leq w$. This proves the proposition when $v \leq w$.

Suppose that $v \not\leq w$. Since $v' \leq w$ by Lemma 3.4, ${v'}^{(d)} \in J_w$ and hence

$$x_{v'^{(d)}} \neq 0 \quad \text{for all } 1 \leq d \leq n-1.$$

By (2.3) we have $x_{\underline{i}} = 0$ for $\underline{i} \in I_{d,n}$ with $\underline{i} \not\leq w^{(d)}$, so the construction of v' shows that the sum $\sum_{k=1}^d a_{v'(k)}$ is smallest among the sum $\sum_{k \in \underline{i}} a_k$ for $\underline{i} \in I_{d,n}$ with $x_{\underline{i}} \neq 0$. This implies the desired identity in the proposition as before. □

The Orbit-Cone correspondence and Proposition 3.6 imply the following.

Corollary 3.7. *The maximal cone in the fan of Y_w corresponding to a fixed point uB $(u \leq w)$ is of the form*

$$C_w(u) := \bigcup_{v \in \mathfrak{S}_n \ s.t. \ v' = u} C(v).$$

Using Proposition 3.6, we can characterize generic points in X_w as follows.

Proposition 3.8. *A point x in X_w is generic in X_w if and only if $x_{\underline{j}} \neq 0$ for any $\underline{j} \in J_w$. (Therefore, there exists a generic point in X_w as explained after* (2.4).*)*

Proof. Suppose that $x_{\underline{j}} \neq 0$ for any $\underline{j} \in J_w$. Then Proposition 3.6 shows that the orbit closure $\overline{T \cdot x}$ contains the fixed point $v'B$ for any v but $v' = v$ if $v \leq w$ by Lemma 3.4 (2). Hence $\overline{T \cdot x}$ contains all the T-fixed points in X_w, which means that x is generic in X_w, proving the "if"part in the proposition.

Suppose that x is a generic point in X_w but $x_{\underline{j}} = 0$ for some $\underline{j} \in J_w$. Then the $\underline{j}$-th Plücker coordinate of the limit point $\lim_{t \to 0} \lambda^{\mathbf{a}}(t) \cdot x$ vanishes for any $\mathbf{a} \in \mathbb{Z}^n$, which means that the $\underline{j}$-th coordinate of any T-fixed point in $\overline{T \cdot x}$ vanishes. On the other hand, there is some $v \leq w$ with $v^{(d)} = \underline{j}$ by definition of J_w where d is the length of $\underline{j}$. Since $v^{(k)}$-th Plücker coordinate of the T-fixed point vB does not vanish for any $1 \leq k \leq n$, this shows that $vB \notin \overline{T \cdot x}$ although $v \leq w$. This contradicts x being generic in X_w. Therefore $x_{\underline{j}} \neq 0$ for any $\underline{j} \in J_w$, proving the "only if "part in the proposition. □

4. Right weak order

In this section, we describe the set $\{v \in \mathfrak{S}_n \mid v' = u\}$ in Corollary 3.7 using the right weak (Bruhat) order on $\mathfrak{S}_n$ (see [6, §3.1]). For $u_1, u_2 \in \mathfrak{S}_n$, $u_1 \leq_R u_2$ means that $u_2 = u_1 s_{i_1} \cdots s_{i_k}$ for some simple reflections $s_{i_1}, \dots, s_{i_k}$ such that $\ell(u_1 s_{i_1} \dots s_{i_j}) = \ell(u_1) + j$ for $0 \leq j \leq k$, where $\ell(v)$ denotes the length of $v \in \mathfrak{S}_n$. This order is called the *right weak order*. The right weak order interval $[u_1, u_2]_R$ is defined to be the set $[u_1, u_2]_R := \{x \in \mathfrak{S}_n \mid u_1 \leq_R x \leq_R u_2\}$.

Lemma 4.1. *$v \geq_R v'$ for any $v \in \mathfrak{S}_n$.*

Proof. There is $0 \leq d \leq n$ such that $v(i) = v'(i)$ for $1 \leq i \leq d$. We shall prove the lemma by downward induction on d starting from n. If $d = n$, then $v = v'$ and the lemma trivially holds in this case. We assume that the lemma holds for v with $v(i) = v'(i)$ for $1 \leq i \leq d$ and prove the lemma for v with $v(i) = v'(i)$ for $1 \leq i \leq d-1$. We may assume $v(d) \neq v'(d)$. Define $j \in [n]$ by $v(j) = v'(d)$. Since $v(i) = v'(i)$ for $1 \leq i \leq d-1$, we have $d < j$ and it follows from the construction of v' that

$$\begin{aligned} &\{v(1), \dots, v(d-1), v(i)\} \uparrow \not\leq w^{(d)} \qquad \text{for } d \leq i < j, \\ &\{v(1), \dots, v(d-1), v(j)\} \uparrow \leq w^{(d)}. \end{aligned} \tag{4.1}$$

In particular

$$v(i) > v(j) \text{ for } d \leq i < j. \tag{4.2}$$

We consider $u \in \mathfrak{S}_n$ defined by

$$u = v(1) \cdots v(d-1) v(j) v(d) \cdots v(j-1) v(j+1) \cdots v(n).$$

Since $v(i) = v'(i)$ for $1 \leq i \leq d-1$, it follows from (4.1) and (4.2) that we have

$$\begin{aligned} &u(i) = v'(i) \quad \text{for } 1 \leq i \leq d, \\ &u' = v', \\ &v = u s_d s_{d+1} \cdots s_{j-1} \quad \text{with } \ell(v) = \ell(u) + (j-d), \text{ so } v \geq_R u. \end{aligned}$$

Because of the first and second identities above, the induction assumption can be applied to u, so that $u \geq_R u'$. This together with the second and last identities above shows $v \geq_R v'$. □

Proposition 4.2. *Let $u \leq w$. Then there exists a (unique) $u_w \in \mathfrak{S}_n$ such that*

$$\{v \in \mathfrak{S}_n \mid v' = u\} = [u, u_w]_R.$$

Proof. We denote the left hand side in the proposition by $S(u)$. It suffices to prove that if

(1) $\ell(v s_p) = \ell(v s_q) = \ell(v) + 1$ for some $p < q$, and,
(2) $v' = (v s_p)' = (v s_q)' = u$,

then there exists $\tilde{v} \in \mathfrak{S}_n$ such that $\{vs_p, vs_q\} \subset [v, \tilde{v}]_R \subset S(u)$. We note that

$$v(p) < v(p+1), \quad v(q) < v(q+1) \tag{4.3}$$

by (1) above. We define $i, j \in [n]$ by

$$v(p) = u(i), \quad v(q) = u(j).$$

Then it follows from (2) above that

$$\begin{aligned} &\{u(1), \dots, u(i-1), v(p)\} \uparrow \leq w^{(i)}, \quad \{u(1), \dots, u(i-1), v(p+1)\} \uparrow \not\leq w^{(i)}, \\ &\{u(1), \dots, u(j-1), v(q)\} \uparrow \leq w^{(j)}, \quad \{u(1), \dots, u(j-1), v(q+1)\} \uparrow \not\leq w^{(j)}. \end{aligned} \tag{4.4}$$

We consider two cases.

Case 1. The case where $q - p \geq 2$. In this case, s_p and s_q commute and we take $\tilde{v} = vs_ps_q$. Then $[v, \tilde{v}]_R = \{v, vs_p, vs_q, \tilde{v} = vs_ps_q\}$, and (2) above implies $\tilde{v}' = u$. Therefore $\{vs_p, vs_q\} \subset [v, \tilde{v}]_R \subset S(u)$.

Case 2. The case where $q-p = 1$, i.e. $q = p+1$. In this case, it follows from (4.3) that

$$v(p) < v(p+1) < v(p+2). \tag{4.5}$$

Note that $i < j$ since $v' = u$ and $u(i) = v(p) < v(p+1) = u(j)$. We take $\tilde{v} = vs_ps_{p+1}s_p$, i.e.

$$\tilde{v} = v(1) \cdots v(p-1)v(p+2)v(p+1)v(p)v(p+3) \cdots v(n). \tag{4.6}$$

Then

$$[v, \tilde{v}]_R = \{v,\ vs_p,\ vs_{p+1},\ vs_ps_{p+1},\ vs_{p+1}s_p,\ \tilde{v} = vs_ps_{p+1}s_p\}.$$

Since $s_ps_{p+1}s_p = s_{p+1}s_ps_{p+1}$ and $q = p+1$, we have $\tilde{v} \geq_R vs_p$ and $\tilde{v} \geq_R vs_q$ by (4.5). By (4.4) and (4.5) we have

$$\{u(1), \dots, u(i-1), v(p+1)\} \uparrow \not\leq w^{(i)}, \quad \{u(1), \dots, u(i-1), v(p+2)\} \uparrow \not\leq w^{(i)}.$$

This together with the first inequality in (4.4), (4.6) and the assumption $v' = u$ show that $\tilde{v}'(i) = v(p) = u(i)$. Then, the conclusion $\tilde{v}' = u$ follows from the second line in (4.4) and the assumption $v' = u$. Since

$$\begin{aligned} vs_ps_{p+1} &= v(1) \cdots v(p-1)v(p+1)v(p+2)v(p)v(p+3) \cdots v(n), \\ vs_{p+1}s_p &= v(1) \cdots v(p-1)v(p+2)v(p)v(p+1)v(p+3) \cdots v(n), \end{aligned}$$

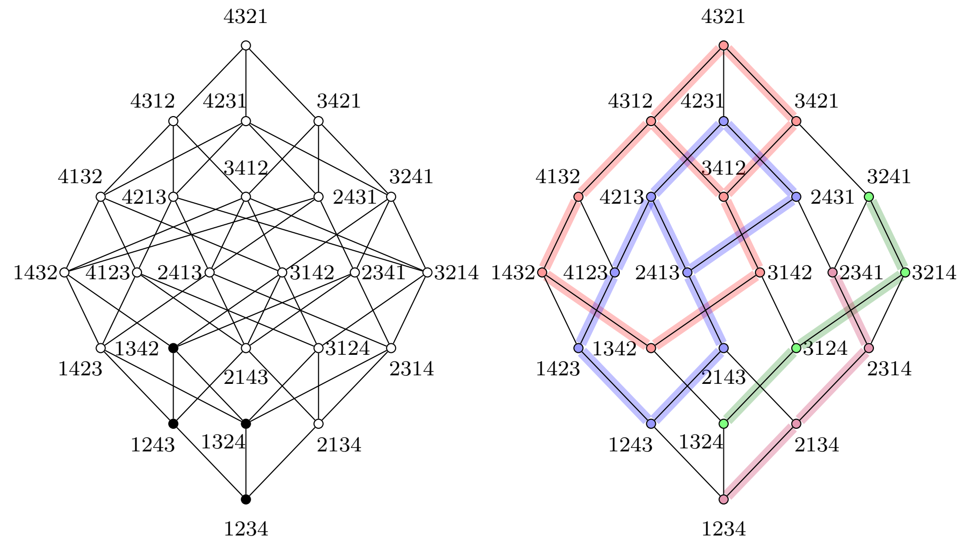
the same observation for $\tilde{v}$ shows that $(vs_ps_{p+1})' = u = (vs_{p+1}s_p)'$. Therefore $\{vs_p, vs_q\} \subset [v, \tilde{v}]_R \subset S(u)$ in this case, too. □

Example 4.3. Take $w = 1342$. Then there are four elements $u \in \mathfrak{S}_4$ such that $u \leq w$ (see Figure 3(1)) and one can check the following (see Figure 3(2)):

u	1342	1243	1324	1234
u_w	4321	4231	3241	2341

TABLE 1. u_w when $w = 1342$.

$$\begin{aligned} [1342, 4321]_R &= \{1342, 3142, 1432, 4132, 3412, 4312, 3421, 4321\}, \\ [1243, 4231]_R &= \{1243, 2143, 1423, 4123, 2413, 4213, 2431, 4231\}, \\ [1324, 3241]_R &= \{1324, 3124, 3214, 3241\}, \\ [1234, 2341]_R &= \{1234, 2134, 2314, 2341\}. \end{aligned}$$



(1) Bruhat order of $\mathfrak{S}_4$ and elements smaller than or equal to 1342 are marked by $\bullet$.

(2) Right weak order of $\mathfrak{S}_4$ and right weak order intervals in Table 1.

FIGURE 3. Bruhat order and right weak order of $\mathfrak{S}_4$.

5. DUAL CONE OF $C_w(w)$

By Corollary 3.7 and Proposition 4.2, the maximal cone in the fan of Y_w corresponding to the fixed point uB ($u \leq w$) is of the form

$$C_w(u) = \bigcup_{v \in [u, u_w]_R} C(v).$$

As noted in Lemma 3.4 $w_0' = w$, so $u_w = w_0$ when $u = w$. Hence the maximal cone corresponding to wB is of the form

$$C_w := C_w(w) = \bigcup_{v \in [w, w_0]_R} C(v).$$

Our purpose of this section is to identify the dual of the maximal cone C_w (Proposition 5.3).

Definition 5.1. For $w \in \mathfrak{S}_n$, we define

$$E_w := \{(w(i), w(j)) \mid 1 \leq i < j \leq n,\ \ell(w) - \ell(t_{w(i),w(j)}w) = 1\}$$

where $t_{a,b}$ denotes the transposition of a and b and $\ell(v)$ denotes the length of a permutation v as before. (Note. The condition $\ell(w) - \ell(t_{w(i),w(j)}w) = 1$ above is equivalent to $w(i) > w(j)$ and $w(k) \notin [w(j), w(i)]$ for $i < \forall k < j$.)

Example 5.2. (1) If $w = 3152674$, then

$$E_w = \{(3,1), (3,2), (5,2), (5,4), (6,4), (7,4)\}.$$

(2) If $w = 3715264$, then

$$E_w = \{(3,1), (3,2), (7,1), (7,5), (7,6), (5,2), (5,4), (6,4)\}.$$

We define

$$D_w := \text{ the cone in } \mathbb{R}^n \text{ spanned by } \{e_b - e_a \mid (a,b) \in E_w\}. \tag{5.1}$$

Since $C(v)$'s are the closed regions divided by the hyperplanes defined by $e_i - e_j$ $(1 \le i < j \le n)$, it follows from (5.1) that the dual cone of D_w in $\mathbb{R}^n$, denoted by $D_w^\vee$, is the union of some of $C(v)$'s.

Proposition 5.3. $D_w^\vee = C_w$.

This proposition follows from the following two lemmas. Remember that $u_w = w_0$ when $u = w$ and hence $v \in [w, w_0]_R$ if and only if $v' = w$ by Proposition 4.2.

Lemma 5.4. *If $v \in [w, w_0]_R$, i.e. $v' = w$, then $C(v)$ is contained in $D_w^\vee$.*

Proof. Since $v \in [w, w_0]_R$, one can write

$$v = ws_{i_1} \cdots s_{i_k}$$

where $\ell(v) = \ell(w) + k$. We prove the lemma by induction on k. If $k = 0$, then $v = w$ and the lemma is obvious. Suppose that $k \ge 1$ and the lemma holds for $k - 1$. We set

$$u = ws_{i_1} \cdots s_{i_{k-1}} \qquad \text{and} \qquad i_k = p.$$

Then, since $v = us_p$, we have

$$\begin{aligned} & v = v(1) \cdots v(p-1)v(p)v(p+1)v(p+2) \cdots v(n), \\ & u = v(1) \cdots v(p-1)v(p+1)v(p)v(p+2) \cdots v(n), \quad \text{and} \\ & \ell(v) = \ell(u) + 1. \end{aligned} \tag{5.2}$$

Take any element (a,b) of E_w. By induction assumption we have

$$\langle e_b - e_a, x \rangle \ge 0 \qquad (\forall x \in C(u))$$

where $\langle\ ,\ \rangle$ denotes the standard inner product on $\mathbb{R}^n$. The above inequality is equivalent to a being ahead of b in the one-line notation for u. What we have to prove is that a is still ahead of b in the one-line notation for v. We take two cases.

Case 1. The case where $\{v(p), v(p+1)\} \ne \{a, b\}$. In this case, it is easy to see from (5.2) that a is still ahead of b in the one-line notation for v since so is for u.

Case 2. The case where $\{v(p), v(p+1)\} = \{a, b\}$. Since a is ahead of b in the one-line notation for u, we have $a = v(p+1)$ and $b = v(p)$ by (5.2) in this case. We note that $a > b$ since $(a,b) \in E_w$. This together with $v = us_p$ shows that $\ell(v) = \ell(u) - 1$ but this contradicts the last identity in (5.2). Therefore Case 2 does not occur. □

Lemma 5.5. *If $v \notin [w, w_0]_R$, i.e. $v' \ne w$, then $C(v)$ is not contained in $D_w^\vee$.*

Proof. Since $v' < w$ by assumption and Lemma 3.4 (1), there exists $i \in [n]$ such that

$$v'(j) = w(j) \quad (1 \le j \le i-1), \qquad v'(i) < w(i). \tag{5.3}$$

Define q and m by

$$v'(i) = v(q) = w(m). \tag{5.4}$$

It follows from (5.3) that

$$i < m \tag{5.5}$$

and from the construction of v', (5.3) and (5.4) that

$$\text{any element in } \{v(1),\dots,v(q-1)\}\backslash\{w(1),\dots,w(i-1)\} > w(i). \tag{5.6}$$

Since $i < m$ by (5.5) but $w(i) > w(m)$ by (5.3) and (5.4), there exists ℓ between i and $m-1$ such that

$$w(i) \geq w(\ell) > w(m) \qquad \text{and} \qquad (w(\ell), w(m)) \in E_w. \tag{5.7}$$

Now we define p by $v(p) = w(\ell)$. Then, since $i \leq \ell$, we have

$$v(p) = w(\ell) \notin \{w(1),\dots,w(i-1)\}. \tag{5.8}$$

It follows from (5.4), (5.6), (5.7) and (5.8) that

$$q < p \qquad \text{and} \qquad v(q) = w(m) < w(\ell) = v(p).$$

Since $q < p$, $e_{v(q)} - e_{v(p)}$ takes a negative value on $C(v)$ (through the inner product) but $(v(p), v(q)) = (w(\ell), w(m)) \in E_w$ by (5.7). This shows that $C(v)$ is not contained in $D_w^\vee$, proving the lemma. □

The two lemmas above imply Proposition 5.3.

6. Simpliciality of D_w and graph Γ_w

Let Γ_w be the graph associated to E_w, i.e., the vertices of Γ_w are the positive integers appearing in E_w (so the vertices of Γ_w are elements in $[n]$) and elements in E_w are edges. We show that Y_w is smooth at wB if and only if Γ_w is a forest (Theorem 6.8).

Example 6.1. Following Example 5.2, we have graphs $\Gamma_{3152674}$ and $\Gamma_{3715264}$ as in Figures 4 and 5.

Lemma 6.2. *If $(a,b) \in E_w$, then $e_b - e_a$ is an edge vector of D_w.*

Proof. Suppose that

$$e_b - e_a = \sum_{i=1}^{k} c_i(e_{b_i} - e_{a_i}) \qquad \text{with } c_i > 0 \tag{6.1}$$

where $(a,b) \neq (a_i,b_i) \in E_w$ for $i = 1,\dots,k$. Suppose $k \geq 2$ and we deduce a contradiction. Since $(a_i,b_i) \in E_w$, we have $a_i > b_i$. Therefore, it follows from (6.1) that $a = \max_{1\leq i\leq k}\{a_i\}$. We may assume $a_1 = a$ if necessary by changing indices. Since

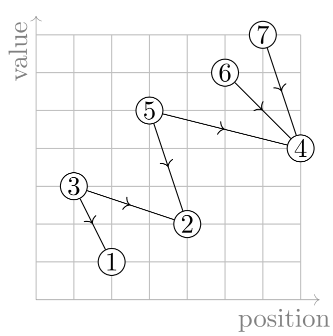


Figure 4. Graph $\Gamma_{3152674}$.

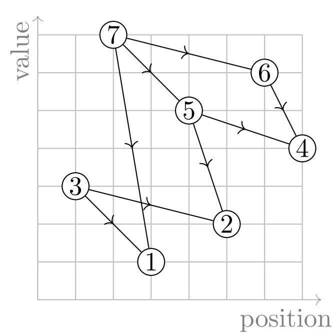


Figure 5. Graph $\Gamma_{3715264}$.

$(a_1,b_1)\neq(a,b)$ and $a_1=a$, we have $b_1\neq b$. In order for (6.1) to hold, e_{b_1} must be killed at the right hand side of (6.1). This means that there exists some $i\neq 1$ such that $a_i=b_1$. We may assume $a_2=b_1$ if necessary by changing indices. If $b_2\neq b$, then we repeat the same argument and may assume that $a_3=b_2$. We repeat this argument. Then we reach $m\geq 2$ such that $b_m=b$, i.e. we obtain the following sequence of pairs in E_w:

$$\begin{aligned}&(a_1,b_1),\ (a_2,b_2),\ \dots,\ (a_m,b_m)\\ &\text{such that } a_1=a,\ b_m=b,\ b_i=a_{i+1} \text{ for } 1\leq i\leq m-1.\end{aligned} \tag{6.2}$$

Remember that since $(a_i,b_i)\in E_w$, a_i is ahead of b_i in the one-line notation for w. Since $m\geq 2$, this together with (6.2) shows that some positive integer x with $a>x>b$ (e.g. $x=b_1=a_2$) appears between a and b in the one-line notation for w. This contradicts (a,b) being in E_w. □

Corollary 6.3. *The dual cone D_w is simplicial if and only if $|E_w|=\dim D_w$.*

We note that unless the identity $|E_w|=\dim D_w$ is satisfied, Y_w is singular by Proposition 5.3. Here is a way to find $\dim D_w$. The idea is to project D_w successively along vectors e_b-e_a for $(a,b)\in E_w$.

Example 6.4. (1) Take $w=3152674$ in Example 5.2(1). The cone D_w is spanned by

$$\{e_1-e_3,\ e_2-e_3,\ e_2-e_5,\ e_4-e_5,\ e_4-e_6,\ e_4-e_7\}.$$

Take any element from the above, say e_4-e_5, and set $e_4=e_5$. Then the resulting vectors are

$$\{e_1-e_3,\ e_2-e_3,\ e_2-e_4,\ e_4-e_6,\ e_4-e_7\}.$$

Next, take e_4-e_6 for example and set $e_4=e_6$. Then the resulting vectors are

$$\{e_1-e_3,\ e_2-e_3,\ e_2-e_4,\ e_4-e_7\}.$$

Continue this procedure. Then all the vectors above vanish after four more times. Therefore, $\dim D_w=6$. Since $|E_w|$ is also 6, D_w is simplicial and hence the merged cone C_w in the fan is also simplicial by Proposition 5.3. The corresponding graph Γ_w is a path graph, see Figure 4.

(2) Take $w=3715264$ in Example 5.2(2). Then $|E_w|=8$ but $\dim D_w=6$. Therefore Y_w is singular. The corresponding graph Γ_w is connected and $b_1(\Gamma_w)=2$ where b_1 denotes the first Betti number, see Figure 5.

(3) Take $w=3412$. Then $|E_w|=4$ but $\dim D_w=3$. Therefore Y_w is singular. The corresponding graph Γ_w is connected and $b_1(\Gamma_w)=1$, see Figure 6.

(4) Take $w=341265$. Then $|E_w|=5$ but $\dim D_w=4$. Therefore Y_w is singular. The corresponding graph Γ_w has two connected components and $b_1(\Gamma_w)=1$, see Figure 7.

Since $|E_w|$ is nothing but the number of the edges in Γ_w, the same argument as in Example 6.4 proves the following.

Lemma 6.5. *We have*

$$\dim_{\mathbb{C}} Y_w=\dim D_w=|\text{vertices of }\Gamma_w|-|\text{connected components of }\Gamma_w|.$$

Therefore

$$|E_w|-\dim D_w=b_1(\Gamma_w)\ \geq 0,$$

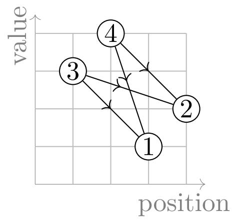


FIGURE 6. Graph Γ_{3412}.

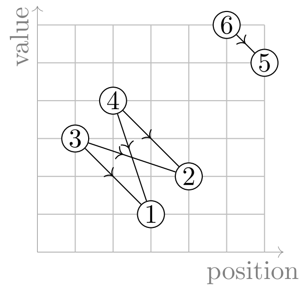


FIGURE 7. Graph Γ_{341265}.

where $b_1(\Gamma_w)$ denotes the first Betti number of Γ_w, and hence D_w is simplicial if and only if Γ_w is a forest (as an undirected graph) by Corollary 6.3.

Remember that our cone D_w has $\{e_b - e_a \mid (a,b) \in E(\Gamma_w)\}$ as edge vectors (Lemma 6.2). They lie in the linear subspace, denoted by H^{n-1}, of $\mathbb{R}^n$ with the sum of the coordinates equals to zero. In general, we have the following.

Lemma 6.6. *Let Γ be a directed graph with vertices in $\{1,\dots,n\}$ and let D be a cone in $\mathbb{R}^n$ with $\{e_b - e_a \mid (a,b) \in E(\Gamma)\}$ as edge vectors, where $E(\Gamma)$ denotes the set of directed edges in Γ. Then the following are equivalent:*

(1) *D is simplicial.*
(2) *Γ is a forest (as an undirected graph).*
(3) *D is non-singular, i.e. the set $\{e_b - e_a \mid (a,b) \in E(\Gamma)\}$ is a part of a $\mathbb{Z}$-basis of $H^{n-1} \cap \mathbb{Z}^n$.*

In particular, D_w is simplicial if and only if D_w is non-singular.

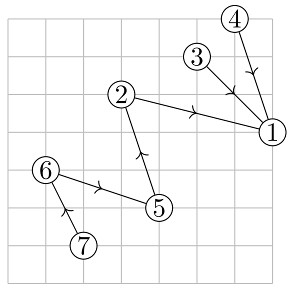
Proof. The equivalence of (1) and (2) can be seen by the same argument as above for D_w. It is obvious that (3) implies (1), so it is enough to show that (2) implies (3). We may assume that Γ is a tree. To prove (3), we may change the order of coordinates of $\mathbb{R}^n$ and note that changing the order of the coordinates is nothing but relabeling the vertices of Γ.

We shall relabel the vertices of Γ. For any two vertices x and y of Γ, there is a unique path connecting them since Γ is a tree. We define the distance $d(x,y)$ to be the number of edges in the path. Choose any vertex of Γ and label it as 1. For any positive integer d, consider the set

$$A_d := \{v \in V(\Gamma) \mid d(v,1) = d\}$$

where $V(\Gamma)$ denotes the set of vertices of Γ. Then we label the vertices in A_1 as $2, 3, \dots, |A_1|+1$ and then label the vertices in A_2 as $|A_1|+2, |A_1|+3, \dots, |A_1|+|A_2|+1$ and so on. We orient each edge (a,b) of this relabeled graph in such a way that $a > b$ and denote the relabeled directed graph by $\bar{\Gamma}$. Then it is not difficult to see that $\{e_b - e_a \mid (a,b) \in E(\bar{\Gamma})\}$ is a $\mathbb{Z}$-basis of the free abelian group generated by $e_1 - e_2, e_2 - e_3, \dots, e_{m-1} - e_m$, where m is the number of vertices of Γ, and hence a part of a $\mathbb{Z}$-basis of $H^{n-1} \cap \mathbb{Z}^n$, proving (3). □

Example 6.7. Take $w = 3152674$ in Example 5.2(1). Then the graph Γ_w is in Figure 4. We can reorder the vertices and change directions on edges as follows.



Then the corresponding edge vectors are

$$\{e_1 - e_2, e_1 - e_3, e_1 - e_4, e_2 - e_5, e_5 - e_6, e_6 - e_7\},$$

which is a part of $\mathbb{Z}$-basis of $H^6 \cap \mathbb{Z}^7$.

Combining Corollary 6.3, Lemmas 6.5 and 6.6, we obtain

Theorem 6.8. *The generic torus orbit closure Y_w is smooth at the fixed point wB if and only if Γ_w is a forest.*

To our surprise, the graph Γ_w has been studied and the following is proven.

Proposition 6.9 ([7])**.** *The graph Γ_w is a forest if and only if w avoids the patterns* 4231 *and* $45\bar{3}12$.

In Appendix A, we will provide a proof different from the proof in [7]. Here, a permutation w *avoids the pattern* 4231 if one cannot find indices $i < j < k < \ell$ such that $w(\ell) < w(j) < w(k) < w(i)$. Similarly w *avoids the pattern* $45\bar{3}12$ if every occurrence of the pattern 4512 is a subsequence of an occurrence of 45312.

Remark 6.10. In [7], they associate a graph G_π to a permutation π and prove that G_π is a forest if and only if π avoids the patterns 1324 and $21\bar{3}54$. In fact, our Γ_w is their G_{w_0w}, so we obtain the statement in Proposition 6.9.

We restate Theorem 6.8 in terms of pattern avoidance using Proposition 6.9.

Theorem 6.11. *The generic torus orbit closure Y_w is smooth at the fixed point wB if and only if w avoids the patterns* 4231 *and* $45\bar{3}12$.

Remark 6.12. According to [7] and [24], Γ_w is a forest if and only if our Schubert variety X_w ($= \overline{BwB/B}$) is factorial. (Note. X_w in [24] is different from our X_w, indeed their X_w is Ω_w in Fulton's book [14], which is the closure of the *dual* Schubert cell indexed by w.)

7. Smoothness of Y_w at other fixed points

In the previous sections, we identified the cone dual to the maximal cone $C_w = C_w(w)$ and associated the graph Γ_w. In this section, we will do the same task to the other maximal cones $C_w(u)$ $(u \le w)$ generalizing the previous case, namely we identify the cone dual to $C_w(u)$ and associate a graph, denoted by $\Gamma_w(u)$, to each $u \le w$. It seems that if the graph Γ_w is a forest, then so is $\Gamma_w(u)$ for any $u \le w$. This means that Y_w is smooth if it is smooth at the fixed point wB. We propose this and a slightly more general statement as a conjecture at the end of this section.

We begin with the generalization of E_w introduced in Definition 5.1.

Definition 7.1. For $u \le w$, we define

$$\widetilde{E}_w(u) := \{(u(i), u(j)) \mid 1 \le i < j \le n,\ t_{u(i),u(j)}u \le w,\ |\ell(u) - \ell(t_{u(i),u(j)}u)| = 1\}$$

where $t_{a,b}$ denotes the transposition of a and b and $\ell(v)$ denotes the length of a permutation v as before.

Remark 7.2.
(1) $\widetilde{E}_w(w) = E_w$ because the condition $\ell(w) - \ell(t_{w(i),w(j)}w) = 1$ in Definition 5.1 is equivalent to

$$t_{w(i),w(j)}w \le w \quad \text{and} \quad |\ell(w) - \ell(t_{w(i),w(j)}w)| = 1.$$

(2) The condition $|\ell(u) - \ell(t_{u(i),u(j)}u)| = 1$ in Definition 7.1 is equivalent to

$$u(k) \notin [u(j), u(i)] \quad \text{when } u(i) > u(j),$$
$$u(k) \notin [u(i), u(j)] \quad \text{when } u(i) < u(j),$$

for $i < \forall k < j$.

Example 7.3. Take $w = 3412$.
(1) If $u = 2143$, then $\widetilde{E}_w(u) = \{(1,4), (2,3), (2,1), (4,3)\}$.
(2) If $u = 2413$, then $\widetilde{E}_w(u) = \{(2,3), (2,1), (4,1), (4,3)\}$.
(3) If $u = 1432$, then $\widetilde{E}_w(u) = \{(1,3), (4,3), (3,2)\}$.

We define $(a,b) + (b,c) = (a,c)$. Then, in (1) in the example above, we have

$$(2,3) = (2,1) + (1,4) + (4,3).$$

We say that an element of $\widetilde{E}_w(u)$ is *decomposable* if it is sum of some other elements in $\widetilde{E}_w(u)$, and *indecomposable* otherwise.

Definition 7.4. We define

$$E_w(u) := \{\text{indecomposable elements in } \widetilde{E}_w(u)\}$$

and

$$D_w(u) := \text{ the cone in } \mathbb{R}^n \text{ spanned by } \{e_b - e_a \mid (a,b) \in E_w(u)\}.$$

Remark 7.5.
(1) $E_w(w) = \widetilde{E}_w(w) = E_w$, so $D_w(w) = D_w$,
(2) $E_{w_0}(u) = \{(u(i), u(i+1)) \mid i = 1, 2, \dots, n-1\}$.
(3) When $u = id$ (the identity permutation), $E_w(id) = \{(a, a+1) \mid t_{a,a+1}id \le w\}$.

Example 7.6. For (1) in Example 7.3, $E_w(u) = \{(1,4), (2,1), (4,3)\}$, and for (2) and (3), $E_w(u) = \widetilde{E}_w(u)$. One can check that $C_w(u) = D_w(u)^\vee$ in these cases.

The purpose of this section is to prove the following proposition.

Proposition 7.7. *$C_w(u) = D_w(u)^\vee$ for any $u \le w$.*

As a first step, we prove the following lemma.

Lemma 7.8. *$C(u) = D_{w_0}(u)^\vee$ for any $u \in \mathfrak{S}_n$.*

Proof. Let (a,b) be an arbitrary element in $E_{w_0}(u)$. Then a appears ahead of b in the one-line notation for u. Therefore $e_b - e_a$ is non-negative on the cone $C(u)$, which means that $C(u) \subset D_{w_0}(u)^\vee$. Suppose $C(u) \subsetneq D_{w_0}(u)^\vee$ (and we deduce a contradiction). Then there is some $v (\neq u) \in \mathfrak{S}_n$ such that $C(v) \subset D_{w_0}(u)^\vee$. Since $v \neq u$, there is a pair (a,b) such that a appears ahead of b in the one-line notation for u while b appears ahead of a in the one-line notation for v. Suppose $a > b$. Then there is a sequence of pairs $(a_{i-1}, a_i) \in E_{w_0}(u)$ $(i = 1, \dots, k)$ such that

$$a = a_0 > a_1 > \cdots > a_k = b. \tag{7.1}$$

Since $C(v) \subset D_{w_0}(u)$ and $(a_{i-1}, a_i) \in E_{w_0}(u)$, $e_{a_i} - e_{a_{i-1}}$ is non-negative on $C(v)$. This means that in the one-line notation for v, a_{i-1} appears ahead of a_i for any $i = 1, \dots, k$ so that a appears ahead of b, which is a contradiction. The same argument works when $a < b$ if the inequalities in (7.1) are reversed. □

Lemma 7.9. *Let $u \le w$. If $v' = u$, then $C(v) \subset D_w(u)^\vee$. Therefore $C_w(u) \subset D_w(u)^\vee$ by Corollary 3.7.*

Proof. Since $v' = u$, we have $v \ge_R u$ by Lemma 4.1. Therefore v is of the form

$$v = us_{i_1} \cdots s_{i_k} \quad \text{with } \ell(v) = \ell(u) + k.$$

We shall prove the lemma by induction on k. When $k = 0$, that is, $v = u$, $C(v) = C(u)$. By Lemma 7.8, $C(u) = D_{w_0}(u)^\vee$. Since $D_{w_0}(u) \supset D_w(u)$, we obtain $C(u) \subset D_w(u)^\vee$ by taking their dual. This proves the lemma when $k = 0$.

Suppose $k \ge 1$ and the lemma holds for v with $\ell(v) \le \ell(u) + k - 1$. We set

$$\bar{v} = us_{i_1} \cdots s_{i_{k-1}} \qquad \text{and} \qquad i_k = p.$$

Then $v = \bar{v}s_p$, i.e.

$$\begin{aligned} \bar{v} &= v(1) \cdots v(p-1)v(p+1)v(p)v(p+2) \cdots v(n), \\ v &= v(1) \cdots v(p-1)v(p)v(p+1)v(p+2) \cdots v(n), \end{aligned} \tag{7.2}$$

and $C(\bar{v}) \subset D_w(u)^\vee$ by induction assumption (note that $\bar{v}' = u$ since $\bar{v} \in [u, v]_R$ and $v' = u$). Let (a, b) be an arbitrary element of $E_w(u)$. Since $e_b - e_a$ is non-negative on $C(\bar{v})$, a appears ahead of b in the one-line notation for $\bar{v}$.

If $\{v(p), v(p+1)\} \ne \{a, b\}$, then the order of a and b in the one-line notation for v is same as that for $\bar{v}$. Therefore, $e_b - e_a$ is non-negative on $C(v)$ as well. Thus it suffices to show that the case where $\{v(p), v(p+1)\} = \{a, b\}$ does not occur. Suppose $\{v(p), v(p+1)\} = \{a, b\}$. Then we have

$$a = v(p+1) < v(p) = b \tag{7.3}$$

because a appears ahead of b in the one-line notation for $\bar{v}$ and $v = \bar{v}s_p$ with $\ell(v) = \ell(\bar{v}) + 1$. On the other hand, since $(a, b) \in E_w(u)$, a appears ahead of b in the one-line notation for u; so $a = u(s)$ and $b = u(t)$ with some $s < t$, i.e.

$$\begin{aligned} u &= u(1) \cdots u(s-1)\ a\ u(s+1) \cdots u(t-1)\ b\ u(t+1) \cdots u(n), \\ t_{a,b}u &= u(1) \cdots u(s-1)\ b\ u(s+1) \cdots u(t-1)\ a\ u(t+1) \cdots u(n). \end{aligned}$$

Since $(a, b) \in E_w(u)$, we have $t_{a,b}u \le w$ by definition. This means that

$$\{u(1), \dots, u(s-1), b\} \uparrow\le w^{(s)}. \tag{7.4}$$

Now remember that $\bar{v}' = u$. This together with (7.2), (7.3) and (7.4) shows that $v'(s) = b$ and hence $v' \ne u$, a contradiction. □

Lemma 7.10. *Let $x \le w$, $y \le w$ and $x \ne y$. If the cones $C_w(x)$ and $C_w(y)$ share a common facet, then they are related by a transposition, i.e. $x = t_{a,b}y$ for some transposition $t_{a,b}$.*

Proof. x and y are fixed points in the flag variety G/B under the torus action and the assumption implies that they are in $\mathbb{C}P^1$ fixed pointwise under the codimension one subtorus corresponding to the facet $C_w(x) \cap C_w(y)$. In other words, x and y are vertices of the GKM graph of G/B and they are joined by an edge. Therefore they are related by a transposition ([9], see also [23, Proposition 2.1]). □

We prepare one more lemma.

Lemma 7.11. *Let $u \leq w$. If $t_{u(j),u(k)}u \leq w$ with $j<k$, then $e_{u(k)}-e_{u(j)} \in D_w(u)$.*

Proof. It suffices to show that

(∗) $(u(j),u(k))$ is a finite sum of elements in $\widetilde{E}_w(u)$.

If there is no $j<m<k$ such that $u(m)$ is in between $u(j)$ and $u(k)$ (we say that $u(j)$ and $u(k)$ are *adjacent* in this case), then $(u(j),u(k)) \in \widetilde{E}_w(u)$ by definition of $\widetilde{E}_w(u)$ and hence $e_{u(k)}-e_{u(j)} \in D_w(u)$ by definition of $D_w(u)$. Thus we may assume that $u(j)$ and $u(k)$ are not adjacent. We consider two cases.

Case 1. The case where $u(j) > u(k)$. In this case, there is a sequence $j < m_1 < m_2 < \cdots < m_p < k$ such that

$u(j) > u(m_1) > u(m_2) > \cdots > u(m_p) > u(k)$ and

$u(m_\ell)$ and $u(m_{\ell+1})$ are adjacent for $\ell = 0,1,\dots,p$, where $m_0 = j, m_{p+1} = k$.

Since $(u(m_\ell),u(m_{\ell+1}))$ is an inversion of u and $u \leq w$, we have $t_{u(m_\ell),u(m_{\ell+1})}u \leq w$ and hence $(u(m_\ell),u(m_{\ell+1})) \in \widetilde{E}_w(u)$ for $\ell = 0,1,\dots,p$. This shows that

$$(u(j),u(k)) = \sum_{\ell=0}^{p} (u(m_\ell),u(m_{\ell+1})) \tag{7.5}$$

proving the assertion (∗).

Case 2. The case where $u(j) < u(k)$. Similarly to Case 1 above, there is a sequence $j < m_1 < m_2 < \cdots < m_p < k$ such that

$u(j) < u(m_1) < u(m_2) < \cdots < u(m_p) < u(k)$ and

$u(m_\ell)$ and $u(m_{\ell+1})$ are adjacent for $\ell = 0,1,\dots,p$, where $m_0 = j, m_{p+1} = k$.

In this case, since $(u(m_\ell),u(m_{\ell+1}))$ is not an inversion of u, it is not immediate that $t_{u(m_\ell),u(m_{\ell+1})}u \leq w$. However, $t_{u(j),u(k)}u \leq w$ by assumption and this will imply $t_{u(m_\ell),u(m_{\ell+1})}u \leq w$. Indeed, setting $u(m_\ell) = a_\ell$ for simplicity, one can see that

$$\begin{aligned} &t_{a_0,a_1}u = t_{a_{p+1},a_1}t_{a_1,a_0}(t_{a_0,a_{p+1}}u),\\ &t_{a_\ell,a_{\ell+1}}u = t_{a_{p+1},a_\ell}t_{a_{\ell+1},a_0}t_{a_\ell,a_0}t_{a_{p+1},a_{\ell+1}}(t_{a_0,a_{p+1}}u) \quad \text{for } \ell = 1,\dots,p. \end{aligned} \tag{7.6}$$

Here $t_{a_0,a_{p+1}}u = t_{u(j),u(k)}u \leq w$ by assumption and all the transpositions $t_{a,b}$ in (7.6) (except $t_{a_0,a_{p+1}}$) have $a > b$ and (a,b) is an inversion of the permutation, say v, which $t_{a,b}$ is applied to; so if $v \leq w$, then $t_{a,b}v \leq w$. Therefore (7.6) shows that $t_{a_\ell,a_{\ell+1}}u \leq w$ for $\ell = 0,1,\dots,p$ and hence $(u(m_\ell),u(m_{\ell+1})) = (a_\ell,a_{\ell+1}) \in \widetilde{E}_w(u)$, proving the assertion (∗) by (7.5). □

Now we are in a position to prove Proposition 7.7.

Proof of Proposition 7.7. We know $C_w(u) \subset D_w(u)^\vee$ by Lemma 7.9. Suppose that $C_w(u) \subsetneq D_w(u)^\vee$. Then there exist $v \in \mathfrak{S}_n$ and a simple reflection s_i such that

$$C(v) \subset C_w(u), \quad C(vs_i) \not\subset C_w(u), \quad C(vs_i) \subset D_w(u)^\vee. \tag{7.7}$$

Claim. $e_{v(i+1)} - e_{v(i)} \in D_w(u)$.

We admit the claim and complete the proof. Since $C(vs_i) \subset D_w(u)^\vee$ by assumption, the claim says that $e_{v(i+1)} - e_{v(i)}$ takes non-negative values on $C(vs_i)$. On the other hand, in the one-line notation of vs_i, $v(i+1)$ appears on the left of $v(i)$ and this means that $e_{v(i+1)} - e_{v(i)}$ takes non-positive values on $C(vs_i)$. Therefore $e_{v(i+1)} - e_{v(i)}$ must vanish on $C(vs_i)$ but this contradicts $C(vs_i)$ being of dimension n. Thus, it suffices to prove the claim above.

By (7.7), $C(vs_i) \subset C_w(x)$ for some $x(\neq u) \leq w$. The intersection $C(v)\cap C(vs_i)$ is a facet of $C(v)$ and $C(vs_i)$, and $C_w(u)\cap C_w(x)$ contains $C(v)\cap C(vs_i)$, so $C_w(u)$ and $C_w(x)$ are adjacent. Therefore $x = t_{a,b}u$ for some transposition $t_{a,b}$ by Lemma 7.10. Since $C(vs_i) \subset C_w(x)$ and $C(v) \subset C_w(u)$, it follows that

$$(vs_i)' = x = t_{a,b}u \quad \text{and} \quad v' = u. \tag{7.8}$$

We define j and k by

$$v(i) = u(j), \quad v(i+1) = u(k). \tag{7.9}$$

We consider two cases.

Case 1. The case where $v(i) < v(i+1)$. Since $v' = u$, we have $j < k$ in this case. This together with $(vs_i)' \neq u$ implies that

$$\begin{aligned} v' = u &= u(1)\cdots u(j-1)u(j)\cdots u(k)\cdots u(n), \\ (vs_i)' &= u(1)\cdots u(j-1)u(k)\cdots\cdots. \end{aligned} \tag{7.10}$$

On the other hand, we know $(vs_i)' = t_{a,b}u$ by (7.8). This together with (7.10) implies that

$$(vs_i)' = t_{u(j),u(k)}u.$$

Since $j < k$ and $(vs_i)' \leq w$, the above together with Lemma 7.11 shows that $e_{u(k)} - e_{u(j)} \in D_w(u)$. Then the claim follows from (7.9).

Case 2. The case where $v(i) > v(i+1)$. We claim $j < k$ in this case, too. Indeed, since $v(i+1)$ appears on the left of $v(i)$ in the one-line notation of vs_i and $v(i+1) < v(i)$, one sees that $v(i+1)(= u(k))$ appears ahead of $v(i)(= u(j))$ in the one line notation of $(vs_i)'$. Therefore, if $k < j$, then $(vs_i)'$ coincides with u which contradicts the assumption $(vs_i)' \neq u$. Therefore $j < k$. Then (7.10) holds in this case too and the claim follows from the same argument as in Case 1. This completes the proof of the claim and the proposition. □

We associate a graph $\Gamma_w(u)$ to $E_w(u)$ as before. Note that $\Gamma_w(id)$ is the disjoint union of path graphs (in particular, a forest) by Remark 7.5(3). Then by Lemma 6.6, we obtain

Corollary 7.12. *The dual cone $D_w(u)$ is smooth if and only if $\Gamma_w(u)$ is a forest. Indeed, Y_w is smooth at the fixed point uB if and only if $\Gamma_w(u)$ is a forest. In particular, Y_w is smooth at the fixed point idB for any w.*

Remark 7.13. Since Y_w has at least one smooth T-fixed point idB, it follows from [11, Proposition 7.2 and Remark 7.3] that Y_w is smooth if and only if the Bruhat graph of Y_w is regular, where regular means that there are exactly $\dim_{\mathbb{C}} Y_w$ T-stable curves at each T-fixed point in Y_w.

Using the fact that idB is a smooth point in Y_w, we can give an alternative proof to the following result which is indeed proved for any Lie type in [17].

Corollary 7.14 ([17]). *A Schubert variety X_w is a toric variety if and only if w is a product of distinct simple reflections.*

Proof. The Schubert variety X_w is a toric variety if and only if $X_w = Y_w$. This is equivalent to $\dim_{\mathbb{C}} X_w = \dim_{\mathbb{C}} Y_w$ since X_w is irreducible and $X_w \supset Y_w$. As is well-known $\dim_{\mathbb{C}} X_w = \ell(w)$. On the other hand, $\dim_{\mathbb{C}} Y_w$ is equal to the dimension of any maximal cone in the fan of Y_w. Since the fixed point idB is smooth in Y_w, the maximal cone corresponding to idB is simplicial and hence its dimension is equal to $|E_w(id)|$ by Proposition 7.7. Here

$$E_w(id) = \{v \in \mathfrak{S}_n \mid \ell(v) = 1,\, v \leq w\}$$

(see Remark 7.5(3)) and the condition $\ell(v) = 1$ and $v \leq w$ is equivalent to v being a simple reflection appearing in a reduced expression of w. This implies the desired result since $\ell(w)$ is the number of simple reflections appearing in a reduced expression of w. □

Remark 7.15. A toric Schubert variety X_w is a Bott manifold (the top space of a Bott tower in [15]). Indeed, since w is a product of simple reflections, the Bruhat graph of X_w is combinatorially same as the 1-skeleton of a cube. This means that the underlying simplicial complex of the fan of X_w is the boundary complex of a cross-polytope. Therefore X_w is a Bott manifold (see [8, Corollary 7.8.11]).

We propose the following conjecture.

Conjecture 7.16. *The graph $\Gamma_w(u)$ is a forest for any $u \leq w$ if Γ_w is a forest.* (*This is equivalent to saying that the generic torus orbit closure Y_w is smooth if it is smooth at the fixed point wB.*)

Here is an example which supports the conjecture.

Example 7.17. Take $w = 32154$. Then $E_w = \{(3,2),(2,1),(5,4)\}$. Using simple transpositions s_i for $1 \leq i \leq 4$, we have that $w = s_1s_2s_1s_4$. There are twelve elements $u \in \mathfrak{S}_5$ such that $u \leq w$, and we have Table 2 for $E_w(u)$. We can see that each $\Gamma_w(u)$ is a forest for $u \leq w$.

u	$E_w(u)$	u	$E_w(u)$
$s_1s_2s_1s_4 = 32154$	$(3,2),(2,1),(5,4)$	$s_1s_2s_1 = 32145$	$(3,2),(2,1),(4,5)$
$s_4s_2s_1 = 31254$	$(3,1),(1,2),(5,4)$	$s_4s_1s_2 = 23154$	$(2,3),(3,1),(5,4)$
$s_2s_1 = 31245$	$(3,1),(1,2),(4,5)$	$s_1s_2 = 23145$	$(2,3),(3,1),(4,5)$
$s_4s_1 = 21354$	$(2,1),(1,3),(5,4)$	$s_4s_2 = 13254$	$(1,3),(3,2),(5,4)$
$s_1 = 21345$	$(2,1),(1,3),(4,5)$	$s_2 = 13245$	$(1,3),(3,2),(4,5)$
$s_4 = 12354$	$(1,2),(2,3),(5,4)$	$e = 12345$	$(1,2),(2,3),(4,5)$

Table 2. $E_w(u)$ for $u \leq w$ when $w = 32154$.

One can check Conjecture 7.16 for $n \leq 5$ with the aid of computers. Indeed, the computer calculation suggests that $b_1(\Gamma_w(u)) \leq b_1(\Gamma_w)$ for any $u \leq w$, where $b_1(\Gamma)$ denotes the first Betti number of a graph Γ. More strongly, it suggests that $b_1(\Gamma_w(u)) \leq b_1(\Gamma_w(t_{a,b}u))$ for $(a,b) \in E_w(u)$ with $a < b$. The latter implies that any two non-simple vertices in the Bruhat interval polytope $Q_{id,w^{-1}}$ (see Section 8 for $Q_{id,w^{-1}}$) would be joined by edges with non-simple vertices as endpoints (see Figures 24 and 26 in Appendix B).

8. Poincaré polynomial of Y_w

The Eulerian number $A(n,k)$ is the number of permutations in $\mathfrak{S}_n$ with k ascents and the Eulerian polynomial $A_n(t)$ is the generating function of the Eulerian numbers, i.e. $A_n(t)=\sum_{k=0}^{n-1}A(n,k)t^k$. As is well-known, the Poincaré polynomial of the permutohedral variety Y_{w_0} agrees with $A_n(t^2)$. In this section we consider a generalization of Eulerian numbers, introduce a polynomial $A_w(t)$ for each $w\in\mathfrak{S}_n$, and show that the Poincaré polynomial of Y_w is given by $A_w(t^2)$ when Y_w is smooth.

We set

$$a_w(u):=\#\{(u(i),u(j))\in E_w(u)\mid u(i)<u(j)\}\quad\text{for } u\le w,$$
$$A_w(n,k):=\#\{u\le w\mid a_w(u)=k\}\quad\text{for } k\ge 0,$$

and define

$$A_w(t):=\sum_{u\le w}t^{a_w(u)}=\sum_{k\ge 0}A_w(n,k)t^k.$$

Example 8.1. Using Figure 3(1) and Table 3, one can find

$$A_w(t)=t^3+11t^2+7t+1\quad\text{when } w=4231,$$
$$A_w(t)=t^3+7t^2+5t+1\quad\text{when } w=3412.$$

In these cases, $A_w(t)$ are not palindromic and Y_w are not smooth. On the other hand, from Table 2 one can easily find

$$A_w(t)=t^3+5t^2+5t+1\quad\text{when } w=32154.$$

In this case, $A_w(t)$ is palindromic and Y_w is smooth.

u	$E_w(u)$	$a_w(u)$
4231	$(4,2),(4,3),(2,1),(3,1)$	0
4132	$(4,1),(4,3),(1,2),(3,2)$	1
4213	$(4,2),(2,1),(1,3)$	1
2431	$(2,4),(4,3),(3,1)$	1
3241	$(3,2),(3,4),(2,1),(4,1)$	1
1432	$(1,4),(4,3),(3,2)$	1
4123	$(4,1),(1,2),(2,3)$	2
2413	$(2,4),(4,1),(1,3)$	2
3142	$(3,1),(3,4),(1,2),(4,2)$	2
3214	$(3,2),(2,1),(1,4)$	1
2341	$(2,3),(3,4),(4,1)$	2
1423	$(1,4),(4,2),(2,3)$	2
1342	$(1,3),(3,4),(4,2)$	2
2143	$(2,1),(1,4),(4,3)$	1
3124	$(3,1),(1,2),(2,4)$	2
2314	$(1,4),(2,3),(3,1)$	2
1243	$(1,2),(2,4),(4,3)$	2
1324	$(1,3),(3,2),(2,4)$	2
2134	$(2,1),(1,3),(3,4)$	2
1234	$(1,2),(2,3),(3,4)$	3

$w = 4231.$

u	$E_w(u)$	$a_w(u)$
3412	$(3,1),(3,2),(4,1),(4,2)$	0
1432	$(1,3),(4,3),(3,2)$	1
2413	$(2,1),(2,3),(4,1),(4,3)$	1
3142	$(3,1),(1,4),(4,2)$	1
3214	$(3,2),(2,1),(2,4)$	1
1423	$(1,2),(4,2),(2,3)$	2
1342	$(1,3),(3,4),(4,2)$	2
2143	$(2,1),(1,4),(4,3)$	1
3124	$(3,1),(1,2),(2,4)$	2
2314	$(2,3),(3,1),(3,4)$	2
1243	$(1,2),(2,4),(4,3)$	2
1324	$(1,3),(3,2),(2,4)$	2
2134	$(2,1),(1,3),(3,4)$	2
1234	$(1,2),(2,3),(3,4)$	3

$w = 3412.$

TABLE 3. $E_w(u)$ and $a_w(u)$.

Remark 8.2. Looking at descents

$$d_w(u) := \#\{(u(i),u(j)) \in E_w(u) \mid u(i) > u(j)\} \quad \text{for any } u \le w,$$

one can define a polynomial

$$\bar{A}_w(t) := \sum_{u \le w} t^{d_w(u)}.$$

When Y_w is smooth, we have $\bar{A}_w(t) = A_w(t)$ but otherwise they may differ. Indeed

$$\bar{A}_w(t) = t^4 + 2t^3 + 6t^2 + 10t + 1 \quad \text{when } w = 4231,$$
$$\bar{A}_w(t) = t^4 + t^3 + 4t^2 + 7t + 1 \quad \text{when } w = 3412.$$

As remarked in Remark 7.5,

$$E_{w_0}(u) = \{(u(i), u(i+1)) \mid i = 1, 2, \dots, n-1\},$$

so $a_{w_0}(u)$ is the number of ascents in u and hence $A_{w_0}(n,k)$ is the number of permutations in $\mathfrak{S}_n$ with k ascents, which is the Eulerian number $A(n,k)$. Therefore, $A_{w_0}(t)$ is the Eulerian polynomial $A_n(t)$. As is well-known, the Poincaré polynomial of the permutohedral variety Y_{w_0} is given by $A_n(t^2) = A_{w_0}(t^2)$. The following theorem generalizes this fact.

Theorem 8.3. *If Y_w is smooth, then its Poincaré polynomial agrees with $A_w(t^2)$ and hence $A_w(t)$ is palindromic and unimodal.*

Remark 8.4. When $w = 4231$ or 3412, Y_w is singular and $A_w(t)$ is not palindromic but one can see that $A_w(t^2)$ still agrees with the Poincaré polynomial of Y_w, see Appendix B.

Proof of Theorem 8.3. The maximal cone $C_w(u)$ in the fan of Y_w corresponds to the fixed point uB and its dual cone $D_w(u)$ is spanned by $e_{u(j)}-e_{u(i)}$'s for $(u(i),u(j)) \in E_w(u)$. Suppose that Y_w is smooth. Then $e_{u(j)}-e_{u(i)}$'s for $(u(i),u(j)) \in E_w(u)$ are weights of the tangential T-representation of Y_w at uB. We choose an element $\mathbf{a} = [a_1,\dots,a_n] \in \mathbb{Z}^n$ such that $a_1 < a_2 < \cdots < a_n$ and consider the $\mathbb{C}^*$-action on Y_w through $\lambda^{\mathbf{a}}\colon \mathbb{C}^* \to T$. Then $Y_w^{\mathbb{C}^*} = Y_w^T$ and the integers $\langle e_{u(j)}-e_{u(i)}, \mathbf{a}\rangle = a_{u(j)} - a_{u(i)}$ are the weights of the $\mathbb{C}^*$-action and the number of positive weights at uB is $a_w(u)$. Since $Y_w^{\mathbb{C}^*} = Y_w^T = \{uB \mid u \le w\}$, this shows that the Poincaré polynomial of Y_w is given by $\sum_{u\le w} t^{2a_w(u)} = A_w(t^2)$.

Since Y_w is smooth and projective, the Poincaré duality and the hard Lefschetz theorem imply the palindromicity and unimodality of the Poincaré polynomial of Y_w, proving the theorem. □

We shall interpret the proof of Theorem 8.3 from the viewpoint of symplectic geometry. There is a map called moment map

$$\mu\colon G/B \to \mathbb{R}^n$$

such that $\mu(uB) = (u^{-1}(1),\dots,u^{-1}(n))$ and $\mu(G/B)$ is the convex hull of $\mu(uB)$'s for all $u \in \mathfrak{S}_n$, that is the permutohedron of dimension $n-1$. The image $\mu(Y_w)$ is the convex hull of $\mu(uB)$'s for all $u \le w$ by [2, Theorem 2]. The vectors $e_{u(j)}-e_{u(i)}$'s for $(u(i),u(j)) \in E_w(u)$ are exactly the edge vectors of $\mu(Y_w)$ emanating from $\mu(uB)$. If Y_w is smooth, then the polytope $\mu(Y_w)$ is simple.

A vector $\mathbf{a} \in \mathbb{Z}^n$ gives a linear map $L^{\mathbf{a}}\colon \mathbb{R}^n \to \mathbb{R}$ defined by the pairing $\langle \mathbf{a},\ \rangle$. If Y_w is smooth and $\mathbf{a}$ is generic as in the proof of Theorem 8.3, then the composition $L^{\mathbf{a}} \circ \mu\colon Y_w \to \mathbb{R}$ is a moment map with respect to the action of the circle in $\mathbb{C}^*$ through $\lambda^{\mathbf{a}}\colon \mathbb{C}^* \to T$, so that $L^{\mathbf{a}}\circ\mu$ is a Morse function with uB's as critical points and $2d_w(u)$ is the Morse index at uB. The function $-L^{\mathbf{a}}\circ\mu\colon Y_w \to \mathbb{R}$ is also a Morse function but the Morse index at uB is $2a_w(u)$. This is the interpretation of the proof of Theorem 8.3 from the viewpoint of symplectic geometry.

Now, since Y_w is smooth if and only if $\Gamma_w(u)$ is a forest for any $u \le w$ (Corollary 7.12), we obtain the following corollary from Theorem 8.3.

Corollary 8.5. *The polynomial $A_w(t)$ is palindromic if the graph $\Gamma_w(u)$ is a forest for any $u \le w$.*

We conclude with remarks and questions about criterion of smoothness of Y_w. The polytope $\mu(Y_w)$ is the Bruhat interval polytope $Q_{id,w^{-1}}$ in [22] for any $w \in \mathfrak{S}_n$, where id denotes the identity permutation, and the vectors $e_{u(j)}-e_{u(i)}$'s for $(u(i),u(j)) \in E_w(u)$ are the edge vectors of the polytope emanating from the vertex $\mu(uB)$. Note that the 1-skeleton of the polytope $Q_{id,w^{-1}}$ is the Bruhat graph of Y_w (where the label of the vertex $\mu(uB)$ in the Bruhat graph is u although $\mu(uB) = (u^{-1}(1),\dots,u^{-1}(n))$). Therefore the following are equivalent:

(1) Y_w is smooth.
(2) The Bruhat interval polytope $Q_{id,w^{-1}}$ is simple, equivalently the Bruhat graph of Y_w is regular.
(3) The graph $\Gamma_w(u)$ is a forest for any $u \le w$.

The former part in Conjecture 7.16 says that the following (4) would imply (3) above:

(4) The graph $\Gamma_w = \Gamma_w(w)$ is a forest.

Therefore, if the conjecture is true, then all four statements above are equivalent.

It is known that the Schubert variety X_w is smooth if and only if its Poincaré polynomial is palindromic (see [5, Theorem 6.0.4, Corollary 6.1.13, Theorem 6.2.4]). We may ask whether the same holds for Y_w, in other words, whether the following (5) is equivalent to (1) above:

(5) The Poincaré polynomial of Y_w is palindromic.

If Y_w is smooth, then $A_w(t)$ is palindromic (Theorem 8.3) but otherwise $A_w(t)$ may not be palindromic. As in Remark 8.4, $A_w(t^2)$ might agree with the Poincaré polynomial of Y_w for any $w \in \mathfrak{S}_n$. Related to (5), we may ask whether the following (6) is equivalent to (1) above:

(6) $A_w(t)$ is palindromic.

When $n \leq 3$, Y_w is smooth for all w. When $n = 4$, Y_w is smooth if and only if w is different from 4231 and 3412. As is in Example 8.1, $A_w(t)$ is not palindromic when $w = 4231$ or 3412. Therefore (1) and (6) above are equivalent for $n \leq 4$.

Appendix A. Acyclicity of graph and pattern avoidance of permutation

In the appendix, we will give a proof of Proposition 6.9 which is different from the proof in [7]. If (a, b) is an inversion in w, namely $a > b$ and $w^{-1}(a) < w^{-1}(b)$, then there is a sequence of pairs $(a_i, b_i) \in E_w$ $(i = 1, \dots, k)$ such that

$$a = a_1 > b_1 = a_2 > b_2 = a_3 > \cdots > b_{k-1} = a_k > b_k = b.$$

This produces a path from a to b in Γ_w and we call such a path a *descending path* from a to b. There is a descending path from a to b if and only if (a, b) is an inversion of w. We denote by $\operatorname{Inv}(w)$ the set of inversions of w.

Proof of Proposition 6.9. The proposition is equivalent to the statement that w has a pattern 4231 or 4512 which is not a subsequence of 45312 if and only if Γ_w has a cycle, and we shall prove this equivalent statement.

The only if part ($\Longrightarrow$). **Case 1**. The case where w has the pattern 4231, see Figure 8(1). In this case there are $a, b, c, d \in [n]$ such that

$$w^{-1}(a) < w^{-1}(b) < w^{-1}(c) < w^{-1}(d) \qquad \text{and} \qquad a > c > b > d.$$

Since $(a, b), (b, d) \in \operatorname{Inv}(w)$, there is a descending path P in Γ_w from a to d via b. Note that P does not contain c because $b \in P$ and $(b, c) \notin \operatorname{Inv}(w)$. Similarly, since $(a, c), (c, d) \in \operatorname{Inv}(w)$, there is a descending path Q in Γ_w from a to d via c and Q does not contain b because $c \in Q$ and $(b, c) \notin \operatorname{Inv}(w)$. Therefore the union $P \cup Q$ contains a cycle.

Case 2. The case where w has the pattern 4512 which is not a subsequence of 45312, see Figure 8(2). The argument is similar to Case 1 but rather subtle. In this case there are $a, b, c, d \in [n]$ such that

$$w^{-1}(a) < w^{-1}(b) < w^{-1}(c) < w^{-1}(d) \qquad \text{and} \qquad b > a > d > c \tag{A.1}$$

and we may assume that

$$\begin{aligned} &\text{there is no } p \text{ such that } a > p > d \text{ and } w^{-1}(a) < w^{-1}(p) < w^{-1}(d), \text{ and} \\ &\text{there is no } q \text{ such that } b > q > c \text{ and } w^{-1}(b) < w^{-1}(q) < w^{-1}(c). \end{aligned} \tag{A.2}$$

(1) w has pattern 4231.

(2) w has pattern 4512 which is not a subsequence of 45312.

FIGURE 8. Graph Γ_w.

Since $(a,d) \in \mathrm{Inv}(w)$, there is a descending path P_1 from a to d and since $(a,c) \in \mathrm{Inv}(w)$, there is a descending path P_2 from a to c. Both P_1 and P_2 do not contain the vertex b because $b > a$. Note that

$$P_1 \cap P_2 = \{a\}, \text{ so } P_1 \cup P_2 \text{ is a path which joins } c \text{ and } d \text{ and does not contain } b.$$

Indeed, the vertex a is an end point of P_1 and P_2 and if P_1 and P_2 has another common vertex, say p, then $a > p > d$ and $w^{-1}(a) < w^{-1}(p) < w^{-1}(c)$ but this contradicts (A.2) because $w^{-1}(c) < w^{-1}(d)$ by (A.1). Similarly, since $(b,c) \in \mathrm{Inv}(w)$, there is a descending path Q_1 from b to c, and since $(b,d) \in \mathrm{Inv}(w)$, there is a descending path Q_2 from b to d. Both Q_1 and Q_2 do not contain the vertex a because they are descending paths and $w^{-1}(a) < w^{-1}(b)$. Note also that

$$Q_1 \cap Q_2 = \{b\}, \text{ so } Q_1 \cup Q_2 \text{ is a path which joins } c \text{ and } d \text{ and does not contain } a.$$

Indeed, the vertex b is an end point of Q_1 and Q_2 and if Q_1 and Q_2 have another common vertex, say q, then $b > q > d$ and $w^{-1}(b) < w^{-1}(q) < w^{-1}(c)$ but this contradicts (A.2) because $d > c$ by (A.1).

Both $P := P_1 \cup P_2$ and $Q := Q_1 \cup Q_2$ are *paths* joining c and d, and $a \in P$ but $a \notin Q$. Therefore, the union $P \cup Q$ contains a cycle.

The if part ($\Longleftarrow$). Suppose that Γ_w contains a cycle S. We may assume that S is a circle.

Claim. The cycle S contains at least four vertices.

Indeed, if it has only three vertices, say x, y, z and $w^{-1}(x) < w^{-1}(y) < w^{-1}(z)$, then since S is a triangle, we must have $(x,y), (y,z), (x,z) \in E_w$ (Figure 9) and the belonging of the first two elements to E_w implies $x > y > z$ but this contradicts (x,z) being in E_w. Therefore the claim follows.

Let a (resp. d) be the vertex of S such that $w^{-1}(a) \le w^{-1}(x)$ (resp. $w^{-1}(x) \le w^{-1}(d)$) for all vertices x of S, in other words,

$$\begin{aligned} &a \text{ is leftmost while } d \text{ is rightmost among vertices of } S \\ &\text{in the one-line notation for } w. \end{aligned}$$

We call the interval $[w^{-1}(a), w^{-1}(d)]$ the *width* of S. We may assume that the width of S is minimal, i.e. there is no cycle S' in Γ_w such that the width of S' is properly contained in the width of S. We say that a descending path P from x to

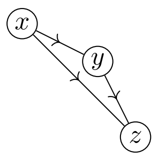


FIGURE 9

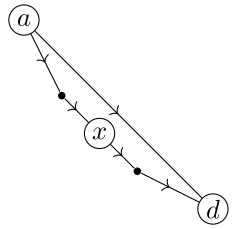


FIGURE 10

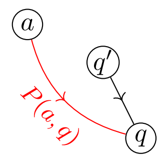


FIGURE 11

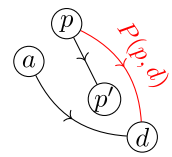


FIGURE 12

y is *emanating* from x or is *terminating* at y. Note that since a (resp. d) is the leftmost (resp. rightmost) vertex of S, the two edges at the vertex a (resp. d) in S are emanating from a (resp. terminating at d).

We consider two cases.

Case I. The case where a and d are joined by an edge, i.e. $(a, d) \in E_w$. In this case we will find vertices b and c of S such that $abcd$ is the pattern 4512 which is not a subsequence of 45312.

Since $(a, d) \in E_w$, we have

$$a > d. \tag{A.3}$$

We note that the path from a to d (outside of the edge (a, d)) is not a descending path because if so, any vertex x in the middle of the path (Figure 10) satisfies $w^{-1}(a) < w^{-1}(x) < w^{-1}(d)$ and $a > x > d$ but this contradicts (a, d) being in E_w. Therefore, there is a vertex q ($q \neq a, d$) of S such that the descending path from a to q is maximal, which means that if q' is the vertex of S next to q but outside of the descending path, then $(q', q) \in E_w$ (Figure 11). We denote this maximal descending path by

$$P(a, q). \tag{A.4}$$

Similarly, there is a vertex p ($\neq a, d, q$) of S such that the path

$$P(p, d) \tag{A.5}$$

from p to d which does not contain a is a maximal descending path terminating at d, which means that if p' is the vertex of S next to p and outside of $P(p, d)$, then $(p, p') \in E_w$ (Figure 12). It follows that

$$\begin{aligned} &a > q, \quad p > d \\ &p > p', \quad w^{-1}(p) < w^{-1}(p'), \\ &q' > q, \quad w^{-1}(q') < w^{-1}(q). \end{aligned} \tag{A.6}$$

(Although we will not use the last two inequalities above for q and q', the argument below will work if we use q and q' instead of p and p'.)

We claim

$$p > a > d > q. \tag{A.7}$$

Indeed, if $a > p$ (Figure 13), then $a > p > d$ by (A.6) and $w^{-1}(a) < w^{-1}(p) < w^{-1}(d)$ since a is the leftmost vertex of S and d is the rightmost vertex of S. This contradicts (a, d) being in E_w. Therefore $p > a$. Similarly, if $q > d$ (Figure 14), then $a > q > d$ by (A.6) and $w^{-1}(a) < w^{-1}(q) < w^{-1}(d)$ but this contradicts (a, d) being in E_w. Therefore $d > q$. These together with (A.3) prove (A.7).

Now we take two cases.

(1) The case where $w^{-1}(p) < w^{-1}(q)$ (Figure 15). In this case, we take $b = p$ and $c = q$. Then $w^{-1}(a) < w^{-1}(b) < w^{-1}(c) < w^{-1}(d)$ by assumption and $b > a > d > c$ by (A.7). Therefore $abcd$ is the pattern 4512. Moreover, since $(a, d) \in E_w$, there is no r such that $a > r > d$ and $w^{-1}(a) < w^{-1}(r) < w^{-1}(d)$, which means that $abcd$ is not a subsequence of 45312.

(2) The case where $w^{-1}(q) < w^{-1}(p)$ (Figure 16). We look at p' defined above (a similar argument will work if we loot at q'). We claim

$$d > p'. \tag{A.8}$$

Indeed, if $p' > d$ (like Figure 16), then $(p', d) \in \mathrm{Inv}(w)$ and hence there is a descending path

$$P(p', d)$$

from p' to d and $P(p', d)$ does not contain p because $w^{-1}(p) < w^{-1}(p')$. Therefore the union

$$(p, p') \cup P(p', d)$$

is a descending path from p to d. On the other hand, $P(p, d)$ in (A.5) is also a descending path from p to d but does not contain p'. Therefore the union

$$\big((p, p') \cup P(p', d)\big) \cup P(p, d)$$

contains a cycle and its width is $[w^{-1}(p), w^{-1}(d)]$. Since this width is properly contained in the width $[w^{-1}(a), w^{-1}(d)]$ of S, this contradicts the minimality of the width of S. Therefore $d > p'$.

By (A.7) and (A.8) we have

$$p > a > d > p' \qquad \text{(Figure 17)}. \tag{A.9}$$

Moreover, since a is the leftmost vertex of the cycle S and d is the rightmost vertex of S, it follows from (A.6) that

$$w^{-1}(a) < w^{-1}(p) < w^{-1}(p') < w^{-1}(d).$$

This together with (A.9) shows that if we take $b = p$ and $c = p'$, then $abcd$ is the pattern 4512. Moreover, since $(a, d) \in E_w$, there is no r such that $a > r > d$ and $w^{-1}(a) < w^{-1}(r) < w^{-1}(d)$, which means that $abcd$ is not a subsequence of 45312.

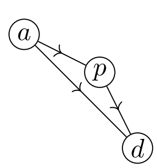


Figure 13

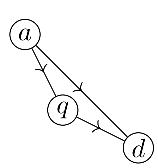


Figure 14

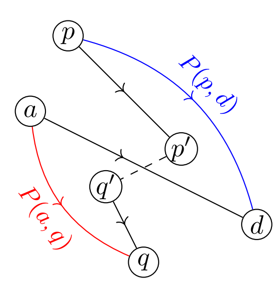


Figure 15

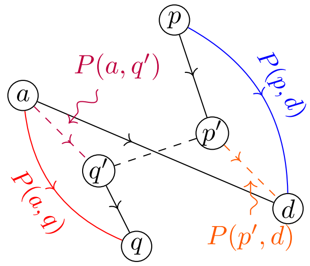


FIGURE 16

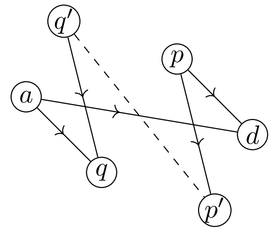


FIGURE 17

Case II. The case where a and d are not joined by an edge, i.e. $(a,d) \notin E_w$. In this case we will find vertices b and c of S such that $abcd$ is the pattern 4231. There are two edges of S emanating from a and we take vertices $p,q \in S$ such that $(a,p),(a,q) \in E_w$. We may assume

$$w^{-1}(p) < w^{-1}(q) \tag{A.10}$$

without loss of generality (Figures 18 and 19). Since $(a,p),(a,q) \in E_w$, we have

$$a > p, \qquad a > q.$$

We claim $q > p$ so that

$$a > q > p \qquad \text{(i.e. Figure 19 does not occur).} \tag{A.11}$$

Indeed, if $p > q$ (Figure 19), then $a > p > q$ and $w^{-1}(a) < w^{-1}(p) < w^{-1}(q)$ by (A.10). But this contradicts (a,q) being in E_w.

Now we take two cases.

(1) The case where $p > d$ (Figure 20). In this case $a > q > p > d$ by (A.11) and $w^{-1}(a) < w^{-1}(p) < w^{-1}(q) < w^{-1}(d)$ by (A.10). Therefore if we take $b = p$ and $c = q$, then $abcd$ is the pattern 4231.

(2) The case where $d > p$. We shall observe that this case does not occur. Let p' be the vertex of S such that the path

$$P(p,p')$$

in S joining p and p' is a maximal descending path from p to p' so that if p'' is the vertex next to p' and outside of $P(p,p')$, then $(p'',p') \in E_w$ (Figures 21 and 22). By the choice of p' and p'' we have

$$w^{-1}(p) \leq w^{-1}(p'), \quad p \geq p' \qquad \text{and} \qquad w^{-1}(p'') < w^{-1}(p'), \quad p'' > p'. \tag{A.12}$$

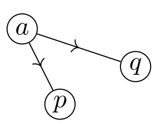


FIGURE 18

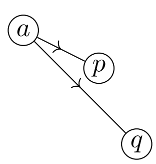


FIGURE 19

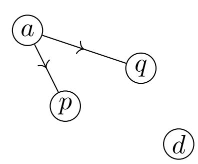


FIGURE 20

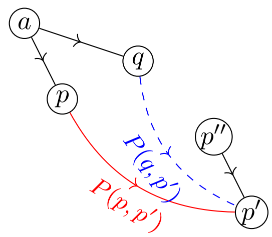


Figure 21

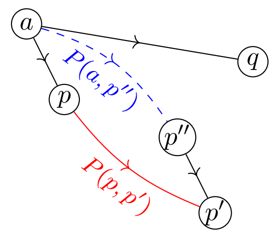


Figure 22

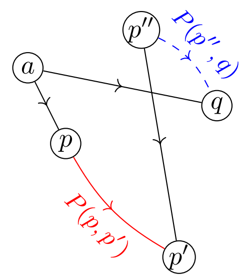


Figure 23

If $w^{-1}(q) < w^{-1}(p')$ (Figure 21), then there is a descending path $P(q, p')$ from q to p' in Γ_w because $q > p'$ by (A.11) and (A.12). Since $P(q, p')$ does not contain p by (A.10), the union $(a, q) \cup P(q, p')$ is a descending path from a to p' which does not contain p. On the other hand, the union $(a, p) \cup P(p, p')$ is also a descending path from a to p' but contains p. Therefore the union

$$\big((a, q) \cup P(q, p')\big) \cup \big((a, p) \cup P(p, p')\big)$$

contains a cycle and its width is $[w^{-1}(a), w^{-1}(p')]$. But this contradicts the minimality of the width of S. Therefore

$$w^{-1}(p') < w^{-1}(q) \qquad \text{(Figure 22)}. \tag{A.13}$$

By (A.12) and (A.13), we have

$$w^{-1}(p'') < w^{-1}(q).$$

If $a > p''$ (like Figure 22), then $(a, p'') \in \mathrm{Inv}(w)$ and hence there is a descending path $P(a, p'')$ from a to p'' in Γ_w. The union $P(a, p'') \cup (p'', p')$ is a descending path from a to p'. On the other hand, the union $(a, p) \cup P(p, p')$ is also a descending path from a to p' but does not contain p'' because p'' is the vertex of S next to p' but outside of $P(p, p')$. Therefore the union

$$\big(P(a, p'') \cup (p'', p')\big) \cup \big((a, p) \cup P(p, p')\big)$$

contains a cycle and its width is $[w^{-1}(a), w^{-1}(p')]$. But this contradicts the minimality of the width of S. Therefore $p'' > a$ and hence

$$p'' > a > q \qquad \text{(Figure 23)}$$

by (A.11). Then since $w^{-1}(p'') < w^{-1}(q)$ by (A.12) and (A.13), we have $(p'', q) \in \mathrm{Inv}(w)$ and hence there is a descending path $P(p'', q)$ from p'' to q in Γ_w. The union

$$P(p'', q) \cup (p'', p') \tag{A.14}$$

is a *path* joining p' and q (note that (p'', p') is an edge). Similarly, the union

$$(a, p) \cup P(p, p') \cup (a, q) \tag{A.15}$$

is also a *path* joining p' and q. However, the path in (A.14) does not contain the vertex a while the path in (A.15) does. Therefore the union of these two paths contains a cycle and its width is $[w^{-1}(a), w^{-1}(q)]$. This again contradicts the minimality of the width of S. Thus Case II (2) does not occur. □

Appendix B. Retraction sequences of a polytope and the Poincaré polynomial

In this appendix, we compute Poincaré polynomials of Y_w when $w = 4231$ and $w = 3412$. We recall the definition of *retraction sequences* of a polytope from [3, 4].

Definition B.1. A *polytopal complex* $\mathcal{C}$ is a finite collection of polytopes in $\mathbb{R}^n$ satisfying:

(1) if E is a face of F and $F \in \mathcal{C}$ then $E \in \mathcal{C}$.
(2) If $E, F \in \mathcal{C}$ then $E \cap F$ is a face of both E and F.

We denote the underlying set of a polytopal complex $\mathcal{C}$ by $|\mathcal{C}| = \bigcup_{F \in \mathcal{C}} F$. Given an n-dimensional polytope Q, the polytopal complex $\mathcal{C}(Q)$ is a collection of all faces of Q. For a polytopal complex $\mathcal{C}$, we call a vertex $v \in |\mathcal{C}|$ is a *free vertex* if it has a neighborhood in $|\mathcal{C}|$ that is diffeomorphic to $\mathbb{R}^N_{\geq 0}$ as manifolds with corners for some integer N.

Definition B.2 ([4, §2])**.** A *retraction sequence* of a polytope Q is a sequence of triples $\{(B_k, E_k, b_k)\}_{1 \leq k \leq \ell}$ defined inductively:

- $B_1 = Q = E_1$ and b_1 is a free vertex in Q, i.e. a simple vertex.
- Given (B_k, E_k, b_k), the next term $(B_{k+1}, E_{k+1}, b_{k+1})$ is defined to be:

$$B_{k+1} := |\{E \in \mathcal{C}(B_k) \mid b_k \notin V(E)\}|,$$

  b_{k+1} is a free vertex in B_{k+1}, and E_{k+1} is the maximal face of B_{k+1} containing b_{k+1}.

Here, ℓ is the number of vertices of Q.

It is known that every simple polytope admits a retraction sequence, see [4, Proposition 2.3]. Suppose that X is a projective toric variety whose underlying polytope is Q. If Q admits a retraction sequence, then we can construct a **q**-CW complex structure on X using the similar argument in the proof of [3, Proposition 4.4] (This observation will be deeply studied in [21]). We note that **q**-CW complex structure is a generalization of CW complex structure using the quotient of a disc by the action of a finite group rather than ordinary cells, see [19]. Since a retraction sequence $\{(B_k, E_k, b_k)\}_{1 \leq k \leq \ell}$ produces only even dimensional "cells" of dimensions $\{2\dim(E_k)\}_{1 \leq k \leq \ell}$, one can show the following:

Proposition B.3. *Suppose that X is a projective toric variety whose underlying polytope is Q. If Q admits a retraction sequence $\{(B_k, E_k, b_k)\}_{k=1}^{\ell}$ then the Poincaré polynomial of X agrees with*

$$\sum_{k=1}^{\ell} t^{2\dim(E_k)}.$$

When $w = 4231$, the Bruhat interval polytope $Q_{id,w^{-1}}$ is given in Figure 24. The vertices are labeled by the corresponding permutations, i.e. $\mu(uB) = (u^{-1}(1), \dots, u^{-1}(n))$ is labeled by u for $u \leq w$. The polytope $Q_{id,w^{-1}}$ is not simple but it admits a retraction sequence as in Figure 25. By Proposition B.3, the Poincaré polynomial of Y_w is $t^6 + 11t^4 + 7t^2 + 1$, which agrees with $A_w(t^2)$ (see Table 4).

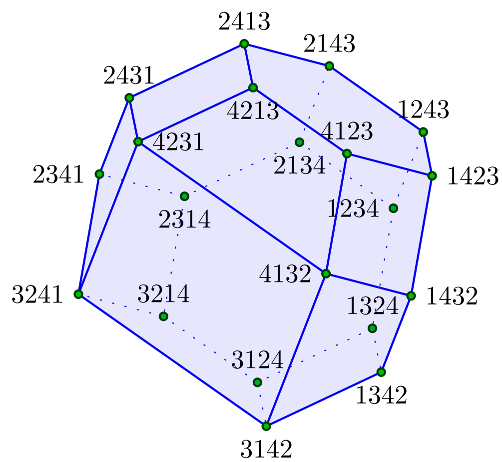


FIGURE 24. The Bruhat interval polytope $Q_{id,4231^{-1}} = Q_{id,4231}$.

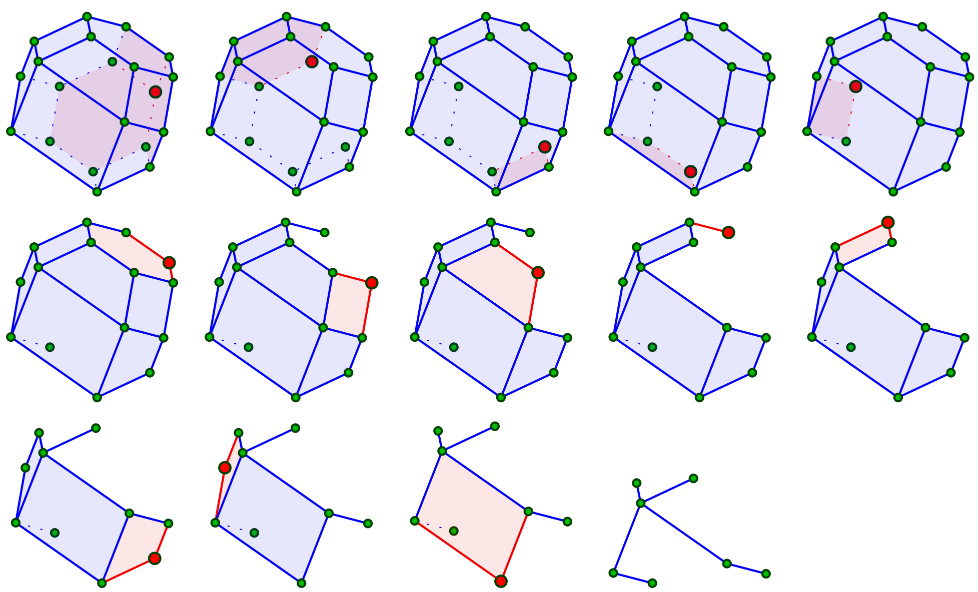


FIGURE 25. A retraction sequence $\{(B_k, E_k, b_k)\}$ of $Q_{id,4231}$.

b_k	1234	2134	1324	3124	2314	1243	1423	4123	2143	2413
$\dim(E_k)$	3	2	2	2	2	2	2	2	1	2
b_k	1342	2341	3142	1432	4132	3214	3241	4213	2431	4231
$\dim(E_k)$	2	2	2	1	1	1	1	1	1	0

TABLE 4. A sequence of vertices b_k and dimensions $\dim(E_k)$.

The Bruhat interval polytope $Q_{id,w^{-1}}$ for $w = 3412$ is given in Figure 26. The vertex $\mu(uB) = (u^{-1}(1), \dots, u^{-1}(n))$ is labeled by u for $u \leq w$. The polytope $Q_{id,w^{-1}}$ is not simple but it admits a retraction sequence as in Figure 27. By Proposition B.3, the Poincaré polynomial of Y_w is $t^6 + 7t^4 + 5t^2 + 1$, which agrees with $A_w(t^2)$ (see Table 5).

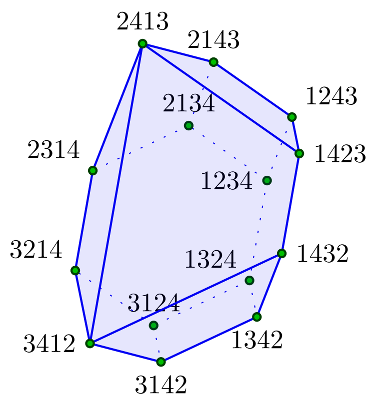


FIGURE 26. The Bruhat interval polytope $Q_{id,3412^{-1}} = Q_{id,3412}$.

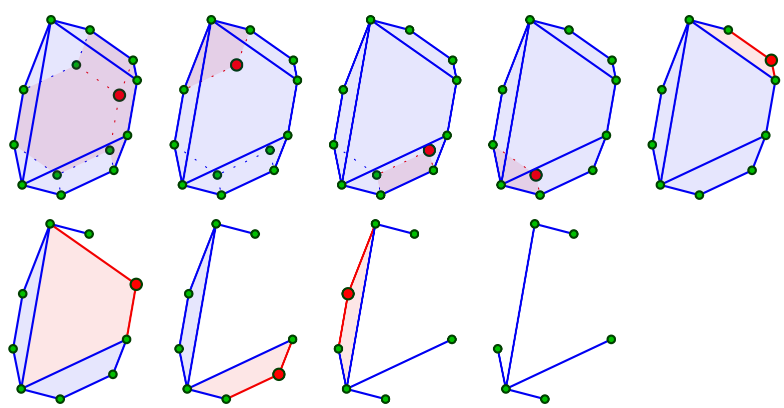


FIGURE 27. A retraction sequence $\{(B_k, E_k, b_k)\}$ of $Q_{id,3412}$.

b_k	1234	2134	1324	3124	1243	1423	1342
$\dim(E_k)$	3	2	2	2	2	2	2
b_k	2314	2143	2413	1432	3142	3214	3412
$\dim(E_k)$	2	1	1	1	1	1	0

TABLE 5. A sequence of vertices b_k and dimensions $\dim(E_k)$.

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